Locating Chromatic Number of Palm Graph

Des Welyyanti, Ibrahim Taufiqurrahman, Dony Permana, Rifda Sasmi Zahra, and Lyra Yulianti

***Abstract*—The locating chromatic number was introduced by Chartrand in 2002. Let $G = (V, E)$ be a connected graph and let c be a coloring of G. If distinct vertices in G have distinct color codes, then c is called the locating coloring of G. The locating chromatic number of a graph, denoted as $\chi_L(G)$, is the minimum number of colors in a locating coloring of G. There have been several studies about the locating chromatic number of certain graphs. In this paper, we discuss the locating chromatic number of palm graph $C_kP_lS_m$ with $k \geq 3, l \geq 2, m \geq 2$.**

***Index Terms*—Color code, connected graph, locating chromatic number, palm graph.**

I. Introduction

THE concept of the locating chromatic number was first introduced by Chartrand et al. [4] in 2002. They determined the locating chromatic number for various graphs. Specifically, for the cycle graph C_n with $n \geq 3$, they found that $\chi_L(C_n) = 3$ for odd n, and $\chi_L(C_n) = 4$ for even n. Similarly, for the path graph P_n where $n \geq 3$, $\chi_L(P_n) = 3$ was established. This topic has since garnered significant interest from many researchers. In 2012, Asmiati et al. [2] determined the locating chromatic number of the firecrackers graph. Concurrently, Asmiati and Baskoro [1] characterized all graphs containing cycles with a locating chromatic number of three. Expanding on this work, in 2014, Welyyanti et al. [7] generalized the definition of the locating chromatic number to encompass all graph types, both connected and disconnected graphs. They also explored the locating chromatic number of disconnected graphs with paths and cycles as its components [9]. In 2018, Welyyanti [6] identified several sufficient conditions for the finite locating chromatic number of disconnected graphs, and further discussions on the locating chromatic number of such graphs including paths, cycles, stars, and double stars were conducted [8]. Additionally, Zikra [10] examined the locating chromatic number of disconnected graphs with fan graphs as components. In 2021, Irawan et al. [3] determined the locating chromatic number for origami graphs.

In this paper, we aim to determine the locating chromatic number of the palm graph $C_kP_lS_m$ for $k \geq 3, l \geq 2$, and $m \geq 2$.

II. Palm Graph

The definition of the graph is sourced from [5]. A palm graph, denoted as $H = C_kP_lS_m$, consists of a cycle C_k, a path P_l, and a star S_m for $k \geq 3$ and $l, m \geq 2$ with two additional edges. The vertex set of H is defined as follows.

$$\begin{aligned} V(H) &= \{u_i | 1 \leq i \leq k\} \cup \{w_j | 1 \leq j \leq l\} \\ &\quad \cup \{v_t | 0 \leq t \leq m\}, \\ E(H) &= \{u_a u_{a+1} | 1 \leq a \leq k-1\} \cup \{u_1 u_k\} \\ &\quad \cup \{w_b w_{b+1} | 1 \leq b \leq l-1\} \\ &\quad \cup \{v_0 v_c | 1 \leq c \leq m\} \cup \{u_1 w_l, w_1 v_0\}. \end{aligned}$$

The palm graph $H = C_kP_lS_m$ for $k \geq 3$ and $l, m \geq 2$ is presented in Figure 1.

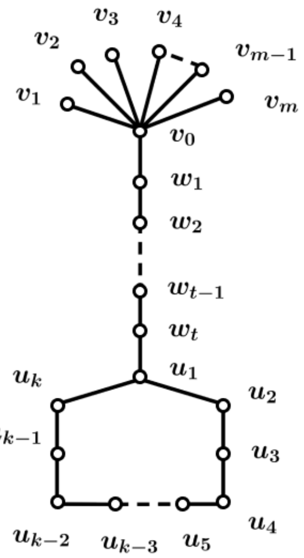

Fig. 1. Palm Graph ($C_kP_lS_m$) for $k \geq 3$ and $l, m \geq 2$.

III. Locating Chromatic Number of a Graph

Let c denote a coloring of a graph G wherein $c(u) \neq c(v)$ for any two adjacent vertices u and v within G. Define O_i as the set of vertices colored with color i, referred to as the color class. Consequently, $\Pi = \{O_1, O_2, \ldots, O_k\}$ represents the set of color classes from $V(G)$. The color code of a vertex v, denoted by $c_\Pi(v)$, is the k-tuple $c_\Pi(v) = (d(v, O_1), d(v, O_2), \ldots, d(v, O_k))$, where $d(v, O_i) = \min\{d(v, x) | x \in O_i\}$ for $1 \leq i \leq k$. If every pair of distinct vertices in G possesses unique color codes, then the coloring c is termed a locating coloring of G.

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The least number of colors required for a locating coloring is known as the locating chromatic number of G, denoted as $\chi_L(G)$. Given that every locating coloring is a proper coloring, it follows that $\chi(G) \leq \chi_L(G)$, as stated in [4]. The subsequent theorem and corollary, sourced from [4], are related to the locating chromatic number of a graph.

Theorem 3.1: [4] Let c be the locating coloring of a connected graph G. If u and v are two distinct vertices in G such that $d(u, w) = d(v, w)$ for every $w \in V(G) - \{u, v\}$, then $c(u) \neq c(v)$. In particular, if u and v are non-adjacent vertices of G such that $N(u) = N(v)$, then $c(u) \neq c(v)$.

Corollary 3.1: [4] Suppose that G is a connected graph with one vertex on k leaves, then $\chi_L(G) \geq k + 1$.

IV. Locating Chromatic Number of Palm Graph

In Theorem 4.1, we explore the locating chromatic number of a palm graph denoted as $H = C_kP_lS_m$, where $k \geq 3$ and $l, m \geq 2$.

Theorem 4.1: Let $H = C_kP_lS_m$ be a palm graph for $k \geq 3$ and $l, m \geq 2$, then

$$\chi_L(H) = \begin{cases} 4, & for\ m = 2, \\ m + 1, & for\ m \geq 3. \end{cases}$$

Proof: Let us consider a palm graph $H = C_kP_lS_m$ for given $k \geq 3$ and $l, m \geq 2$. The proof is divided into two principal cases.

Case 1. For $m = 2$

First, we determine the lower bound of $\chi_L(C_kP_lS_m)$ for $k \geq 3, l \geq 2$, and $m = 2$. Suppose $H = C_kP_lS_m$ for $m = 2$ and $k = 3$, with a 3-locating coloring. Since graph C_3 is a complete graph, then u_1, u_2, u_3 in graph C_3 represent the dominant vertices. Given that H has two vertices with maximum degrees of 3, namely v_0 and u_1, where v_0 is considered a dominant vertex. Thus, H is found to have four dominant vertices. This leads to the conclusion that there must be at least two dominant vertices sharing the same color, and consequently, the same color code. Hence, $\chi_L(H) \geq 4$.

Next, we determine the upper bound of $\chi_L(C_kP_lS_m)$ for $k \geq 3, l \geq 2$, and $m = 2$ and divide it into several cases.

Case 1.1. For odd k and odd l

Define a coloring $c_1 : V(H) \to \{1, 2, 3, 4\}$, for $1 \leq s \leq k$ and $1 \leq t \leq l$ where k, l are odd such that

$$\begin{aligned} c_1(v_0) &= 1, \\ c_1(v_1) &= 2, \\ c_1(v_2) &= 3, \\ c_1(u_s) &= \begin{cases} 2, & for\ s = 1, \\ 3, & for\ even\ s, \\ 1, & otherwise, \end{cases} \\ c_1(w_t) &= \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases} \end{aligned}$$

This coloring for a palm graph $H = C_kP_lS_m$ with $m = 2$ and odd values for k and l, is illustrated in Figure 2.

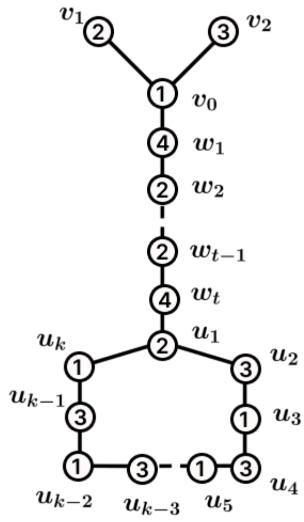

Fig. 2. Palm Graph ($C_kP_lS_m$) for $m = 2$, odd k, and odd l.

As shown in Figure 2, consider the vertex v_0 where $c(v_0) = 1$. The shortest distance from v_0 to vertices of other colors, excluding its own color (color 1), is $d(v_0, v_1) = d(v_0, v_2) = d(v_0, w_1) = 1$. This indicates that the shortest distance from v_0 to colors 2, 3, and 4 is the same, which is 1. Thus, the color code for vertex v_0 is $c_\Pi(v_0) = (0, 1, 1, 1)$. Therefore, we can say that v_0 is a dominant vertex with color 1. This method also applies to the other vertices as well. Next, let $\Pi_1 = \{O_1, O_2, O_3, O_4\}$ be a partition of $V(H)$. Based on Figure 2, we can generalize the color codes for each vertex as follows:

$$\begin{aligned} c_\Pi(v_0) &= (0, 1, 1, 1), \\ c_\Pi(v_1) &= (1, 0, 2, 2), \\ c_\Pi(v_2) &= (1, 2, 0, 2), \\ c_\Pi(u_1) &= (1, 0, 1, 1), \\ c_\Pi(u_s) &= (1 - s\ mod\ 2, s - 1, s\ mod\ 2, s), \\ &\quad for\ 1 < s \leq \frac{k+1}{2} \\ c_\Pi(u_s) &= (1 - s\ mod\ 2, k + 1 - s, s\ mod\ 2, k + 2 - s), \\ &\quad for\ s > \frac{k+1}{2} \\ c_\Pi(w_t) &= (t, t\ mod\ 2, t + 1, 1 - t\ mod\ 2),\ for\ t \leq \frac{l+1}{2} \\ c_\Pi(w_t) &= (l + 2 - t, t\ mod\ 2, l + 2 - t, 1 - t\ mod\ 2), \\ &\quad for\ t > \frac{l+1}{2}, \end{aligned}$$

Based on the color codes above, all vertices of $H = C_kP_lS_m$ for $m = 2$ and odd values for k and l have different color codes. Therefore, it is established that $\chi_L(H) \leq 4$.

Case 1.2. For odd k and even l

Define a coloring $c_2 : V(H) \to \{1, 2, 3, 4\}$, for $1 \leq s \leq k$ and $1 \leq t \leq l$ where k is odd and l is even such that

$$\begin{aligned} c_2(v_0) &= 1, \\ c_2(v_1) &= 2, \\ c_2(v_2) &= 3, \end{aligned}$$

$$c_2(u_s) = \begin{cases} 4, & for\ s = 1, \\ 2, & for\ even\ s, \\ 3, & otherwise, \end{cases}$$

$$c_2(w_t) = \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_k P_l S_m$ for $m = 2$, odd k, and even l, is illustrated in Figure 3.

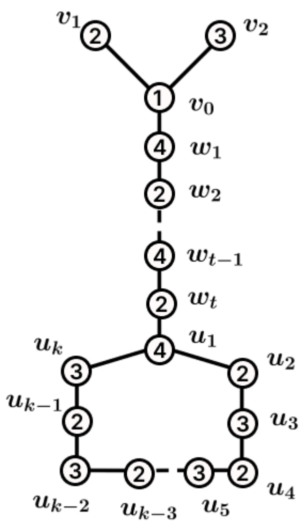

Fig. 3. Palm Graph ($C_k P_l S_m$) for $m = 2$, odd k, and even l.

Now, let $\Pi_2 = \{O_1, O_2, O_3, O_4\}$ be a partition of $V(H)$. From Figure 3, we can derive the generalized color codes for each vertex as follows:

$$\begin{aligned}
&c_\Pi(v_0) = (0, 1, 1, 1),\\
&c_\Pi(v_1) = (1, 0, 2, 2),\\
&c_\Pi(v_2) = (1, 2, 0, 2),\\
&c_\Pi(u_1) = (l + 1, 1, 1, 0),\\
&c_\Pi(u_s) = (l + s, s\ mod\ 2, 1 - s\ mod\ 2, s - 1),\\
&\qquad for\ 1 < s \le \frac{k+1}{2}\\
&c_\Pi(u_s) = (l + (k + 2 - s), s\ mod\ 2, 1 - s\ mod\ 2,\\
&\qquad k + 1 - s), for\ s > \frac{k+1}{2}\\
&c_\Pi(w_t) = (t, t\ mod\ 2, t + 1, 1 - t\ mod\ 2),\ \ for\ t \le \frac{l}{2}\\
&c_\Pi(w_t) = (t, t\ mod\ 2, l + 2 - t, 1 - t\ mod\ 2),\\
&\qquad for\ t > \frac{l}{2},
\end{aligned}$$

Given the color codes described above, all vertices of $H = C_k P_l S_m$ with $m = 2$, odd values of k, and even values of l have distinct color codes. Consequently, it is determined that $\chi_L(H) \le 4$.

Case 1.3. For even k and odd l

Define a coloring $c_3 : V(H) \to \{1, 2, 3, 4\}$, for $1 \le s \le k$ and $1 \le t \le l$ where k is even and l is odd such that

$$\begin{aligned}
&c_3(v_0) = 1,\\
&c_3(v_1) = 2,\\
&c_3(v_2) = 3,
\end{aligned}$$

$$c_3(u_s) = \begin{cases} 2, & for\ s \in \{1, \frac{k}{2} + 1\}, \\ 3, & for\ odd\ s\ and\ s > \frac{k}{2} + 1 \\ & \quad or\ even\ s\ and\ s < \frac{k}{2} + 1, \\ 1, & for\ even\ s\ and\ s > \frac{k}{2} + 1 \\ & \quad or\ odd\ s\ and\ 1 < s < \frac{k}{2} + 1, \end{cases}$$

$$c_3(w_t) = \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_k P_l S_m$ for $m = 2$, even k, and odd l, is illustrated in Figure 4.

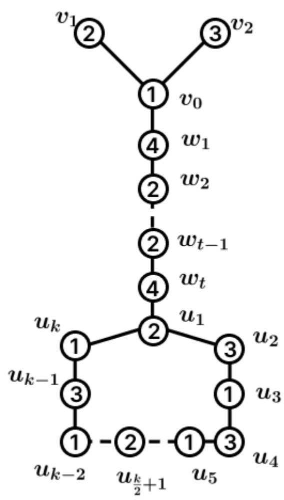

Fig. 4. Palm Graph ($C_k P_l S_m$) for $m = 2$, even k, and odd l.

Now, consider the partition $\Pi_3 = \{O_1, O_2, O_3, O_4\}$ of the vertex set $V(H)$. From Figure 4, we can establish the generalized color codes for each vertex as follows:

$$\begin{aligned}
&c_\Pi(v_0) = (0, 1, 1, 1),\\
&c_\Pi(v_1) = (1, 0, 2, 2),\\
&c_\Pi(v_2) = (1, 2, 0, 2),\\
&c_\Pi(u_1) = (1, 0, 1, 1),\\
&c_\Pi(u_{\frac{k}{2}+1}) = (1, 0, 1, \frac{k}{2} + 1),\\
&c_\Pi(u_s) = (1 - s\ mod\ 2, s - 1, s\ mod\ 2, s),\\
&\qquad for\ 1 < s \le \left\lfloor \frac{k}{4} \right\rfloor + 1,\\
&c_\Pi(u_s) = (1 - s\ mod\ 2, \frac{k}{2} + 1 - s, s\ mod\ 2, s),\\
&\qquad for\ \left\lfloor \frac{k}{4} \right\rfloor + 1 < s < \frac{k}{2} + 1,\\
&c_\Pi(u_s) = (s\ mod\ 2, s - \frac{k}{2} + 1, 1 - s\ mod\ 2, k + 2 - s),\\
&\qquad for\ \frac{k}{2} + 1 < s < \left\lceil \frac{3k}{4} \right\rceil + 1,\\
&c_\Pi(u_s) = (s\ mod\ 2, k + 1 - s, 1 - s\ mod\ 2, k + 2 - s),
\end{aligned}$$

$$for\ s \geq \left\lceil \frac{3k}{4} \right\rceil + 1,$$
$$c_\Pi(w_t) = (t, t\ mod\ 2, t+1, 1 - t\ mod\ 2),$$
$$for\ t \leq \frac{l+1}{2}$$
$$c_\Pi(w_t) = (l+2-t, t\ mod\ 2, l+2-t, 1 - t\ mod\ 2),$$
$$for\ t > \frac{l+1}{2},$$

According to the color codes provided above, all vertices of $H = C_k P_l S_m$ with $m = 2$, even values of k, and odd values of l possess distinct color codes. Therefore, it is established that $\chi_L(H) \leq 4$.

Case 1.4. For k and l are even

Define a coloring $c_4 : V(H) \to \{1, 2, 3, 4\}$, for $1 \leq s \leq k$ and $1 \leq t \leq l$ where k and l are even such that

$$c_4(v_0) = 1,$$
$$c_4(v_1) = 2,$$
$$c_4(v_2) = 3,$$
$$c_4(u_s) = \begin{cases} 4, & for\ s \in \{1, \frac{k}{2}+1\}, \\ 3, & for\ odd\ s\ and\ 1 < s < \frac{k}{2}+1 \\ & or\ even\ s\ and\ s > \frac{k}{2}+1, \\ 2, & for\ even\ s\ and\ s < \frac{k}{2}+1 \\ & or\ odd\ s\ and\ s > \frac{k}{2}+1, \end{cases}$$
$$c_3(w_t) = \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_k P_l S_m$ with $m = 2$ and even values for k and l, is illustrated in Figure 5.

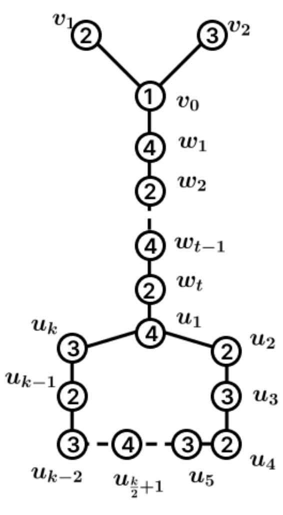

Fig. 5. Palm Graph ($C_k P_l S_m$) for $m = 2$, even k, and even l.

Now, let $\Pi_4 = \{O_1, O_2, O_3, O_4\}$ be a partition of $V(H)$. From Figure 5, we can derive the generalized color codes for each vertex as follows:

$$c_\Pi(v_0) = (0, 1, 1, 1),$$
$$c_\Pi(v_1) = (1, 0, 2, 2),$$
$$c_\Pi(v_2) = (1, 2, 0, 2),$$
$$c_\Pi(u_1) = (l+1, 1, 1, 0),$$
$$c_\Pi(u_{\frac{k}{2}+1}) = (l + \frac{k}{2} + 1, 1, 1, 0),$$
$$c_\Pi(u_s) = (l+s, s\ mod\ 2, 1 - s\ mod\ 2, s-1),$$
$$for\ 1 < s \leq \left\lfloor \frac{k}{4} \right\rfloor + 1,$$
$$c_\Pi(u_s) = (l+s, s\ mod\ 2, 1 - s\ mod\ 2, \frac{k}{2} + 1 - s),$$
$$for\ \left\lfloor \frac{k}{4} \right\rfloor + 1 < s < \frac{k}{2} + 1,$$
$$c_\Pi(u_s) = (k+l-s+2, 1 - s\ mod\ 2, s\ mod\ 2,$$
$$s - \frac{k}{2} + 1),\ for\ \frac{k}{2} + 1 < s < \left\lceil \frac{3k}{4} \right\rceil + 1,$$
$$c_\Pi(u_s) = (k+l-s+2, 1 - s\ mod\ 2, s\ mod\ 2,$$
$$k+1-s),\ for\ s \geq \left\lceil \frac{3k}{4} \right\rceil + 1,$$
$$c_\Pi(w_t) = (t, t\ mod\ 2, t+1, 1 - t\ mod\ 2),\ for\ t \leq \frac{l}{2}$$
$$c_\Pi(w_t) = (t, t\ mod\ 2, l+2-t, 1 - t\ mod\ 2),$$
$$for\ t > \frac{l}{2},$$

Based on the color codes above, all vertices of $H = C_k P_l S_m$ for $m = 2$, even k, and even l have different color codes. Therefore, it is established that $\chi_L(H) \leq 4$.

Consider $H = C_k P_l S_m$ for $k \geq 3, l \geq 2$, and $m = 2$. Based on Case 1, since we have $\chi_L(H) \geq 4$ and $\chi_L(H) \leq 4$, then $\chi_L(H) = 4$.

Case 2. For $m \geq 3$

First, we determine the lower bound of $\chi_L(C_k P_l S_m)$ for $k \geq 3, l \geq 2$, and $m \geq 3$. Let $H = C_k P_l S_m$ for $m \geq 3$. This graph includes vertices adjacent to m leaves. Based on Corollary 3.1, it follows that $\chi_L(H) \geq m + 1$.

Next, we determine the upper bound of $\chi_L(C_k P_l S_m)$ for $k \geq 3, l \geq 2$, and $m \geq 3$ and divide it into several cases.

Case 2.1. For k and l are odd

Define a coloring $c_5 : V(H) \to \{1, 2, 3, \ldots, m+1\}$, for $0 \leq r \leq m$, $1 \leq s \leq k$, and $1 \leq t \leq l$ where k and l are odd such that

$$c_5(v_r) = r + 1,$$
$$c_5(u_s) = \begin{cases} 2, & for\ s = 1, \\ 3, & for\ even\ s, \\ 1, & otherwise, \end{cases}$$
$$c_5(w_t) = \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_k P_l S_m$ with $m \geq 3$, and odd values for k and l, is illustrated in Figure 6.

As illustrated in Figure 6, consider the vertex v_0 where $c(v_0) = 1$. The shortest distance from v_0 to vertices of different colors, except for its own color (color 1), is $d(v_0, v_1) = d(v_0, v_2) = \ldots = d(v_0, v_m) = 1$. This shows that the shortest distance from v_0 to colors 2, 3, ..., m where $m \geq 3$ is 1. Consequently, the color code for vertex v_0 is $c_\Pi(v_0) = (0, 1, 1, \ldots, 1)$. Hence, v_0 is a dominant vertex with color 1. This method is also applicable to other vertices. Next, let $\Pi_5 = \{O_1, O_2, \ldots, O_{m+1}\}$ be a partition of $V(H)$.

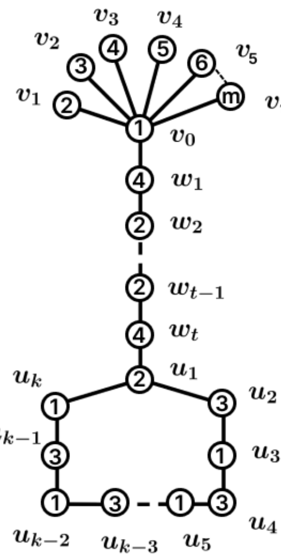

Fig. 6. Palm Graph ($C_kP_lS_m$) for $m \geq 3$, odd k, and odd l.

Referring to Figure 6, we can generalize the color codes for each vertex as follows:

$$\begin{aligned}
c_\Pi(v_0) &= (0, 1, 1, 1, \ldots, 1),\\
c_\Pi(v_1) &= (1, 0, 2, 2, \ldots, 2),\\
c_\Pi(v_2) &= (1, 2, 0, 2, \ldots, 2),\\
&\vdots\\
c_\Pi(v_{m+1}) &= (1, 2, 2, \ldots, 2, 0),\\
c_\Pi(u_1) &= (1, 0, 1, 1, l + 2, \ldots, l + 2),\\
c_\Pi(u_s) &= (1 - s \ mod\ 2, s - 1, s \ mod\ 2, s, l + 1 + s,\\
&\quad \ldots, l + 1 + s), \ for\ 1 < s \leq \frac{k+1}{2}\\
c_\Pi(u_s) &= (1 - s \ mod\ 2, k + 1 - s, s \ mod\ 2, k + 2 - s,\\
&\quad l + k + 3 - s, \ldots, l + k + 3 - s),\\
&\quad for\ s > \frac{k+1}{2}\\
c_\Pi(w_t) &= (t, t \ mod\ 2, t + 1, 1 - t \ mod\ 2, t + 1, \ldots,\\
&\quad t + 1), \ for\ t \leq \frac{l+1}{2}\\
c_\Pi(w_t) &= (l + 2 - t, t \ mod\ 2, l + 2 - t, 1 - t \ mod\ 2,\\
&\quad t + 1, \ldots, t + 1), \ for\ t > \frac{l+1}{2},
\end{aligned}$$

According to the color codes given above, all vertices of $H = C_kP_lS_m$ for $m \geq 3$, with odd k and odd l, have distinct color codes. Consequently, it is determined that $\chi_L(H) \leq m + 1$.

Case 2.2. For odd k and even l

Define a coloring $c_6 : V(H) \to \{1, 2, 3, \ldots, m + 1\}$, for $0 \leq r \leq m$, $1 \leq s \leq k$, and $1 \leq t \leq l$ where k and l are even such that

$$c_6(v_r) = r + 1,$$

$$c_6(u_s) = \begin{cases} 4, \ for\ s = 1,\\ 2, \ for\ even\ s,\\ 3, \ otherwise, \end{cases}$$

$$c_6(w_t) = \begin{cases} 4, \ for\ odd\ t,\\ 2, \ for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_kP_lS_m$ with $m \geq 3$, odd k, and even l, is illustrated in Figure 7.

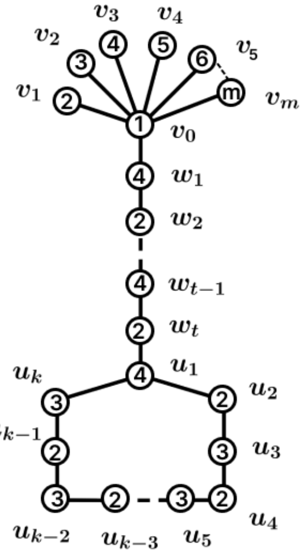

Fig. 7. Palm Graph ($C_kP_lS_m$) for $m \geq 3$, odd k, and even l.

Next, let $\Pi_6 = \{O_1, O_2, \ldots, O_{m+1}\}$ be a partition of $V(H)$. Referring to Figure 7, we can generalize the color codes for each vertex as follows:

$$\begin{aligned}
c_\Pi(v_0) &= (0, 1, 1, 1, \ldots, 1),\\
c_\Pi(v_1) &= (1, 0, 2, 2, \ldots, 2),\\
c_\Pi(v_2) &= (1, 2, 0, 2, \ldots, 2),\\
&\vdots\\
c_\Pi(v_{m+1}) &= (1, 2, 2, \ldots, 2, 0),\\
c_\Pi(u_1) &= (l + 1, 1, 1, 0, l + 2, \ldots, l + 2),\\
c_\Pi(u_s) &= (l + s, s \ mod\ 2, 1 - s \ mod\ 2, s - 1, l + 1 + s,\\
&\quad \ldots, l + 1 + s), \ for\ 1 < s \leq \frac{k+1}{2}\\
c_\Pi(u_s) &= (l + (k + 2 - s), s \ mod\ 2, 1 - s \ mod\ 2,\\
&\quad k + 1 - s, l + (k + 3 - s), \ldots,\\
&\quad l + (k + 3 - s)), \ for\ s > \frac{k+1}{2}\\
c_\Pi(w_t) &= (t, t \ mod\ 2, t + 1, 1 - t \ mod\ 2, t + 1, \ldots,\\
&\quad t + 1), \ for\ t \leq \frac{l}{2}\\
c_\Pi(w_t) &= (t, t \ mod\ 2, l + 2 - t, 1 - t \ mod\ 2, t + 1,\\
&\quad \ldots, t + 1), for\ t > \frac{l}{2},
\end{aligned}$$

According to the provided color codes, every vertex of $H = C_kP_lS_m$ with $m \geq 3$, odd k, and even l have unique color codes. Hence, it's concluded that $\chi_L(H) \leq m + 1$.

Case 2.3. For even k and odd l

Define a coloring $c_7 : V(H) \to \{1, 2, 3, \ldots, m + 1\}$, for $0 \leq r \leq m, 1 \leq s \leq k$ and $1 \leq t \leq l$ where k is even and l

is odd such that

$$c_7(v_r) = r+1,$$
$$c_7(u_s) = \begin{cases} 2, & for\ s \in \{1, \frac{k}{2}+1\}, \\ 3, & for\ odd\ s\ and\ s > \frac{k}{2}+1 \\ & or\ even\ s\ and\ s < \frac{k}{2}+1, \\ 1, & for\ even\ s\ and\ s > \frac{k}{2}+1 \\ & or\ odd\ s\ and\ 1 < s < \frac{k}{2}+1, \end{cases}$$
$$c_7(w_t) = \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_kP_lS_m$ with $m \geq 3$, even k, and odd l, is illustrated in Figure 8.

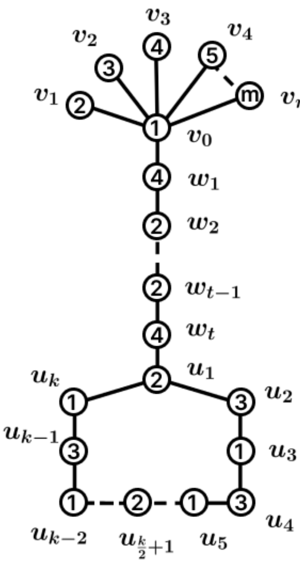

Fig. 8. Palm Graph ($C_kP_lS_m$) for $m \geq 3$, even k, and odd l.

Next, consider the partition $\Pi_7 = \{O_1, O_2, \ldots, O_{m+1}\}$ of the vertex set $V(H)$. Based on Figure 8, we can derive the generalized color codes for each vertex as follows:

$$c_\Pi(v_0) = (0, 1, 1, 1, \ldots, 1),$$
$$c_\Pi(v_1) = (1, 0, 2, 2, \ldots, 2),$$
$$c_\Pi(v_2) = (1, 2, 0, 2, \ldots, 2),$$
$$\vdots$$
$$c_\Pi(v_{m+1}) = (1, 2, 2, \ldots, 2, 0),$$
$$c_\Pi(u_1) = (1, 0, 1, 1, l+2, \ldots, l+2),$$
$$c_\Pi(u_{\frac{k}{2}+1}) = (1, 0, 1, \frac{k}{2}+1, l+\frac{k}{2}+2, \ldots, l+\frac{k}{2}+2),$$
$$c_\Pi(u_s) = (1 - s\ mod\ 2, s-1, s\ mod\ 2, s, l+1+s, \ldots, l+1+s),\ for\ 1 < s \leq \left\lfloor \frac{k}{4} \right\rfloor + 1,$$
$$c_\Pi(u_s) = (1 - s\ mod\ 2, \frac{k}{2}+1-s, s\ mod\ 2, s, l+1+s, \ldots, l+1+s,),\ for\ \left\lfloor \frac{k}{4} \right\rfloor + 1 < s < \frac{k}{2}+1,$$
$$c_\Pi(u_s) = (s\ mod\ 2, s - \frac{k}{2}+1, 1 - s\ mod\ 2, k+2-s, k+l-s+2, \ldots, k+l-s+2),\ for\ \frac{k}{2}+1 < s < \left\lfloor \frac{3k}{4} \right\rceil + 1,$$
$$c_\Pi(u_s) = (s\ mod\ 2, k+1-s, 1 - s\ mod\ 2, k+2-s, k+l-s+2, \ldots, k+l-s+2),\ for\ s \geq \left\lfloor \frac{3k}{4} \right\rceil + 1,$$
$$c_\Pi(w_t) = (t, t\ mod\ 2, t+1, 1 - t\ mod\ 2, t+1, \ldots, t+1),\ for\ t \leq \frac{l+1}{2}$$
$$c_\Pi(w_t) = (l+2-t, t\ mod\ 2, l+2-t, 1 - t\ mod\ 2, t+1, \ldots, t+1),\ for\ t > \frac{l+1}{2},$$

Based on the color codes above, all vertices of $H = C_kP_lS_m$ for $m \geq 3$, even k, and odd l have different color codes. Therefore, it is established that $\chi_L(H) \leq m+1$.

Case 2.4. For k and l are even

Define a coloring $c : V(H) \rightarrow \{1, 2, 3, \ldots, m+1\}$, for $0 \leq r \leq m, 1 \leq s \leq k$ and $1 \leq t \leq l$ where k and l are even such that

$$c_8(v_r) = r+1,$$
$$c_8(u_s) = \begin{cases} 4, & for\ s \in \{1, \frac{k}{2}+1\}, \\ 3, & for\ odd\ s\ and\ 1 < s < \frac{k}{2}+1 \\ & or\ even\ s\ and\ s > \frac{k}{2}+1, \\ 2, & for\ even\ s\ and\ s < \frac{k}{2}+1 \\ & or\ odd\ s\ and\ s > \frac{k}{2}+1, \end{cases}$$
$$c_8(w_t) = \begin{cases} 4, & for\ odd\ t, \\ 2, & for\ even\ t. \end{cases}$$

This coloring for a palm graph $H = C_kP_lS_m$ with $m \geq 3$, and odd values for k and l is illustrated in Figure 9.

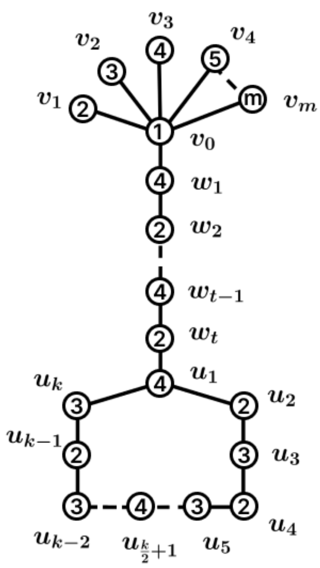

Fig. 9. Palm Graph ($C_kP_lS_m$) for $m \geq 3$, even k, and even l.

Next, let $\Pi_8 = \{O_1, O_2, \ldots, O_{m+1}\}$ be a partition of the vertex set $V(H)$. From Figure 9, we can generalize the color codes for each vertex as follows:

$$
\begin{aligned}
c_\Pi(v_0) &= (0, 1, 1, 1, \ldots, 1),\\
c_\Pi(v_1) &= (1, 0, 2, 2, \ldots, 2),\\
c_\Pi(v_2) &= (1, 2, 0, 2, \ldots, 2),\\
&\vdots\\
c_\Pi(v_{m+1}) &= (1, 2, 2, \ldots, 2, 0),\\
c_\Pi(u_1) &= (l+1, 1, 1, 0, l+2, \ldots, l+2),\\
c_\Pi(u_{\frac{k}{2}+1}) &= (l + \frac{k}{2} + 1, 1, 1, 0, l + \frac{k}{2} + 2, \ldots,\\
&\quad l + \frac{k}{2} + 2),\\
c_\Pi(u_s) &= (l+s, s \ mod\ 2, 1 - s \ mod\ 2, s-1,\\
&\quad l+s+1, \ldots, l+s+1),\\
&\quad for\ 1 < s \leq \left\lfloor \frac{k}{4} \right\rfloor + 1,\\
c_\Pi(u_s) &= (l+s, s \ mod\ 2, 1 - s \ mod\ 2, \frac{k}{2} + 1 - s,\\
&\quad l+s+1, \ldots, l+s+1),\\
&\quad for\ \left\lfloor \frac{k}{4} \right\rfloor + 1 < s < \frac{k}{2} + 1,\\
c_\Pi(u_s) &= (k + l - s + 2, 1 - s \ mod\ 2, s \ mod\ 2,\\
&\quad s - \frac{k}{2} + 1, k+1-s, \ldots, k+1-s),\\
&\quad for\ \frac{k}{2} + 1 < s < \left\lfloor \frac{3k}{4} \right\rceil + 1,\\
c_\Pi(u_s) &= (k + l - s + 2, 1 - s \ mod\ 2, s \ mod\ 2,\\
&\quad k+1-s, k+1-s, \ldots, k+1-s),\\
&\quad for\ s \geq \left\lfloor \frac{3k}{4} \right\rceil + 1,\\
c_\Pi(w_t) &= (t, t \ mod\ 2, t+1, 1 - t \ mod\ 2, t+1, \ldots,\\
&\quad t+1),\ for\ t \leq \frac{l}{2}\\
c_\Pi(w_t) &= (t, t \ mod\ 2, l+2-t, 1 - t \ mod\ 2, t+1,\\
&\quad \ldots, t+1),\ for\ t > \frac{l}{2},
\end{aligned}
$$

According to the color codes provided above, all vertices of $H = C_k P_l S_m$ for $m \geq 3$, with even k and even l, possess distinct color codes. Hence, it is concluded that $\chi_L(H) \leq m + 1$.

Consider $H = C_k P_l S_m$ for $k \geq 3, l \geq 2$, and $m \geq 3$. Based on Case 2, since we have $\chi_L(H) \geq m + 1$ and $\chi_L(H) \leq m + 1$, then $\chi_L(H) = m + 1$. ■

V. Conclusion

In this paper, we have examined the locating chromatic number of the palm graph $H = C_k P_l S_m$, where $k \geq 3$ and $l, m \geq 2$. Here, C_k represents a cycle with at least 3 vertices, P_l denotes a path with at least 2 vertices, and S_m signifies a star with $m+1$ vertices. Our findings reveals that the locating chromatic number of H equals 4 when $m = 2$ and $m + 1$ when $m \geq 3$.

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