Fundamental Theorems in Discrete Fractional Calculus using Nabla Operator

Abisha M, Sathinathan T, Saraswathi D and Britto Antony Xavier G

***Abstract*—The theory of discrete versions of the fundamental nabla integration theorems is being developed in this work. Through ∞-order nabla-integrable function, this theory has been developed. Afterwards, a number of basic theorems and examples on fractional order sums in the context of discrete fractional calculus are derived using this theory. A few definitions and a summation formula derived from the inverse of ∇_ℓ have been provided, along with theorems on integer order. Furthermore, we have obtained theorems regarding fractional order napla integration, supported by appropriate examples.**

***Index Terms*—Closed form, Summation form, Newton's formula, Discrete integration, Discrete ℓ-Nabla fractional calculus, Fractional sum.**

I. Introduction

DIFFERENCE equations are meant for discrete process where as the differential equations deals with continuous system. The certain phenomena of their evolution is usually describe over the course of time ([1], [2], [3], [4]). A difference equation is an operator where the differences between successive values of a function of an integer variable has involved. The properties which has been often qualitative, such solutions of difference equation are rather difference solutions of the correponding differential equations. Riccati's, Duffing's, Mathieu's, Euler's, Verhulst's, Clairaut's, Bernoulli's and Volterra's are several well known difference equations ([5], [6]). Authors in [7] studied a boundary value problem of a class of linear singular systems of fractional nabla difference equations whose coefficients are constant matrices.

In 1837, the Irish mathematician and physicist William Rowan Hamilton who called it ∇ received its full exposition at the hands of P.G. Tait. Infact Smith gave a suggestion to tait and James Clerk Maxwell refered to the operator as nabla in their extensive private correspondence, most of these references are of a humorous character. Sir W. R. Hamliton is credited with inventing this symbolic operator ∇ which is currently widely used. The monosyllable del is so short and easy to say that it causes no discomfort to the speaker or listener even when it appears multiple times in difficult equations. Most commonly nabla is used to simplify expressions for the gradient, curl, divergence, derivative and laplacian.

In [8], the ν^{th} order fractional sum of given function f based at a is defined as

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s+1-\nu)} f(s), \qquad (1)$$

where $\nu > 0$ and a real valued function f is defined for $s \in a + \mathbb{N}(1)$ and $\Delta^{-\nu} f$ is defined for $t = a + \nu + \mathbb{N}(1)$. The authors in ([9], [10], [11], [12], [13], [14]) have developed several theorems based on equation (1) and also the generalized ℓ-delta and nabla operator denoted as Δ_ℓ and ∇_ℓ. Authors in [15] developed the discrete fundamental theorems using a new strategy known as delta integration method and extended to $\ell-$delta integration and its sum. This motivates us to develop some fundamental theorems in discrete fractional calculus in the case of backward difference operator denoted as ∇_ℓ.

In this article, section II provides the preliminaries of nabla operator and its inverse. In section III, we develop the integer order nabla integration and its sum and in section IV we covered the fractional order nabla integration and its sum.

II. Preliminaries of ℓ-Nabla Operator

In this section, we present basic definitions of nabla operator, falling factorial, summation formula and its inverse of nabla operator. For $a \in \mathbb{R} = (-\infty, \infty)$, we use the notation $N(a) = a, a+1, a+2, \cdots$.

Definition II.1. Let $N(a)$ be a subset of $\mathbb{R}$ which satisfies the condition that $\zeta \in N(a)$ implies $\zeta \pm 1 \in N(a)$ and $f : N(a) \to \mathbb{R}$. Then the ℓ-nabla operator on f is defined as

$$\nabla_\ell f(\zeta) = f(\zeta) - f(\zeta - \ell), \quad \zeta \in N(a) \qquad (2)$$

If there exists $f_1 : N(a) \to \mathbb{R}$, then its inverse operator on f is defined as

$$\nabla_\ell f_1(\zeta) = f(\zeta) \Leftrightarrow f_1(\zeta) + k = \nabla_\ell^{-1} f(\zeta) \qquad (3)$$

where k is the arbitrary constants.

Example II.2. *Taking $f(\zeta) = e^\zeta$ in equation (2), we get*

$$\nabla_\ell \, e^\zeta = e^\zeta - e^{\zeta-\ell} = e^\zeta(1 - e^{-\ell})$$

$$\nabla_\ell^{-1} \, e^\zeta = \frac{e^\zeta}{(1 - e^{-\ell})}$$

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Abisha M is a research scholar at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: mabisha1996@gmail.com)

Sathinathan T is an Assistant Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: sathinathan@shctpt.edu)

Saraswathi D is a research scholar at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: dswathisaranya@gmail.com)

Britto Antony Xavier G is an Associate Professor at the Department of Mathematics, Sacred Heart College (Autonomous), Tirupattur-635 601, Tamil Nadu, India. Affiliated to Thiruvalluvar University, Serkaddu, Vellore-632 115, Tamil Nadu, India. (e-mail: brittoshc@gmail.com)

Lemma II.3. *For the function* $(\zeta-a)_\ell^{(k)}$ *where* $\zeta\in\mathbb{R},\ k\in\mathbb{N}$*, we have*

$$\nabla_\ell(\zeta-a)_\ell^{(k)}=k\ell(\zeta-\ell-a)_\ell^{(k-1)} \quad (4)$$

Proof: For $k=0$, $\nabla_\ell(\zeta-a)_\ell^{(0)}=0$;
Applying $f(\zeta)=(\zeta-a)_\ell^{(1)}$ in equation (2), we get
$\nabla_\ell(\zeta-a)_\ell^{(1)}=(\zeta-a)_\ell^{(1)}-(\zeta-\ell-a)_\ell^{(1)}=\zeta-a-\zeta+\ell+a=\ell$;
Applying $f(\zeta)=(\zeta-a)_\ell^{(2)}$ in equation (2), we get
$\nabla_\ell(\zeta-a)_\ell^{(2)}=(\zeta-a)_\ell^{(2)}-(\zeta-\ell-a)_\ell^{2}=2\ell(\zeta-\ell-a)_\ell^{(1)}$;
Continuing this process, we get equation (4). ■

Definition II.4. For $\alpha\in\mathbb{R}-\{0,-1,-2,-3,...\}$, the Gamma function is defined as

$$\Gamma(\alpha)=\int_0^\infty \zeta^{\alpha-1}e^{-\zeta}d\zeta \quad (5)$$

$$Also\ \ \Gamma(\alpha)=(\alpha-1)\Gamma(\alpha-1),\alpha\notin\{0,-1,-2,...\}\,. \quad (6)$$

Definition II.5. For $\zeta\in\mathbb{R}$ and $n\in\mathbb{N}(1)$, the n^{th} falling factorial of ζ, denoted as $\zeta_\ell^{(n)}$ is defined by

$$\zeta_\ell^{(n)}=\prod_{r=0}^{n-1}(\zeta-r\ell)\quad and\quad \zeta_\ell^{(0)}=1. \quad (7)$$

If γ, $\zeta\in\mathbb{R}$, the γ^{th} order of gamma falling factorial is defined as

$$\zeta_\ell^{(\gamma)}=\frac{\Gamma\left(\frac{\zeta}{\ell}-1\right)}{\Gamma\left(\frac{\zeta}{\ell}-(\gamma-1)\right)}\ell^\gamma,\ \frac{\zeta}{\ell}-\gamma+1\notin\{0,-1,-2,-3,...\}\,. \quad (8)$$

Results II.6. Some special cases: (i) $0_\ell^{(0)}=1$, (ii) $n_1^{(n)}=n!$ (iii) $n_1^{(n+r)}=0$ for $r=1,2,3,...$ and $n\in\mathbb{N}(0)$ (iv) $\zeta_\ell^{(n)}\neq 0$, for $\zeta\in\mathbb{R}-\mathbb{N}(0)$ and $n\in\mathbb{N}(0)$.

Theorem II.7. *If* $\nabla_\ell f_1(\zeta)=f(\zeta)$, $\frac{\zeta-a}{\ell}\in N(1)$ *for* ζ *and* $a\in N(a)$*, then the first order anti-difference principle of* ∇_ℓ *operator is given by*

$$f_1(\zeta)-f_1(a)=\sum_{s=1}^{m}f(a+s\ell). \quad (9)$$

Proof: The given condition $\nabla_\ell f_1=f$, and equation (2) yields

$$f_1(\zeta)=f(\zeta)+f_1(\zeta-\ell) \quad (10)$$

Substituting $\zeta=\zeta-\ell$ in equation (10), we get

$$f_1(\zeta-\ell)=f(\zeta-\ell)+f_1(\zeta-2\ell),$$

Now putting this value of $f_1(\zeta-\ell)$ in equation (10), we get

$$f_1(\zeta)=f(\zeta)+f(\zeta-\ell)+f_1(\zeta-2\ell) \quad (11)$$

Again, Substituting $\zeta=\zeta-2\ell$ in equation (10), we get $f_1(\zeta-2\ell)=f(\zeta-2\ell)+f_1(\zeta-3\ell)$, and then applying in equation (11), we get

$$f_1(\zeta)=f(\zeta)+f(\zeta-\ell)+f(\zeta-2\ell)+f_1(\zeta-3\ell) \quad (12)$$

Continuing like this, equation (12) modifies to

$$f_1(\zeta)=f(\zeta)+f(\zeta-\ell)+\cdots+f(\zeta-(m-1)\ell)+f_1(\zeta-m\ell),$$

$$f_1(\zeta)-f_1(\zeta-m\ell)=f(\zeta)+f(\zeta-\ell)+\cdots+f(\zeta-(m-1)\ell) \quad (13)$$

Now, equation (9) follows by taking $\zeta-ml=a$ and $\frac{\zeta-a}{\ell}=m\in N(1)$ in (13). ■

Example II.8. *Let* $f(\zeta)=e^x$*, then equation* (9) *becomes*

$$f_1(\zeta)-f_1(a)=\sum_{s=1}^{m}e^{a+s\ell}$$

Taking $\zeta=6$, $m=4$, $\ell=1$ *and* $a=2$*, we get*

$$f_1(\zeta)-f_1(a)=\nabla_\ell^{-1}f(\zeta)-\nabla_\ell^{-1}f(a)=\nabla_\ell^{-1}e^\zeta-\nabla_\ell^{-1}e^a$$

From Example II.2, we obtain

$$\begin{aligned}f_1(\zeta)-f_1(a)&=\frac{e^\zeta}{(1-e^{-\ell})}-\frac{e^a}{(1-e^{-\ell})}\\&=\frac{1}{(1-e^{-\ell})}\left[e^\zeta-e^a\right]=\frac{1}{(1-e^{-1})}\left[e^6-e^2\right]\\&=626.525639.\end{aligned}$$

$$\sum_{s=1}^{m}e^{a+s\ell}\ =\ \sum_{s=1}^{4}e^{2+s}=e^3+e^4+e^5+e^6=626.52563955.$$

Corollary II.9. *Let* ζ, $a\in N(a)$ *such that* $m\in\mathbb{N}(1)$ *and* $\nabla_\ell^{-1}f(\zeta)=f_1(\zeta)$*. Then*

$$\nabla_\ell^{-1}f(\zeta)-\nabla_\ell^{-1}f(a)=\sum_{s=1}^{m}f(a+sl)=f_1(\zeta)\Big|_a^\zeta. \quad (14)$$

Proof: The proof follows by taking $\nabla_\ell^{-1}f(\zeta)=f_1(\zeta)$ and replacing ζ by $\zeta-l$ in theorem II.7. ■

III. Integer Order Nabla Integration

The relation (14) is a fundamental theorem of nabla integration. The relations (9) as well as (13) can be considered as first order nabla integration of f. So in this section we derived a main theorem for integer order nabla integration, which is a generalization of the relation (14).

Definition III.1. A function $f:N(a)\to\mathbb{R}$ is called an n^{th} order nabla integrable function if there exists a sequence of functions, say $(f_1,f_2,\cdots,f_n)$ such that

$$\nabla_\ell f_r=f,r=1,2,3,...,n \quad (15)$$

Thus, the sequence $(f_1,f_2,\cdots,f_n)$ is be called as nabla integrating sequence of f.

Example III.2. *The function* $f(\zeta)=2^\zeta$, $\zeta\in N(a)=\mathbb{R}$*, is an* n^{th} *order nabla integrable function having the sequence* $(2^\zeta,2^\zeta,...2^\zeta)$*, since*

$$f(\zeta)=2^\zeta=\frac{\nabla_\ell 2^\zeta}{(1-2^{-\ell})}=\frac{\nabla_\ell^2 2^\zeta}{(1-2^{-\ell})^2}=\frac{\nabla_\ell^n 2^\zeta}{(1-2^{-\ell})^n}=2^\zeta,n\in N(1). \quad (16)$$

Definition III.3. Let $f:N(a)\longrightarrow\mathbb{R}$ be an nabla integrable function having nabla integrating sequence $(f_1,f_2,\cdots,f_n)$. Assume that ζ, $a\in N(a)$ and $n\in\mathbb{N}(1)$ such that $m-n\in\mathbb{N}(0)$. The n^{th} order nabla integration of f is at a is defined by

$$F_a^n(\zeta)=f_n(\zeta)-\sum_{r=0}^{n-1}\frac{f_{n-r}(a)(m+r-1)^{(r)}}{r!}. \quad (17)$$

Definition III.4. Let $f : N(a) \to \mathbb{R}$ be a function $n > 0$ and $\zeta, a \in N(a)$ such that $m - n \in \mathbb{N}(0)$. The integer order (n^{th} order) nabla-ℓ sum of f based at a is defined by,

$$ {}_a\nabla_\ell^{-n} f(\zeta) = \frac{1}{\Gamma(n)} \sum_{r=0}^{m-1} \frac{\Gamma(n+r)}{\Gamma(r+1)} f(\zeta - r\ell). \quad (18) $$

Theorem III.5. *Assume that $f : N(a) \longrightarrow \mathbb{R}$ is having nabla integrating sequence $(f_1, f_2, \cdots, f_n)$. Let $\zeta, a \in N(a)$ such that $m - n \in \mathbb{N}(0)$ and $F_a^n(\zeta)$ be the n^{th} order nabla integration of f based at a as defined in equation* (17)*. Then*

$$ F_a^n(\zeta) = \sum_{r=0}^{m-1} \frac{(n-1+r)^{(n-1)}}{(n-1)!} f(\zeta - r\ell). \quad (19) $$

and

$$ F_a^n(\zeta) := f_n(\zeta) - \sum_{r=0}^{n-1} f_{n-r}(a) \frac{(m+r-1)^{(r)}}{r!} $$
$$ = \sum_{r=0}^{m-1} \frac{(n-1+r)^{(n-1)}}{(n-1)!} f(\zeta - r\ell). \quad (20) $$

Proof: The proof is followed by induction method on n.
If $n = 1$, then Corallary II.9 yields that equation (19), i.e.

$$ F_a^1(\zeta) := f_1(\zeta) - f_1(a) = \sum_{r=1}^{m-1} f(\zeta - r\ell) \quad (21) $$

If $n = 2$, then replacing f by $\nabla_\ell^{-1} f$ in equation (13), we get $\nabla_\ell^{-2} f(\zeta) - \nabla_\ell^{-2} f(a) = \nabla_\ell^{-1} f(\zeta) + \nabla_\ell^{-1} f(\zeta - \ell) + \cdots + \nabla_\ell^{-1} f(\zeta - (m-1)\ell)$,

which gives

$$ f_2(\zeta) - f_2(a) = f_1(\zeta) + f_1(\zeta - \ell) + \cdots + f_1(\zeta - (m-1)\ell) $$

Applying equation(13) for each term of R.H.S. in the above equation, we get,

$$ f_2(\zeta) - f_2(a) = f(\zeta) + 2f(\zeta - \ell) + \cdots + mf(\zeta - (m-1)\ell) + mf_1(\zeta - m\ell) $$
$$ f_2(\zeta) - f_2(a) - mf_1(a) = f(\zeta) + 2(\zeta - \ell) + \cdots + mf(\zeta - (m-1)\ell) $$

which is same as

$$ F_a^2(\zeta) := f_2(\zeta) - \sum_{r=0}^{1} \frac{f_{2-r}(a)(m+r-1)^{(r)}}{r!} $$
$$ = \sum_{r=1}^{m-1} \frac{(1+r)^{(1)}}{1!} f(\zeta - r\ell) \quad (22) $$

Continuing this way, we arrive for $(n-1)^{th}$ case as

$$ F_a^{n-1}(\zeta) := f_{n-1}(\zeta) - \sum_{r=0}^{n-2} f_{n-1-r}(a) \frac{(m+r-1)^{(r)}}{r!} $$
$$ = \sum_{r=0}^{m-1} \frac{(n-2+r)^{(n-2)}}{(n-2)!} f(\zeta - r\ell) \quad (23) $$

By rewritting the terms of left hand side using nabla-ℓ inverse operator and expanding the terms of right hand side of $(n-1)^{th}$ case, we get

$$ \nabla_\ell^{-(n-1)} f(\zeta) - \nabla_\ell^{-(n-1)} f(a) - \frac{m}{1!} \nabla_\ell^{-(n-2)} f(a) - \cdots - \frac{(m+n-3)^{(n-2)}}{(n-2)!} \nabla_\ell^{(-1)} f(a) $$

$$ = \begin{cases} \frac{(n-2)^{(n-2)}}{(n-2)!} f(\zeta) + \frac{(n-1)^{(n-2)}}{(n-2)!} f(\zeta - \ell) + \\ \frac{(n)^{(n-2)}}{(n-2)!} f(\zeta - 2\ell) + \cdots + \\ \frac{(n+m-4)^{(n-2)}}{(n-2)!} f(\zeta - (m-2)\ell) + \\ \frac{(n+m-3)^{(n-2)}}{(n-2)!} f(\zeta - (m-1)\ell) \end{cases} $$

Replacing f by $\nabla_\ell^{-1} f$, we get

$$ \nabla_\ell^{-n} f(\zeta) - \nabla_\ell^{-n} f(a) - \frac{m}{1!} \nabla_\ell^{-(n-1)} f(a) - \cdots - \frac{(m+n-3)^{(n-2)}}{(n-2)!} \nabla_\ell^{-2} f(a) $$

$$ = \begin{cases} \frac{(n-2)^{(n-2)}}{(n-2)!} \nabla_\ell^{-1} f(\zeta) + \frac{(n-1)^{(n-2)}}{(n-2)!} \nabla_\ell^{-1} f(\zeta - \ell) + \\ \frac{(n)^{(n-2)}}{(n-2)!} \nabla_\ell^{-1} f(\zeta - 2\ell) + \cdots + \\ \frac{(n+m-4)^{(n-2)}}{(n-2)!} \nabla_\ell^{-1} f(\zeta - (m-2)\ell) + \\ \frac{(n+m-3)^{(n-2)}}{(n-2)!} \nabla_\ell^{-1} f(\zeta - (m-1)\ell) \end{cases} $$

which modifies as

$$ f_n(\zeta) - f_n(a) - \frac{m}{1!} f_{n-1}(a) - \frac{(m+1)^{(2)}}{2!} f_{n-2}(a) - \cdots - \frac{(m+n-3)^{(n-2)}}{(n-2)!} f_2(a) $$
$$ = \begin{cases} \frac{(n-2)^{(n-2)}}{(n-2)!} f_1(\zeta) + \frac{(n-1)^{(n-2)}}{(n-2)!} f_1(\zeta - \ell) + \\ \frac{(n)^{(n-2)}}{(n-2)!} f_1(\zeta - 2\ell) + \cdots + \\ \frac{(n+m-4)^{(n-2)}}{(n-2)!} f_1(\zeta - (m-2)\ell) + \\ \frac{(n+m-3)^{(n-2)}}{(n-2)!} f_1(\zeta - (m-1)\ell) \end{cases} $$

Applying equation(13) for each term of right side in the above equation, we get,

$$ f_n(\zeta) - f_n(a) - \frac{m}{1!} f_{n-1}(a) - \frac{(m+1)^{(2)}}{2!} f_{n-2}(a) - \cdots - \frac{(m+n-2)^{(n-1)}}{(n-1)!} f_1(a) $$
$$ = \begin{cases} f(\zeta) + \frac{(n)^{(n-1)}}{(n-1)!} f(\zeta - \ell) + \frac{(n+1)^{(n-1)}}{(n-1)!} f(\zeta - 2\ell) + \cdots \\ \qquad + \frac{(m+n-2)^{(n-1)}}{(n-1)!} f(\zeta - (m-1)\ell) \end{cases} $$

The above equation is same as (19) and thus the proof is completed by induction on n. ■

Example III.6. *Taking $f(\zeta) = 2^\zeta$, $a = 2$ and $n = 2$ in equation* (20)*, we get*

$$ F_2^2(\zeta) := f_2(\zeta) - f_2(a) - mf_1(a) = \sum_{r=0}^{m-1} \frac{(1+r)^{(1)}}{(1)!} f(\zeta - r\ell) \quad (24) $$

By taking $\zeta = 6$ and $\ell = 2$ in equation (16) *and inserting*

$m = 2$ *in* (24)*, we get*
$f_2(\zeta) - f_2(a) - mf_1(a) =$

$$\frac{2^6}{(1-2^{-2})^2} - \frac{2^2}{(1-2^{-2})^2} - 2\frac{2^2}{(1-2^{-2})} = 96.$$

$$\sum_{r=0}^{m-1} \frac{(1+r)^{(1)}}{1!} f(\zeta - r\ell) = \sum_{r=0}^{1} \frac{(1+r)^{(1)}}{1!} 2^{6-r\ell}$$
$$= \frac{(1)^{(1)}}{1!}2^6 + \frac{(2)^{(1)}}{1!}2^4 = 96.$$

Corollary III.7. *Let $n \in \mathbb{N}(1), m - n \in \mathbb{N}(0)$. If f is n^{th} order nabla integrable function based at a, then*

$$F_a^n(\zeta) = \ {}_a\nabla_\ell^{-n} f(\zeta) \tag{25}$$

Proof: The proof follows from Theorem III.5 and from the Definition III.4. ■

Remark III.8. If f is n^{th} order nabla integrable function based at $a \in N(a)$, then from corollary III.7 and Definition III.3 and III.4, we obtain

$$ {}_a\nabla_\ell^{-n} f(\zeta) = f_n(\zeta) - \sum_{r=0}^{n-1} \frac{f_{n-r}(a)}{\Gamma(r+1)} \frac{\Gamma(m+r)}{\Gamma m}$$
$$= \frac{1}{\Gamma(n)} \sum_{r=0}^{m-1} \frac{\Gamma(n+r)}{\Gamma(r+1)} f(\zeta - r\ell) \tag{26}$$

Example III.9. *Taking $f(\zeta) = 2^\zeta$, $a = 2.5$, $n = 2$ in equation* (27)*, we get*

$$ {}_2\nabla_\ell^{-2} 2^\zeta = f_2(\zeta) - \sum_{r=0}^{1} \frac{f_{2-r}(a)}{\Gamma(r+1)} \frac{\Gamma(m+r)}{\Gamma m}$$
$$= \frac{1}{\Gamma(2)} \sum_{r=0}^{m-1} \frac{\Gamma(2+r)}{\Gamma(r+1)} f(\zeta - r\ell) \tag{27}$$

By taking $\zeta = 4.5$, $\ell = 2$ in equation(16) *and inserting $m = \dfrac{4.5-2.5}{2} = 1$ in* (27)*, we get*

$f_2(\zeta) - f_2(a) - mf_1(a) =$

$$\frac{2^{4.5}}{(1-2^{-2})^2} - \frac{2^{2.5}}{(1-2^{-2})^2} - \frac{2^{2.5}}{(1-2^{-2})} = 22.62729.$$

$$\frac{1}{\Gamma(2)} \sum_{r=0}^{m-1} \frac{\Gamma(2+r)}{\Gamma(r+1)} f(\zeta - r\ell) = \sum_{r=0}^{1-1} \frac{\Gamma(2+r)}{\Gamma(r+1)} 2^{\zeta - rl}$$
$$= \frac{\Gamma(2)}{\Gamma(1)} 2^{4.5-2(0)} = 2^{4.5} = 22.6274$$

IV. Fractional Order Nabla Integration

The relation (27) in Remark III.8 lead us to form a conjecture in fractional order nabla integration. In this section, we enlarge the infinite and ν^{th} order nabla integration value equal to ν^{th} order fractional sum of f based at a respectively.

Definition IV.1. If $f : N(a) \to \mathbb{R}$ is the n^{th} order nabla integrable function based at a for every $n \in \mathbb{N}(1)$, then f is said to be ∞- order nabla intergrable function.

Remark IV.2. The function mentioned in Example III.2 is ∞- order nabla intergrable function.

Definition IV.3. Let $f : N(a) \to \mathbb{R}$ be a function for $\nu > 0$ and $\zeta,\ a \in N(a)$ such that $m - \nu \in \mathbb{N}(0)$. The fractional order ($\nu^{th}$ order) nabla sum of f based at a is defined by,

$$\nabla_a^{-\nu} f(\zeta) = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{m-1} \frac{\Gamma(\nu+r)}{\Gamma(r+1)} f(\zeta - r\ell). \tag{28}$$

Definition IV.4. Let $f : N(a) \to \mathbb{R}$, $\nu > 0$ and $m - \nu \in \mathbb{N}(0)$. If there exists a function, say $f_a^\nu : a + \nu + \mathbb{N}(0) \to \mathbb{R}$ whose value will be same as to $\nabla_a^{-\nu} f(\zeta)$ i.e.,

$$f_a^\nu(\zeta) = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{m-1} \frac{\Gamma(\nu+r)}{\Gamma(r+1)} f(\zeta - r\ell), \tag{29}$$

then the function f_a^ν is called as ν-order nabla-ℓ integration of f based at a.

V. Applications in Number Theory

In this section, we obtain several formula in number theory using fractional order nabla integration with examples.

Theorem V.1. *For the function $f(\zeta) = 2^\zeta$, if $m - \nu \in \mathbb{N}(1)$, where $\nu > 0$, the k^{th} integer integral of 2^ζ is obtained as*

$$\nabla_\ell^{-k} 2^\zeta \Big|_a^\zeta = \frac{2^a}{k!.\ell^k} (\zeta + (k-1)\ell - a)_\ell^{(k)}$$
$$+ \sum_{s=1}^{\infty} \frac{2^{a+s\ell}(1-2^{-\ell})^s}{(k+s)!.\ell^{k+s}} (\zeta + k\ell - a)_\ell^{(k+s)} \tag{30}$$

Proof: Let us assume the Maclaurin series as

$$f(\zeta) = a_0 + a_1 \frac{(\zeta - a)_\ell^{(1)}}{1!} + a_2 \frac{(\zeta - a)_\ell^{(2)}}{2!} + a_3 \frac{(\zeta - a)_\ell^{(3)}}{3!} + \cdots \tag{31}$$

where $a_i^{'}$ s to be determined.
Taking $\zeta = a$ in equation (31), we get $f(a) = a_0$.
Applying the ∇_ℓ operator on both sides of equation (31), we get

$$\nabla_\ell f(\zeta) = a_0 \nabla_\ell (\zeta - a)_\ell^{(0)} + \frac{a_1}{1!} \nabla_\ell (\zeta - a)_\ell^{(1)} +$$
$$\frac{a_2}{2!} \nabla_\ell (\zeta - a)_\ell^{(2)} + \frac{a_3}{3!} \nabla_\ell (\zeta - a)_\ell^{(3)} + \cdots \tag{32}$$

Applying Lemma II.3 in equation (32), we obtain

$$\nabla_\ell f(\zeta) = \frac{a_1}{1!}\ell + \frac{a_2}{2!} 2\ell(\zeta - \ell - a)_\ell^{(1)} + \frac{a_3}{3!} 3\ell(\zeta - \ell - a)_\ell^{(2)} + \cdots \tag{33}$$

Now taking $\zeta = a + \ell$; in equation (33), we get $a_1 = \dfrac{\nabla_\ell f(a+l)}{\ell}$

Again on applying ∇_ℓ operator on equation (33) we get,

$$\nabla_\ell^2 f(\zeta) = a_2\ell^2 + \frac{a_3\ell^2}{1!}(\zeta - 2\ell - a)_\ell^{(1)} + \frac{a_4\ell^2}{2!}(\zeta - 2\ell - a)_\ell^{(2)} + \cdots \tag{34}$$

Again taking $\zeta = a + 2\ell$ in equation (34), we have
$a_2 = \dfrac{\nabla_\ell^2 f(a + 2\ell)}{\ell^2}$

Applying nabla-ℓ operator on equation (34) we arrive on

$$\nabla_\ell^3 f(\zeta) = a_3\ell^3 + \frac{a_4\ell^3}{1!}(\zeta - 3\ell - a)_\ell^{(1)} + \frac{a_5\ell^3}{2!}(\zeta - 3\ell - a)_\ell^{(2)} + \cdots \tag{35}$$

Continuing the same process, in general taking $\zeta = a + k\ell$, we get $a_k = \dfrac{\nabla_\ell^k f(a+k\ell)}{\ell^k}$.

Thus equation (31) modifies as

$$f(\zeta) = f(a) + \frac{\nabla_\ell f(a+\ell)}{\ell}\frac{(\zeta-a)_\ell^{(1)}}{1!} + \frac{\nabla_\ell^2 f(a+2\ell)}{\ell^2}\frac{(\zeta-a)_\ell^{(2)}}{2!} + \frac{\nabla_\ell^3 f(a+3\ell)}{\ell^3}\frac{(\zeta-a)_\ell^{(3)}}{3!} + \cdots \quad (36)$$

Let $f(\zeta) = 2^\zeta$ in equation (36) and applying equation (16), we get

$$2^\zeta = 2^a + \frac{2^{a+\ell}(1-2^{-\ell})}{\ell}\frac{(\zeta-a)_\ell^{(1)}}{1!} + \frac{2^{a+2\ell}(1-2^{-\ell})^2}{\ell^2}\frac{(\zeta-a)_\ell^{(2)}}{2!} + \cdots \quad (37)$$

Applying ∇_ℓ^{-1} operator on both sides of equation (37) and using lemma II.3, we arrive on

$$\nabla_\ell^{-1} 2^\zeta = 2^a \frac{(\zeta-a)_\ell^{(1)}}{\ell} + \frac{2^{a+\ell}(1-2^{-\ell})}{\ell.1!}\frac{(\zeta+\ell-a)_\ell^{(2)}}{2\ell} + \frac{2^{a+2\ell}(1-2^{-\ell})^2}{\ell^2.2!}\frac{(\zeta+\ell-a)_\ell^{(3)}}{3l} + \cdots \quad (38)$$

Again Applying ∇_ℓ^{-1} operator on equation (38) and using lemma II.3, we get

$$\nabla_\ell^{-2} 2^\zeta\Big|_a^\zeta = \frac{2^a}{2!.\ell^2}(\zeta+\ell-a)_\ell^2 + \frac{2^{a+\ell}(1-2^{-\ell})}{3!.\ell^3}(\zeta+2\ell-a)_\ell^3 + \frac{2^{a+2\ell}(1-2^{-\ell})^2}{4!.\ell^4}(\zeta+2\ell-a)_\ell^4 + \cdots \quad (39)$$

Again repeating the process in (39), we get

$$\nabla_\ell^{-3} 2^\zeta\Big|_a^\zeta = \frac{2^a}{3!.\ell^3}(\zeta+2\ell-a)_\ell^3 + \frac{2^{a+\ell}(1-2^{-\ell})}{4!.\ell^4}(\zeta+3\ell-a)_\ell^4 + \frac{2^{a+2\ell}(1-2^{-\ell})^2}{5!.\ell^5}(\zeta+3\ell-a)_\ell^5 + \cdots \quad (40)$$

Proceeding like this, we get the general form as

$$\nabla_\ell^{-k} 2^\zeta\Big|_a^\zeta = \frac{2^a}{k!.\ell^k}(\zeta+(k-1)\ell-a)_\ell^{(k)} + \frac{2^{a+\ell}(1-2^{-\ell})}{(k+1)!.\ell^(k+1)}(\zeta+k\ell-a)_\ell^{(k+1)} + \frac{2^{a+2\ell}(1-2^{-\ell})^2}{(k+2)!.\ell^{k+2}}(\zeta+k\ell-a)_\ell^{(k+2)} + \cdots$$

Thus we arrive on equation (30). ∎

Corollary V.2. *For $\nu > 0$, if $a \in (-\infty, \infty)$, then the gamma function for the Theorem V.1is given by*

$$\nabla_\ell^{-\nu} 2^\zeta\Big|_a^\zeta = \frac{2^a}{\Gamma(\nu+1)}\frac{\Gamma(m+\nu)}{\Gamma(m)} + \sum_{s=1}^{\infty}\frac{2^{a+s\ell}(1-2^{-\ell})^s}{\Gamma(\nu+s+1)}\frac{\Gamma(m+\nu+1)}{\Gamma(m-s+1)} \quad (41)$$

Proof: Expanding equation (30) in the form of gamma function by using equation (8), we get,

$$\nabla_\ell^{-k} 2^\zeta\Big|_a^\zeta = \frac{2^a}{\Gamma(k+1)\ell^k}\frac{\Gamma\Big(\frac{\zeta+(k-1)\ell-a}{\ell}+1\Big)}{\Gamma\Big(\frac{\zeta+(k-1)\ell-a}{\ell}-k+1\Big)}\ell^k + \sum_{s=1}^{\infty}\frac{2^{a+s\ell}(1-2^{-\ell})^s}{\Gamma(k+s+1)\ell^{k+s}}\frac{\Gamma\Big(\frac{\zeta+k\ell-a}{\ell}+1\Big)}{\Gamma\Big(\frac{\zeta+k\ell-a}{\ell}-k-s+1\Big)}\ell^{k+s}$$

Replacing $\zeta = a + m\ell$ in the above equation, we get equation (41). ∎

Example V.3. *Let $k = 2,\ \zeta = 4; a = 2; \ell = 1$ in equation* (30)*, we get*

$$\nabla_\ell^{-2} 2^\zeta\Big|_a^\zeta = \frac{2^2}{2!.(1)}(4+1-2)_1^{(2)} + \frac{2^{2+1}(1-2^{-1})}{3!.(1)}(4+2-2)_1^{(3)} + \frac{2^{2+2}(1-2^{-1})^2}{4!.(1)}(4+2-2)_1^{(4)} + \cdots \quad (42)$$

Applying the equation (16) *in the left side of equation* (42)*, we get* $\nabla_\ell^{-2} 2^\zeta\Big|_a^\zeta =$

$$\frac{2^4}{(1-2^{-1})^2} - \frac{2^2}{(1-2^{-1})^2} - \frac{2^2}{1-2^{-1}}\frac{(4-2)_1^{(1)}}{1} = 32$$

The expansion of the right side of equation (42)*, we obtain*

$$\frac{4}{2}(3)^{(2)} + \frac{8(1/2)}{6}(4)^{(3)} + \frac{16(1/2)^2}{24}(4)^{(4)} + 0 = 32.$$

Corollary V.4. *For $\nu > 0$, if $\zeta, a \in (-\infty, \infty)$ such that $m \in \mathbb{N}(1)$, then*

$$\sum_{r=0}^{m-1}\frac{1}{\Gamma(\nu)}\frac{\Gamma(\nu+r)}{\Gamma(r+1)}2^{\zeta-r\ell} = \frac{2^a}{\Gamma(\zeta+1)}\frac{\Gamma(m+\nu)}{\Gamma(m)} + \sum_{s=1}^{\infty}\frac{2^{a+s\ell}(1-2^{-\ell})^s}{\Gamma(\nu+s+1)}\frac{\Gamma(m+\nu+1)}{\Gamma(m-s+1)} \quad (43)$$

Example V.5. *Taking $\nu = 2.5, \zeta = 6, a = 4, \ell = 1$ in equation* (43) *where,*
$m = \frac{\zeta-a}{\ell} = \frac{6-4}{1} = 2$*, we get*

$$\sum_{r=0}^{2-1}\frac{1}{\Gamma(2.5)}\frac{\Gamma(2.5+r)}{\Gamma(r+1)}2^{6-r} = \frac{2^4}{\Gamma(3.5)}\frac{\Gamma(4.5)}{\Gamma(2)} + \sum_{s=1}^{\infty}\frac{2^{4+s}(1-2^{-1})^s}{\Gamma(2.5+s+1)}\frac{\Gamma(2+2.5+1)}{\Gamma(2-s+1)} \quad (44)$$

On expanding the left side of the equation (44)*, we get*

$$\frac{1}{\Gamma(2.5)}\frac{\Gamma(2.5)}{\Gamma(1)}2^6 + \frac{1}{\Gamma(2.5)}\frac{\Gamma(3.5)}{\Gamma(2)}2^5 = 144$$

Now on expanding the right side of the equation (44)*, we get*

$$2^4(3.5) + \frac{2^5(\frac{1}{2})}{\Gamma(4.5)}\frac{\Gamma(5.5)}{\Gamma(2)} + \frac{2^6(\frac{1}{2})^2}{\Gamma(5.5)}\frac{\Gamma(5.5)}{\Gamma(1)} + 0 = 144$$

VI. Conclusion

Even though the fractional order ℓ-nabla sum of given function f based at a is availabe in the literature, hardly anyone has attempted to obtain the fractional order ℓ-nabla integration of f. We have developed this discrete fractional integration for factorials and geometric functions, and the functions having discrete Taylor's series expansion. This results generate several identities and formulae.

References

[1] Atici F. M. and Eloe P. W., *A Transform Method in Discrete Fractional Calculus*, International Journal of Difference Equations, Vol.2, Issue 2, pp. 165 - 176, 2007.

[2] Atici F. M. and Eloe P. W., *Discrete Fractional Calculus with the Nabla Operator*, Electronic Journal of Qualitative Theory of Differential Equations, Spec. Ed., pp.1 - 12, 2009.

[3] I. Podlubry, *Fractional differential equations: An Introduction to Fractional Derivatives, Fractional Differential Equations to methods of their Solution and Some of their Applications*, Academic press, California, 1998.

[4] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[5] Henry L Gray and N. Zhang, *On a New Definition of the Fractional Difference*, Mathematics of Computation, Vol.50, pp. 513 - 529, 1988.

[6] Hilfer R., *Applications of fractional Calculus in physics*, world Scientific, Singapore, 2000.

[7] Dassios, Ioannis K and Baleanu Dumitru I, *On a singular system of fractional nabla difference equations with boundary conditions*, Boundary Value Problems, Springer Open, pp. 1-24, 2013.

[8] Goodrich C., Peterson A.C., *Discrete Fractional Calculus*, Springer International Publishing Switzerland, (e-book: DOI: 10.1007/978-3-319-25562-0), 2015.

[9] Haewon Byeon, M. Abisha, V. Rexma Sherine, G. Britto Antony Xavier, S. Prema, Vediyappan Govindan, Hijaz Ahmad, D. Piriadarshan, and Salwa El-Morsy, *Discrete version of fundamental theorems of fractional order integration for nabla operator*, Journal of Mathematics and Computer Science, Vol. 34, Issue 4, pp. 381-393, 10 April 2024.

[10] M. Maria Susai Manuel, G. Britto Antony Xavier and E. Thandapani, *Theory of Generalized Difference Operator and Its Applications*, Far East Journal of Mathematical Sciences, Vol.20, Issue 2, pp. 163 - 171, 2006.

[11] M. Abisha, D. Saraswathi, G. Britto Antony Xavier, S. Don Richard, *Modelling With Fractional Order Continuous and Discrete (Delta) Integration*, Asia Pacific Journal of Mathematics, Vol. 11 , pp.34, 11 March 2024.

[12] M. Maria Susai Manuel, G. Britto Antony Xavier and V. Chandrasekar, *Theory and Application of the Generalized Difference Operator of the* n^{th} *Kind (Part - I)*, Demonstratio Mathematica, Vol. 45, Issue 1, pp. 95 - 106, 2012.

[13] M. Maria Susai Manuel, Adem Kilicman, G. Britto Antony Xavier, R. Pugalarasu and D. S. Dilip, *On the solutions of Second order Generalized Difference Equation*, Advances in Difference Equations, (DOI:1186/1687-1847-2012-105), 2012.

[14] M. Maria Susai Manuel, Adem Kilicman, G. Britto Antony Xavier, R. Pugalarasu and D. S. Dilip, *An Application on the Second order Generalized Difference Equations*, Advances in Difference Equation, Vol. 35, (DOI:10.1186/1687-1847-2013-35), 2013.

[15] Mohammed M. Al-Shamiri, V. Rexma Sherine, G. Britto Antony Xavier, D. Saraswathi, T.G. Gerly, P. Chellamani, Manal ZM Abdalla, N. Avinash, and M. Abisha. *A New Approach to Discrete Integration and Its Implications for Delta Integrable Functions,* Mathematics, Vol. 11, Issue 18, pp. 3872, 11 September, 2023.