Strong Convergence Theorems of Generalized Viscosity Implicit Rules for Fixed Points of Total Asymptotically Nonexpansive Mappings in Hilbert Spaces

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***Abstract*—In this paper, we consider the generalized viscosity implicit rules for fixed points of total asymptotically nonexpansive mapping in Hilbert spaces, and obtain some strong convergence theorems under certain assumptions imposed on the parameters. The results presented in this paper extend and improve varieties of results in the recent literature.**

***Index Terms*—Total asymptotically nonexpansive mapping; Fixed points; Generalized viscosity implicit rules; Hilbert spaces.**

I. Introduction

THROUGHOUT this paper, we assume that C is a nonempty subset of real Hilbert space H. Let $T : C \to C$ be a mapping and $F(T)$ be the set of fixed points of T. Now we recall the following basic definitions.

Definition 1.1 A nonlinear mapping $T : C \to C$ is said to be

(i) contraction if there exists a constant $\alpha \in [0, 1)$ such that

$$\| T(x) - T(y) \| \leq \alpha \|x - y\|, \quad \forall x, y \in C;$$

when $\alpha = 1$, then T is called nonexpansive;

(ii) asymptotically nonexpansive if there exists a real number sequence $\{\mu_n\} \subseteq [0, +\infty)$ with $\lim_{n\to\infty} \mu_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\|, \quad \forall x, y \in C, n \geq 1;$$

(iii) asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0;$$

If we let $\xi_n = max\{0, \sup_{x,y\in C}(\|T^n x - T^n y\| - \|x - y\|)\}$, then $\lim_{n\to\infty} \xi_n = 0$, and relational expression of (iii) is reduced to

$$\|T^n x - T^n y\| \leq \|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1;$$

(iv) generalized asymptotically nonexpansive if there exist two real number sequences $\{\mu_n\}, \{\xi_n\} \subseteq [0, +\infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1;$$

(v) $(\{\mu_n\}, \{\xi_n\}, \zeta)$-total asymptotically nonexpansive if there exist two nonnegative real number sequences$\{\mu_n\}$ and $\{\xi_n\}$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\zeta : R^+ \to R^+$ with $\zeta(0) = 0$ such that, for any $x, y \in C$ and $n \geq 1$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \zeta(\|x - y\|) + \xi_n;$$

(vi) uniformly $L-$Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|, \quad \forall x, y \in C, n \geq 1.$$

Remark 1.1 If $\zeta(x) = x$, total asymptotically nonexpansive mappings coincides with generalized asymptotically nonexpansive mappings. In addition, if $\mu_n = 0$ for all $n \in N$, then generalized asymptotically nonexpansive mappings coincides with asymptotically nonexpansive mappings in the intermediate sense; if $\xi_n = 0$, then generalized asymptotically nonexpansive mappings coincides with asymptotically nonexpansive mappings; if $\mu_n = 0$ and $\xi_n = 0$, then we obtain nonexpansive mappings.

The interest and importance of construction of fixed points of nonlinear operators stem mainly from the fact that it have been widely applied to signal processing, imagine recovery, equilibrium problem, optimization problem and so on, see [1-5] and the references therein. Recently, the viscosity iterative algorithms have become an important tool for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings, an effective approach for finding the solutions of variational inequality problems, and they have been investigated by many authors; see [6-11] and the references therein. For instance, Xu [6] introduced the explicit viscosity method for nonexpansive mappings:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T x_n, \quad n \in N, \tag{1}$$

where $\alpha_n \in (0, 1)$ and f is a contraction. Under some suitable conditions on $\{\alpha_n\}$, he proved that the sequence

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$\{x_n\}$ generated by (1) converges strongly to $q \in F(T)$ in Hilbert spaces or uniformly smooth Banach spaces, which also solves the variational inequality:

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in F(T). \tag{2}$$

On the other hand, the implicit midpoint rule is one of the powerful numerical methods for solving ordinary differential equations and differential algebraic equations, see [12-16] and the references therein.

In 2015, Xu et al. [17] applied the viscosity technique to the implicit midpoint rule for nonexpansive mappings and presented the following viscosity implicit midpoint rule(VIMR):

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(\frac{x_n + x_{n+1}}{2}), \quad n \in N, \tag{3}$$

where $\alpha_n \in (0,1)$ and f is a contraction. They also proved that VIMR converges strongly to a fixed point of T, which is also the unique solution of the variational inequality (2).

In the same year, Yao et al. [18] presented a modified semi-implicit midpoint rule with the viscosity technique for nonexpansive mappings:

$$\begin{cases} w_n = \frac{1}{2}(x_n + x_{n+1}), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T w_n, \quad n \in N, \end{cases} \tag{4}$$

where $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1)$ and $\{\gamma_n\} \subset (0,1)$ are three sequences satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 0$. They proved that the suggested algorithm (4) converges strongly to a special fixed point of nonexpansive mappings under some different conditions.

Later on, Ke and Ma [19] developed the following generalized viscosity implicit scheme to approximate the fixed point of a nonexpansive mapping T in a Hilbert space:

$$\begin{cases} w_n = s_n x_n + (1-s_n)x_{n+1}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T w_n, \quad n \in N, \end{cases} \tag{5}$$

where f is a contraction, and sequences $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$, $\{s_n\}$ are in $(0,1)$ for all $n \in N$. With appropriate assumptions on control sequences, they established the strong convergence results for (5), and solved the variational inequality (2).

In 2018, Yan and Cai [20] introduced the following viscosity implicit midpoint scheme in a Hilbert space:

$$\begin{cases} w_n = \frac{1}{2}(x_n + x_{n+1}), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n w_n, \quad n \in N, \end{cases} \tag{6}$$

where $\{\alpha_n\}$,$\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$, f is a contractive mapping and T is an asymptotically nonexpansive mapping in the intermediate sense. They also proved that the sequence $\{x_n\}$ generated by (6) converges strongly to a point $p \in F(T)$, which is also the unique solution of the variational inequality (2).

Recently, Sang B Mendy et al. [21] studied the following implicit iterative algorithm in Hilbert space:

$$\begin{cases} w_n = s_n x_n + (1-s_n)x_{n+1}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n w_n, \quad n \in N, \end{cases} \tag{7}$$

where $\{\alpha_n\}$,$\{\beta_n\}$,$\{\gamma_n\}$ and $\{s_n\}$ are in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, f is a contractive mapping and T is an asymptotically nonexpansive mapping. Under suitable conditions, they proved that the sequence $\{x_n\}$ converge strongly to a fixed point of T, which also solves the variational inequality (2).

Motivated and inspired by the above work, in this paper we investigate the general viscosity implicit iteration generated by (7) for a total asymptotically nonexpansive mapping in Hilbert spaces. Under suitable assumptions imposed on the parameters, we obtain some strong convergence theorems for finding a fixed point of the total asymptotically nonexpansive mapping. The results we presented extend and improve the corresponding results of [20], [21] and others.

II. Preliminaries

Set H be a Hilbert space with inner product $\langle , \rangle$ and norm $\|\cdot\|$, respectively, and let C be a nonempty, closed, and convex subset of H. Then we have the nearest point projection from H onto C, P_C, defined by

$$P_C(x) := \arg\min_{z\in C} \|x-z\|^2, \quad x \in H.$$

Namely, $P_C(x)$ is the only point in C that minimizes the objective $\|x-z\|^2$ over $z \in C$. Note that $P_C(x)$ is characterized as follows:

$$P_C(x) \in C \quad and \quad \langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall y \in C.$$

In order to prove our results, we need the following lemmas and results.

Lemma 2.1 [22] Let E be a reflexive Banach space with weakly continuous normalised duality. Let C be a closed convex subset of E and $T : C \to C$ be a uniformly continuous total asymptotically nonexpansive mapping with bounded orbit, then $I - T$ is demiclosed at zero,where I is the identity mapping of E.

Lemma 2.2 [23] Assume that $\{a_n\}$ is a sequence of non-negative real numbers such that

$$a_{n+1} \leq (1-\delta_n)a_n + v_n, \quad \forall n \geq 0,$$

where $\{\delta_n\} \subseteq (0,1)$ and $\{v_n\} \subseteq R$ are two sequences such that:

(1) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{v_n}{\delta_n} \leq 0$ or $\sum_{n=1}^{\infty} |v_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

III. Main results

Theorem 3.1 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a uniformly Lipschitzian and $(\{\mu_n\}, \{\xi_n\}, \zeta)$-total asymptotically nonexpansive mapping with two sequences $\{\mu_n\}, \{\xi_n\} \subseteq [0, +\infty)$, and $f : C \to C$ a contraction with coefficient $\alpha \in [0,1)$. Pick any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by (7), where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and $F(T) \neq \emptyset$, and the following conditions hold:
(C1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $\lim_{n\to\infty} \gamma_n = 1$;
(C3) $\lim_{n\to\infty} \frac{\mu_n}{\alpha_n} = 0$;
(C4) $\sum_{n=1}^{\infty} \xi_n < +\infty$;
(C5) $s_n \in (0,1]$ for all $n \geq 0$ and $\lim_{n\to\infty} s_n = s \in (0,1]$;
(C6) there exists a constant $M^* > 0$ such that $\zeta(x) \leq M^* x$

for each $x \geq 0$.
Then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle (I-f)z, x-z\rangle \geq 0, \quad \forall x \in F(T).$$

Proof. We divide the proof into five steps as follows.

(I) we show that sequence $\{x_n\}$ is bounded.

For any $q \in F(T)$, from (7) and (C6) we have

$$\begin{aligned}
&\|x_{n+1}-q\| \\
= \;& \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n T^n w_n - q\| \\
\leq \;& \alpha_n\|f(x_n)-q\|+\beta_n\|x_n-q\|+\gamma_n\|T^n w_n-q\| \\
\leq \;& \alpha_n\|f(x_n)-f(q)\|+\alpha_n\|f(q)-q\|+\beta_n\|x_n-q\| \\
& +\gamma_n[\|w_n-q\|+\mu_n\zeta(\|w_n-q\|)+\xi_n] \\
\leq \;& \alpha_n\alpha\|x_n-q\|+\alpha_n\|f(q)-q\|+\beta_n\|x_n-q\| \\
& +\gamma_n[\|w_n-q\|+\mu_n M^*\|w_n-q\|+\xi_n] \\
\leq \;& \alpha_n\alpha\|x_n-q\|+\beta_n\|x_n-q\|+\gamma_n s_n\|x_n-q\| \\
& +\gamma_n(1-s_n)\|x_{n+1}-q\|+\gamma_n\mu_n M^* s_n\|x_n-q\| \\
& +\gamma_n\mu_n M^*(1-s_n)\|x_{n+1}-q\|+\alpha_n\|f(q)-q\| \\
& +\gamma_n\xi_n \\
= \;& [\alpha_n\alpha+\beta_n+\gamma_n s_n(1+\mu_n M^*)]\|x_n-q\| \\
& +\gamma_n(1-s_n)(1+\mu_n M^*)\|x_{n+1}-q\| \\
& +\alpha_n\|f(q)-q\|+\gamma_n\xi_n,
\end{aligned}$$

that is

$$\begin{aligned}
& [1-\gamma_n(1-s_n)(1+\mu_n M^*)]\|x_{n+1}-q\| \\
\leq \;& [\alpha_n\alpha+\beta_n+\gamma_n s_n(1+\mu_n M^*)]\|x_n-q\| \\
& +\alpha_n\|f(q)-q\|+\gamma_n\xi_n. \qquad (8)
\end{aligned}$$

From $\alpha_n+\beta_n+\gamma_n=1$ and (C2), we obtain that

$$\lim_{n\to\infty}\alpha_n=0; \qquad (9)$$

and

$$\lim_{n\to\infty}\beta_n=0. \qquad (10)$$

Since M^* is a constant, by conditions (C2), (C3), (C5) and (9), for any given positive number $\epsilon(0<\epsilon<1-\alpha)$, there exists a sufficiently large positive integer N such that, for any $n>N$,

$$\gamma_n\mu_n M^* \leq \mu_n M^* \leq \epsilon\alpha_n \qquad (11)$$

and

$$\frac{\gamma_n}{1-\gamma_n(1-s_n)(1+\mu_n M^*)} \leq \frac{2}{s}, \qquad (12)$$

where $\lim\limits_{n\to\infty}\frac{\gamma_n}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}=\frac{1}{s}$. From (8), (11) and (12), for any $n>N$, we have

$$\begin{aligned}
&\|x_{n+1}-q\| \\
\leq \;& \frac{\alpha_n\alpha+\beta_n+\gamma_n s_n(1+\mu_n M^*)}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}\|x_n-q\| \\
& +\frac{\alpha_n\|f(q)-q\|}{1-\gamma_n(1-s_n)(1+\mu_n M^*)} \\
& +\frac{\gamma_n\xi_n}{1-\gamma_n(1-s_n)(1+\mu_n M^*)} \\
= \;& [1+\frac{\alpha_n\alpha-\alpha_n+\gamma_n\mu_n M^*}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}]\|x_n-q\| \\
& +\frac{\alpha_n\|f(q)-q\|}{1-\gamma_n(1-s_n)(1+\mu_n M^*)} \\
& +\frac{\gamma_n}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}\xi_n \\
\leq \;& [1+\frac{\alpha_n\alpha-\alpha_n+\epsilon\alpha_n}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}]\|x_n-q\| \\
& +\frac{\alpha_n\|f(q)-q\|}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}+\frac{2}{s}\xi_n \\
= \;& [1-\frac{\alpha_n(1-\alpha-\epsilon)}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}]\|x_n-q\| \\
& +\frac{\alpha_n(1-\alpha-\epsilon)}{1-\gamma_n(1-s_n)(1+\mu_n M^*)}\frac{\|f(q)-q\|}{1-\alpha-\epsilon} \\
& +\frac{2}{s}\xi_n \\
\leq \;& max\{\|x_n-q\|,\frac{\|f(q)-q\|}{1-\alpha-\epsilon}\}+\frac{2}{s}\xi_n.
\end{aligned}$$

By induction, it follows that

$$\begin{aligned}
&\|x_n-q\| \\
\leq \;& max\{\|x_1-q\|,\frac{\|f(q)-q\|}{1-\alpha-\epsilon}\} \\
& +\frac{2}{s}(\xi_1+\xi_2+\ldots+\xi_{n-1}) \\
\leq \;& max\{\|x_1-q\|,\frac{\|f(q)-q\|}{1-\alpha-\epsilon}\}+\frac{2}{s}\sum_{n=1}^{\infty}\xi_n. \qquad (13)
\end{aligned}$$

From (C4) and (13), we know that the sequence $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{T^n(x_n)\}$.

(II) we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Indeed, it follows from (7) that

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
\leq \;& \|x_{n+1}-T^n x_n\|+\|T^n x_n-x_n\| \\
\leq \;& \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n T^n w_n-T^n x_n\| \\
& +\|T^n x_n-x_n\| \\
\leq \;& \alpha_n\|f(x_n)-T^n x_n\|+\beta_n\|x_n-T^n x_n\| \\
& +\gamma_n\|T^n w_n-T^n x_n\|+\|T^n x_n-x_n\| \\
\leq \;& \alpha_n\|f(x_n)-T^n x_n\|+(1+\beta_n)\|x_n-T^n x_n\| \\
& +\gamma_n[\|w_n-x_n\|+\mu_n\zeta(\|w_n-x_n\|)+\xi_n] \\
\leq \;& \alpha_n\|f(x_n)-T^n x_n\|+(1+\beta_n)\|x_n-T^n x_n\| \\
& +\gamma_n(1-s_n)\|x_{n+1}-x_n\|+\gamma_n\mu_n M^*(1-s_n) \\
& \times\|x_{n+1}-x_n\|+\gamma_n\xi_n \\
\leq \;& \alpha_n\|f(x_n)-T^n x_n\|+(1+\beta_n)\|x_n-T^n x_n\| \\
& +\gamma_n(1-s_n)(1+\mu_n M^*)\|x_{n+1}-x_n\|+\xi_n.
\end{aligned}$$

Since $\{f(x_n)\}$ and $\{T^n x_n\}$ are bounded, there exists $K>0$

such that $\sup_{n\geq 1}\|f(x_n)-T^nx_n\|\leq K$. Thus,

$$\begin{aligned}&\|x_{n+1}-x_n\|\\ \leq\;& \alpha_nK+(1+\beta_n)\|x_n-T^nx_n\|\\ &+\gamma_n(1-s_n)(1+\mu_nM^*)\|x_{n+1}-x_n\|+\xi_n.\end{aligned}$$

It tours out that

$$\begin{aligned}\|x_{n+1}-x_n\| \leq\;& \frac{\alpha_nK}{1-\gamma_n(1-s_n)(1+\mu_nM^*)}\\ &+\frac{(1+\beta_n)\|x_n-T^nx_n\|}{1-\gamma_n(1-s_n)(1+\mu_nM^*)}\\ &+\frac{\xi_n}{1-\gamma_n(1-s_n)(1+\mu_nM^*)}. \quad (14)\end{aligned}$$

By condition (C4), we can see

$$\lim_{n\to\infty}\xi_n=0. \quad (15)$$

From (9), (14), (15) and $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$, we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \quad (16)$$

(III) we claim that $\lim_{n\to\infty}\|Tx_n-x_n\|=0$.

From (7) we get

$$\begin{aligned}&\|x_{n+1}-T^nx_{n+1}\|\\ \leq\;& \alpha_n\|f(x_n)-T^nx_{n+1}\|+\beta_n\|x_n-T^nx_{n+1}\|\\ &+\gamma_n\|T^nw_n-T^nx_{n+1}\|\\ \leq\;& \alpha_n\|f(x_n)-T^nx_{n+1}\|+\beta_n\|x_n-T^nx_{n+1}\|\\ &+\gamma_n\|w_n-x_{n+1}\|+\gamma_n\mu_n\zeta(\|w_n-x_{n+1}\|)+\gamma_n\xi_n\\ \leq\;& \alpha_nK+\beta_n\|x_n-T^nx_{n+1}\|+\gamma_ns_n(1+\mu_nM^*)\\ &\times\|x_n-x_{n+1}\|+\xi_n. \quad (17)\end{aligned}$$

By (15)-(17) and (9)-(10) we have

$$\lim_{n\to\infty}\|x_{n+1}-T^nx_{n+1}\|=0. \quad (18)$$

Since T is uniformly continuous mapping, we obtain that

$$\lim_{n\to\infty}\|Tx_{n+1}-T^{n+1}x_{n+1}\|=0. \quad (19)$$

Moreover,

$$\begin{aligned}\|Tx_{n+1}-x_{n+1}\| \leq\;& \|Tx_{n+1}-T^{n+1}x_{n+1}\|\\ &+\|T^{n+1}x_{n+1}-x_{n+1}\|. \quad (20)\end{aligned}$$

From (19), (20) and $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$, we get that $\lim_{n\to\infty}\|Tx_{n+1}-x_{n+1}\|=0$, which implies that

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0. \quad (21)$$

(IV) we show that

$$\langle z-f(z),z-x_n\rangle\leq 0, \quad (22)$$

where $z=P_{F(T)}f(z)$.

Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle z-f(z),z-x_n\rangle=\limsup_{i\to\infty}\langle z-f(z),z-x_{n_i}\rangle$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to x^*. Without loss of generality, we may assume that $x_{n_i}\rightharpoonup x^*$ as $i\to\infty$. From (21) we have $\lim_{i\to\infty}\|Tx_{n_i}-x_{n_i}\|=0$, and by using Lemma 2.1 we obtain that $x^*=Tx^*$, that is, $x^*\in F(T)$. This together with the property of the metric projection implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle z-f(z),z-x_n\rangle &= \limsup_{i\to\infty}\langle z-f(z),z-x_{n_i}\rangle\\ &= \langle z-f(z),z-x^*\rangle\leq 0.\end{aligned}$$

(V) We prove that $x_n\to z$ as $n\to\infty$, where $z\in F(T)$ is the unique fixed point of contraction $P_{F(T)}f$, that is, $z=P_{F(T)}f(z)$.

Since $\{x_n\}$ is bounded, there exists $M>0$ such that $\sup_{n\geq 1}\|x_n-z\|\leq M$, and from (7) we have

$$\begin{aligned}&\|x_{n+1}-z\|^2\\ =\;& \langle x_{n+1}-z,x_{n+1}-z\rangle\\ =\;& \langle\alpha_nf(x_n)+\beta_nx_n+\gamma_nT^nw_n-z,x_{n+1}-z\rangle\\ =\;& \alpha_n\langle f(x_n)-f(z),x_{n+1}-z\rangle+\alpha_n\langle f(z)-z,x_{n+1}\\ &-z\rangle+\beta_n\langle x_n-z,x_{n+1}-z\rangle+\gamma_n\langle T^nw_n-z,\\ &x_{n+1}-z\rangle\\ \leq\;& \alpha_n\|f(x_n)-f(z)\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z,\\ &x_{n+1}-z\rangle+\beta_n\|x_n-z\|\|x_{n+1}-z\|\\ &+\gamma_n\|T^nw_n-z\|\|x_{n+1}-z\|\\ \leq\;& \alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &+\beta_n\|x_n-z\|\|x_{n+1}-z\|+\gamma_n[\|w_n-z\|\\ &+\mu_n\zeta(\|w_n-z\|)+\xi_n]\|x_{n+1}-z\|\\ \leq\;& \alpha_n\alpha\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &+\beta_n\|x_n-z\|\|x_{n+1}-z\|+\gamma_ns_n\|x_n-z\|\\ &\times\|x_{n+1}-z\|+\gamma_n(1-s_n)\|x_{n+1}-z\|^2\\ &+\gamma_n\mu_nM^*\|s_nx_n+(1-s_n)x_{n+1}-z\|\|x_{n+1}-z\|\\ &+\gamma_n\xi_n\|x_{n+1}-z\|\\ \leq\;& (\alpha_n\alpha+\beta_n+\gamma_ns_n+\gamma_ns_n\mu_nM^*)\|x_n-z\|\\ &\times\|x_{n+1}-z\|+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &+[\gamma_n(1-s_n)(1+\mu_nM^*)]\|x_{n+1}-z\|^2\\ &+\gamma_n\xi_n\|x_{n+1}-z\|\\ \leq\;& \frac{\alpha_n\alpha+\beta_n+\gamma_ns_n(1+\mu_nM^*)}{2}(\|x_n-z\|^2\\ &+\|x_{n+1}-z\|^2)+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle\\ &+[\gamma_n(1-s_n)(1+\mu_nM^*)]\|x_{n+1}-z\|^2\\ &+\gamma_n\xi_n\|x_{n+1}-z\|\\ \leq\;& \frac{\alpha_n\alpha+\beta_n+\gamma_ns_n(1+\mu_nM^*)}{2}\|x_n-z\|^2\\ &+\frac{\alpha_n\alpha+\beta_n+\gamma_n(2-s_n)(1+\mu_nM^*)}{2}\|x_{n+1}-z\|^2\\ &+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle+\gamma_n\xi_n\|x_{n+1}-z\|\\ \leq\;& \frac{\alpha_n\alpha+\beta_n+\gamma_ns_n(1+\mu_nM^*)}{2}\|x_n-z\|^2\\ &+\frac{\alpha_n\alpha+\beta_n+\gamma_n(2-s_n)(1+\mu_nM^*)}{2}\|x_{n+1}-z\|^2\\ &+\alpha_n\langle f(z)-z,x_{n+1}-z\rangle+M\xi_n,\end{aligned}$$

which implies

$$\begin{aligned}&\|x_{n+1}-z\|^2\\ \leq\;& \frac{\alpha_n\alpha+\beta_n+\gamma_ns_n(1+\mu_nM^*)}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}\|x_n-z\|^2\\ &+\frac{2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle+2M\xi_n}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)} \quad (23)\end{aligned}$$

From conditions (C2), (C3), (C5) and (9), it follows that $\lim_{n\to\infty}[2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)]=s>0$, and for sufficiently large $n>N$, we have

$$2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)>0,$$

By (11) and (23) we know that

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
\le\ & [1+\frac{2\alpha_n\alpha+2\beta_n+2\gamma_n-2+2\gamma_n\mu_nM^*}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}]\\
&\times\|x_n-z\|^2\\
&+\frac{2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle+2M\xi_n}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}\\
=\ & [1+\frac{2\alpha_n\alpha-2\alpha_n+2\gamma_n\mu_nM^*}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}]\\
&\times\|x_n-z\|^2\\
&+\frac{2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle+2M\xi_n}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}\\
\le\ & [1+\frac{2\alpha_n\alpha-2\alpha_n+2\epsilon\alpha_n}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}]\\
&\times\|x_n-z\|^2\\
&+\frac{2\alpha_n\langle f(z)-z,x_{n+1}-z\rangle+2M\xi_n}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}\\
=\ & [1-\frac{2\alpha_n(1-\alpha-\epsilon)}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}]\\
&\times\|x_n-z\|^2\\
&+\frac{2\alpha_n(\langle f(z)-z,x_{n+1}-z\rangle+M\frac{\xi_n}{\alpha_n})}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}\\
=\ & (1-\delta_n)\|x_n-z\|^2+v_n,
\end{aligned}\tag{24}$$

where

$$\delta_n=\frac{2\alpha_n(1-\alpha-\epsilon)}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)},$$

$$v_n=\frac{2\alpha_n(\langle f(z)-z,x_{n+1}-z\rangle+M\frac{\xi_n}{\alpha_n})}{2-\alpha_n\alpha-\beta_n-\gamma_n(2-s_n)(1+\mu_nM^*)}.$$

By conditions (C1) and (C4), we get that

$$\lim_{n\to\infty}\frac{\xi_n}{\alpha_n}=0. \tag{25}$$

From (C1)-(C5), (22) and (25), we have $\{\delta_n\}\subset(0,1)$, $\sum_{n=1}^{\infty}\delta_n=\infty$, and

$$\limsup_{n\to\infty}\frac{v_n}{\delta_n}=\limsup_{n\to\infty}\frac{\langle f(z)-z,x_{n+1}-z\rangle+M\frac{\xi_n}{\alpha_n}}{1-\alpha-\epsilon}\le 0.$$

It following form (24) and Lemma 2.2, we obtain that $x_n\to z=P_{F(T)}f(z)$, which solves the following variational inequality:

$$\langle z-f(z),x-z\rangle\ge 0,\quad \forall x\in F(T).$$

The proof is completed.

Corollary 3.2 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T:C\to C$ be a uniformly Lipschitzian and $(\{\mu_n\},\{\xi_n\},\zeta)$-total asymptotically nonexpansive mapping with two sequences $\{\mu_n\},\{\xi_n\}\subseteq[0,+\infty)$, and $f:C\to C$ a contraction with coefficient $\alpha\in[0,1)$. Pick any $x_1\in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n=\frac{1}{2}(x_n+x_{n+1}),\\ x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_nT^nw_n,\quad n\in N,\end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$. If $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$ and $F(T)\neq\emptyset$, and conditions (C1)-(C4) and (C6) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle(I-f)z,x-z\rangle\ge 0,\quad \forall x\in F(T).$$

Proof. Take $s_n=\frac{1}{2}$ for any $n\ge 1$ in Theorem 3.1, then condition (C5) in Theorem 3.1 holds. From Theorem 3.1, the proof is completed.

If $T:C\to C$ is a generalized asymptotically nonexpansive mapping, we can obtain the following two results from Theorem 3.1.

Corollary 3.3 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T:C\to C$ be a uniformly Lipschitzian and generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\},\{\xi_n\}\subseteq[0,+\infty)$, and $f:C\to C$ a contraction with coefficient $\alpha\in[0,1)$. Pick any $x_1\in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n=s_nx_n+(1-s_n)x_{n+1},\\ x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_nT^nw_n,\quad n\in N,\end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$. If $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$ and $F(T)\neq\emptyset$, and conditions (C1)-(C5) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle(I-f)z,x-z\rangle\ge 0,\quad \forall x\in F(T).$$

Proof. Take $\zeta(x)=x(x\ge 0)$ in Theorem 3.1, then condition (C6) in Theorem 3.1 is satisfied automatically. Hence the conclusion of Corollary 3.3 can be obtained from Theorem 3.1 immediately.

Corollary 3.4 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T:C\to C$ be a uniformly Lipschitzian and generalized asymptotically nonexpansive mapping with sequences $\{\mu_n\},\{\xi_n\}\subseteq[0,+\infty)$, and $f:C\to C$ a contraction with coefficient $\alpha\in[0,1)$. Pick any $x_1\in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n=\frac{1}{2}(x_n+x_{n+1}),\\ x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+\gamma_nT^nw_n,\quad n\in N,\end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ are three sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n=1$. If $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$ and $F(T)\neq\emptyset$, and conditions (C1)-(C4) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle(I-f)z,x-z\rangle\ge 0,\quad \forall x\in F(T).$$

Proof. Take $s_n=\frac{1}{2}$ in Corollary 3.3, then condition (C5) in Corollary 3.3 holds. From Corollary 3.3, this completes the proof.

If $T : C \to C$ is an asymptotically nonexpansive mapping in the intermediate, we can obtain the following two corollaries.

Corollary 3.5 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a asymptotically nonexpansive mapping in the intermediate sense with sequence $\{\xi_n\} \subseteq [0, +\infty)$, and $f : C \to C$ a contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = s_n x_n + (1 - s_n)x_{n+1}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n w_n, \quad n \in N, \end{cases} \tag{26}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and $F(T) \neq \emptyset$, and conditions (C1), (C2), (C4) and (C5) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle (I - f)z, x - z\rangle \geq 0, \quad \forall x \in F(T).$$

Proof. Take $\mu_n = 0$ and $\zeta(x) = x(x \geq 0)$, then conditions (C3) and (C6) in Theorem 3.1 are satisfied automatically. Hence the conclusion of Corollary 3.5 can be obtained from Theorem 3.1.

Remark 3.1 Corollary 3.2-3.5 still are new consequences.

Corollary 3.6 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a asymptotically nonexpansive mapping in the intermediate sense with sequence $\{\xi_n\} \subseteq [0, +\infty)$, and $f : C \to C$ a contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = \frac{1}{2}(x_n + x_{n+1}), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n w_n, \quad n \in N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and $F(T) \neq \emptyset$, and conditions (C1), (C2) and (C4) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle (I - f)z, x - z\rangle \geq 0, \quad \forall x \in F(T).$$

Proof. Take $s_n = \frac{1}{2}$, then condition (C4) in Corollary 3.5 holds. From conclusion of Corollary 3.6 can be obtained from Corollary 3.5 immediately.

Remark 3.2 Corollary 3.6 improves and extends the main results of [20] in regard to parameter α_n.

If $T : C \to C$ is an asymptotically nonexpansive mapping, we have the following two results.

Corollary 3.7 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a asymptotically nonexpansive mapping with sequence $\{\mu_n\} \subseteq [0, +\infty)$, and $f : C \to C$ a contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = s_n x_n + (1 - s_n)x_{n+1}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n w_n, \quad n \in N, \end{cases} \tag{27}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and $F(T) \neq \emptyset$, and conditions (C1), (C2) (C3) and (C5) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle (I - f)z, x - z\rangle \geq 0, \quad \forall x \in F(T).$$

Proof. Take $\xi_n = 0$ in Corollary 3.3, then condition (C4) in Corollary 3.3 is satisfied automatically, this completes the proof.

Remark 3.3 Corollary 3.7 studied the strongly convergence theorem without the monotonic increase of sequence $\{s_n\}$, and so improves and extends the main results in [21].

Corollary 3.8 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $T : C \to C$ be a asymptotically nonexpansive mapping with sequence $\{\mu_n\} \subseteq [0, +\infty)$, and $f : C \to C$ a contraction with coefficient $\alpha \in [0, 1)$. Pick any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = \frac{1}{2}(x_n + x_{n+1}), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T^n w_n, \quad n \in N, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and $F(T) \neq \emptyset$, and conditions (C1), (C2) and (C3) in Theorem 3.1 hold, then $\{x_n\}$ converges strongly to a fixed point z of T, which is the unique solution of the variational inequality

$$\langle (I - f)z, x - z\rangle \geq 0, \quad \forall x \in F(T).$$

Proof. Take $s_n = \frac{1}{2}$, then condition (C5) in Corollary 3.7 holds. From Corollary 3.7 we can be obtained from Corollary 3.5 immediately.

IV. Application to variational inequalities

Assume that C is a nonempty closed convex subset of a real Hilbert space H. Let $A : H \to H$ be a single-valued monotone operator such that $C \subset dom(A)$. Next we consider the following variational inequality (VI):

$$\langle Ax_0, x - x_0\rangle \geq 0, \quad x \in C. \tag{28}$$

Notice that VI (28) is equivalent to the fixed point problem, for any $\lambda > 0$,

$$P_C(I - \lambda A)x_0 = x_0. \tag{29}$$

Definition 4.1 A nonlinear mapping $A : H \to H$ is L-Lipschitzian for some $L > 0$, if

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in H.$$

Definition 4.2 A nonlinear mapping $A : H \to H$ is η-inverse-strongly monotone, for some $\eta > 0$, if

$$\langle Ax - Ay, x - y\rangle \eta \|Ax - Ay\|^2, \quad \forall x, y \in H.$$

If A is Lipschitzian and η-inverse-strongly monotone, it is well known [24] that the operator $T = P_C(I - \lambda A)$ is nonexpansive provided $0 < \lambda < 2\eta$. Thus, we can get the following theorem.

Theorem 4.1 Assume that C is a nonempty closed convex subset of the real Hilbert space H. Let $A : H \to H$ be a L-Lipschitzian and η-inverse-strongly monotone mapping and $f : C \to C$ a contraction with coefficient $\alpha \in [0, 1)$. Assume VI(28) is solvable. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = s_n x_n + (1 - s_n)x_{n+1}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C(I - \lambda A)w_n, \end{cases} \quad (30)$$

where $0 < \lambda < 2\eta$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. If $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$ and (C1), (C2) and (C5) in Theorem 3.1 hold. Then $\{x_n\}$ converges strongly to a solution x_0 of VI (28), which is also a solution to the variational inequality

$$\langle (I - f)x_0, x - x_0\rangle \geq 0, \quad x \in A^{-1}(0). \quad (31)$$

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