Perturbation Methods for Solving Non-Linear Ordinary Differential Equations

Malak Hijazi, Mahmood Shareef Ajeel, Kamel Al-Khaled, Hala Al-Khalid

***Abstract*—The goal of this paper is to use general perturbation theory that is, singular perturbation theory and regular perturbation theory to solve both linear and non-linear differential equations. We initially demonstrate the regular perturbation method's application to the classification of nonlinear differential equations in order to demonstrate its efficacy. Next, in the event that normal perturbation approaches prove fruitless, we employ and demonstrate Lindstedt-Poincaré and singular perturbation methods to approximate periodic solutions to nonlinear ordinary differential equations. Upon computing the approximate solutions by Lindstedt-Poincaré, or by singular perturbation methods, we compare it with the exact solution found using MATHEMATICA software, we determine the absolute error and demonstrate the accuracy of our approach.**

***Index Terms*—Non-linear Differential Equations, Perturbation Methods, Periodic Solutions, Approximate Solutions.**

I. Introduction

NONLINEAR ordinary differential equations with perturbation are used to model a wide range of applied problems. These models are widely used in scientific fields such as biology, economics, and physics, and the perturbation method is used in the modeling of various engineering and scientific problems. Ordinary differential equations (ODE) are widely used in physics, electrical engineering (characteristics of voltage/current variation with time in a circuit), mechanical engineering (vibration in mass spring systems), and a variety of other fields. Conventional methods for solving (ODE) are effective for first and second order problems. The problem's complexity increases with increasing order. Thus, rather than attempting an exact solution, determining an approximate solution is also advantageous, solving (ODE) is viewed as an optimization problem.

Analytical techniques for solving non-linear differential equations include the perturbation technique. In order to address certain real-world issues, engineers frequently utilize this method. Using this technique, we typically get a number of noteworthy and intriguing results. But each perturbation method has its own set of drawbacks. Firstly, at least one unknown must be expressed in a sequence of small parameters because all perturbation techniques are dependent on small or large parameters. Unfortunately, small parameters are not present in all non-linear differential equations. Sometimes certain initial or boundary conditions are unnecessary for the simplified linear equations, and most of the time the simplified linear equations differ from the original non-linear differential equation in terms of their properties. Therefore, it's possible that the corresponding initial approximations are not very accurate.

In [1] the well-known Fourier series expansion serves as an approximation function. Finding solutions for nonlinear differential equations to a wide range of model of physical phenomena in scientific applications is critical due to their widespread use in scientific research. Because exact solutions cannot always be determined, approximate solutions to these equations are frequently evaluated. An effective and rapidly convergent analytical technique for nonlinear differential equations is introduced in [2]. As a result, various analytical and numerical solutions to these problems have been proposed.

Closed-form solutions to differential equation models on physics and engineering problems are difficult to obtain, particularly for nonlinear ones. In most cases, only approximate solutions (analytical or numerical) can be anticipated. The perturbation method is a well-known analytical approach to nonlinear problems. According to [3], it is predicated on the existence of small and large parameters, referred to as perturbation quantities. In [4], they showed that the vast majority of oscillation problems in engineering sciences are nonlinear, making analytical solutions difficult. Nonlinear oscillator models have recently gained popularity in the physical and chemical sciences. Due to the limitations of existing exact solutions, many researchers have spent the last few decades developing various analytical methods for solving nonlinear oscillation systems. Nonlinear differential equations can be used to solve a variety of real-world problems in both pure and applied sciences.

Studies in [5] and [6], discussed nonlinear ordinary differential equations (ODE) that are crucial for practical applications, as they describe most nonlinear dynamical phenomena in the real world. In recent decades, there has been a great deal of interest in the periodic solutions of nonlinear oscillators, which deal with oscillatory phenomena, not only in mechanics and physics but also in other engineering fields. According to [7], study the period and periodic solution of nonlinear oscillators that are key topics in nonlinear physical problem research.

Initially, almost all perturbation techniques rely on the assumption that the equations have a small parameter [8].

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The use of perturbation techniques is severely restricted by what is known as the small parameter assumption. The primary goal of this paper is to present techniques for solving non-linear differential equations. In [9], perturbation theory, which is used in a variety of fields, is an effective tool for determining approximate solutions to complex problems by beginning with the exact solution. To solve nonlinear problems, the study in [10] proposes the perturbation methods that are used to obtain an approximate solution to Gelfand differential equation, which governs the dynamics of combustible gases.

As shown in [11], the perturbation method is a pioneering technique for dealing with various types of nonlinear problems. J. H. Poincare method first appeared in the early nineteenth century. In [12], perturbation theory is used systematically to create root-finding algorithms. In [13], perturbation methods are used to deal with nonlinear ordinary differential equations. While authors of [14], introduces a novel perturbation method and they use linearizing techniques to the nonlinear ordinary differential equation with respect to a zeroth order solution. The most flexible tools for nonlinear analysis of physical problems are provided by the perturbation method, according to the authors of [15]. These tools are second-order strongly nonlinear differential equations in the form of

$$y''(t)+y(t)+\epsilon f(y(t),y'(t))=0,\ y(0)=1,y'(0)=0 \quad (1)$$

where ϵ is a sufficiently small parameter, so the nonlinear term $\epsilon f(y(t),y'(t))$ is relatively small. Perturbation techniques are used in [16] to find an approximate solution for nonlinear equations that are not amenable to exact solutions. The question in [17] is how to apply the Lindstedt-Poincaré perturbation method to get reasonable asymptotic expansions for second-order nonlinear differential equation periodic solutions. As per [18], approximation methods were applied to particular problems or ranges of physical parameters because of the nature of nonlinear phenomena. According to [19], the standard Lindstedt-Poincaré method was used for studying nonlinear oscillations. However, the method is limited to solving problems with small parameters. The small parameter limits the applications of the standard Lindstedt-Poincaré method. In [20], Homotopy Perturbation uses traditional perturbation methods and Homotopy techniques to reduce nonlinear problems to linear problems and generate rapid convergent series solutions in most cases.

In [21], the Chapman-Jouguet model for the Chaplygin gas's perturbed initial value problem was examined. A novel method for employing point transformation to transform nonlinear third-order ordinary differential equations into linear forms was created by the authors of [22]. Haar wavelet techniques are developed in [23] to solve ordinary differential equations with initial or boundary conditions. The optimal Homotopy asymptotic method is used in [24] to obtain both analytical and numerical solutions for the velocity profiles of the resulting differential equations.

In [25] and [26] homotopy perturbation method is used to solve linear and nonlinear second-order differential equations with non-constant coefficients. The method produces solutions in convergent series forms with easily computed terms. Authors of [27], they propose a homotopy perturbation method for finding exact solutions to linear and nonlinear partial differential equations. The homotopy perturbation method can be used to find either an exact or approximate solution to the problem. According to [28], the homotopy perturbation method was used a semi-analytical approach for solving nonlinear differential equations.

This paper will primarily focus on the application of perturbation methods to solve nonlinear ordinary differential equations, specifically regular and singular perturbation techniques, which will be defined and explained later, or readers may refer to the original work as in [29]. There are numerous effective approaches to solving nonlinear differential equations. The first is regular perturbation, which is a simple method for finding analytic approximate solutions to nonlinear ordinary differential equations. The basic idea behind this method is to replace a particular type of perturbation series.

$$y(t)=y_0(t)+\epsilon y_1(t)+\epsilon^2 y_2(t)+...$$

into the differential equation $F(t,y,y',y'',.......,\epsilon)=0$, where t is independent variable, and y is the dependent variable, and parameter $0<\epsilon\ll 1$. If this method is successful, then $y_0(t)$ should be a solution of the unperturbed equation $F(t,y,y',y'',.....)=0$. If regular perturbation methods fail, we will use the Lindstedt-Poincaré method [30], and singular perturbation method to find solutions to nonlinear ordinary differential equations.

As shown in [31], the Taylor series model of function behavior is not always effective, the same is true for model systems: a regular perturbation expansion will not always accurately represent our system's behavior. Here are some possible warning signs that something is wrong:

1) An nth order ODE should have n linearly independent solutions, but the equation at $\epsilon=0$ yields the incorrect number of solutions. This approach won't provide all of the potential answers to the entire equation if the equation that results from setting $\epsilon=0$ has fewer solutions. This occurs when $\epsilon=0$ and the coefficient of the highest derivative is zero.
2) The coefficients of ϵ can grow without bound: in the case of an expansion $f(x)=f_0(x)+\epsilon f_1(x)+\epsilon^2 f_2(x)+.....$, the series may not be valid for some values of x if some or all of the $f_i(x)$ become very large. Say, for example, that $f_2(x)\to\infty$ while $f_1(x)$ remains finite, then $\epsilon f_1(x)$ is no longer strictly larger than $\epsilon^2 f_2(x)$ and there is serious trouble.

One more important application for the use of perturbation, we aims to present an analytical approximation study of periodic solutions to second-order nonlinear differential equation systems. Despite the fact that our analysis is based on the Lindstedt-Poincaré method, the chosen development differs from the standard approach. As a result, we see an improvement in the approximation process.

II. Background Materials

This section introduces some of the applied mathematics topics covered in this paper, as well as definitions of big "$\mathcal{O}$", little "o", asymptotic series, dominate balance, the Duffing equation, and others.

Definition 1: Little o, let $f(\epsilon)$ and $g(\epsilon)$ be given function of ϵ, we say $f(\epsilon)$ is a little "o" of $g(\epsilon)$ as $\epsilon \to 0$ and write $f(\epsilon) = o(g(\epsilon))$ as $\epsilon \to 0$ if $\lim_{\epsilon\to 0} \dfrac{f(\epsilon)}{g(\epsilon)} = 0$.

Definition 2: Big $\mathcal{O}$, we say $f(\epsilon) = \mathcal{O}(g(\epsilon))$ as $\epsilon \to 0$ if and only if $|f(\epsilon)| \leq \mu|g(\epsilon)|$ $\forall\epsilon$ with $0 < |\epsilon| < \epsilon_0$ i.e : $|\frac{f(\epsilon)}{g(\epsilon)}| \leq \mu$.

Definition 3: A formal sequence of functions known as an asymptotic series is one that, when terminated after a finite number of terms, approximates a given function as its argument gets closer to a particular, frequently infinite point. The asymptotic series

$$\sum_{n=0}^{N} a_n Q_n(\epsilon)$$

is said to be asymptotic expansion to N terms of $f(\epsilon)$ as $\epsilon \to 0$ if

$$\lim_{\epsilon\to 0} f(\epsilon) \sim \sum_{n=0}^{N} a_n Q_n(\epsilon) + \mathcal{O}(Q_{N+1}(\epsilon)).$$

Asymptotic series also can be expressed in terms of Taylor's approximation of finite terms, as

$$f(\epsilon) \sim \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!}\epsilon^n$$

In order to ascertain the asymptotic behavior of solutions to an ordinary differential equation without actually solving the equation, mathematicians employ the dominant balance method. The research presented in [32] demonstrates that, without solving the equation entirely, the dominant balance technique establishes the asymptotic behavior of solutions to ordinary differential equations. No term in an equation can be greater than any other, according to the dominant balance theory. Even though there could be more than two comparable terms, it's more likely that two terms have a dominant balance and the rest are minor. The initial stage involves presuming that certain terms in the equation can be disregarded, resulting in the "dominant balance" being formed by the remaining terms. We have to go back to our initial strategy and make sure it's consistent if the reduced equation can be solved. Every possible dominant balance in the original equation goes through the same procedure again. The singular perturbation method applies dominant balance to inner solutions, as described in [33].

The Duffing equation describes the Duffing oscillator, which is a second-order, nonlinear differential equation of the form

$$my''(t) + cy'(t) + ky(t) + hy^3 = f(t). \quad (2)$$

The restoring force consists of a linear spring and a cubic nonlinear displacement function. In a mechanical oscillator, m represents the system mass, c is the viscous damping coefficient, and k controls the restoring force size, and h controls the degree of non-linearity in the restoring process. The $y(t)$ function represents displacement at time t. $y'(t)$ represents velocity, and $y''(t)$ denotes acceleration. The Duffing equation describes an oscillator with a cubic nonlinear $y^3(t)$, as shown in [34]. The equation can be approximated by finding a series solution and extracting only a few terms from it. These approximate solutions are extremely useful for determining the oscillator's free vibration when set to zero, as well as the system's response to a non-zero force. Many researchers use this equation [35] and [36] to demonstrate the behavior of nonlinear dynamical systems.

III. Perturbation Theory

We will examine perturbation theory in this section of the paper. We also study singular and regular perturbation theory. In perturbation theory, problems are expressed using a formal power series, also called a perturbation series, in small parameters ϵ. In a power series, the first term gives the exact answer to a problem, and the remaining terms explain any variations from the answer. There are many physical processes with mathematical models that have equations that are not amenable to analytical solution. There are two basic methods for handling these equations:

- numerical methods and
- analytic approximations.

We address analytical approximations and the process of developing a systematic approximation of the solution to an apparently unsolvable problem in this paper.

The perturbation method is one of many mathematical strategies for estimating a solution to a problem that is impossible to solve exactly. Pinchation theory may be used if the problem can be stated by appending a "small" term to the mathematical formulation of an exactly solvable problem. The final solution is expressed using the perturbation method, and this expression is then converted into a formal power series in a "small" parameter called a perturbation series, which measures the deviation from the precisely solvable problem. The leading term in a power series indicates the solution to the exactly solvable problem, and the terms that follow explain how the solution deviated from the original problem due to the deviation. [37].

IV. Regular Perturbation Expansions

An approach to solving a perturbed problem that uses a convergent expansion that combines higher-order corrections and the unperturbed solution is known as a regular perturbation expansion. Promptly solving mathematical problems is often a challenge, and when it is possible, the solution heavily depends on the parameters. $\epsilon = 0$ can have an easy solution if a parameter, such as ϵ, is identified. If ϵ is small but non-zero, one might wonder how the solution changes. These inquiries, which involve differential equations, have a methodical response thanks to perturbation theory.

A. Solutions of Differential Equations

We are attempting to solve the following differential equation in x, $(x \geq 0)$

Example 1: Consider the differential equation

$$\frac{df}{dx} + f(x) - \epsilon f^2(x) = 0, \quad f(0) = 2. \tag{3}$$

We can't solve this differential equation directly. However, let's look at the ODE, for the value of $\epsilon = 0$, to get

$$\frac{df}{dx} + f(x) = 0, \quad f(0) = 2.$$

Which is a simple first order ODE and has solution given by

$$f(x) = 2e^{-x}.$$

or,

$$f(x) = 2e^{-x} + \epsilon f_1(x) + \epsilon^2 f_2(x) + \dots .$$

Substituting this into the original differential equation (3), we obtain $(-2e^{-x}+\epsilon f_1'(x)+\epsilon^2 f_2'(x)+\epsilon^3 f_3'(x)+\dots)+(2e^{-x}+\epsilon f_1(x)+\epsilon^2 f_2(x)+\epsilon^3 f_3(x)+\dots) - \epsilon(2e^{-x}+\epsilon f_1(x)+\epsilon^2 f_2(x)+\epsilon^3 f_3(x)+\dots)^2 = 0$, and we can collect powers of ϵ, to obtain:-

Order ϵ^0 terms :

$$\mathcal{O}(1): \ -2e^{-x} + 2e^{-x} = 0$$

Order ϵ terms :

$$\mathcal{O}(\epsilon): \ f_1'(x) + f_1(x) = 4e^{-2x}, \quad f_1(0) = 0.$$

This is a linear differential equation of first type, we may use the idea of integrating factor technique, then the solution is given by

$$f_1(x) = 4(e^{-x} - e^{-2x}).$$

Order ϵ^2 terms :

$$\mathcal{O}(\epsilon^2): \ f_2'(x) + f_2(x) = 4e^{-x} f_1(x), \quad f_2(0) = 0. \tag{4}$$

Once more, we have a first-type linear differential equation. Using the concept of the integrating factor technique, we can solve Equation (4) by

$$f_2(x) = 8(e^{-x} - 2e^{-2x} + e^{-3x}).$$

This demonstrates how information about the complete solution can be obtained through a perturbation expansion. Keep in mind that, for the terms that we've computed,

$$f_n(x) = 2^{n+1} e^{-x}(1 - e^{-x})^n.$$

Suggesting found solution has the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \epsilon^n 2^{n+1} e^{-x}(1-e^{-x})^n \\ &= 2e^{-x} \sum_{n=0}^{\infty} (2\epsilon(1-e^{-x}))^n = \frac{2e^{-x}}{1 - 2\epsilon(1-e^{-x})}. \end{aligned}$$

Example 2: Consider the second order ordinary differential equation of the form.

$$y''(t) + (1+\epsilon)y(t) = 0, \ y(0) = 1, \ y'(0) = 0. \tag{5}$$

Which has exact solution as

$$y(t) = \cos(\sqrt{1+\epsilon})t.$$

To find an approximate solution, we use the regular perturbation method. Consider the perturbation series

$$y(t) = y_0(t) + \epsilon y_1(t) + \dots,$$

We plugin the perturbation series into Equation (5) to get the ODE.

$$(y_0''(t) + \epsilon y_1''(t) + \dots) + (1+\epsilon)(y_0(t) + \epsilon y_1(t) + \dots) = 0, \tag{6}$$

$$y_0(0) + \epsilon y_1(0) + \dots = 1$$

$$y_0'(0) + \epsilon y_1'(0) + \dots = 0.$$

The coefficients of different powers of ϵ are then equated to zero, yielding

$$\mathcal{O}(1): y_0''(t) + y_0(t) = 0, \quad y_0(0) = 1, \ y_0'(0) = 0. \tag{7}$$

Solving Equation (7), then we get

$$y_0(t) = \cos t.$$

Also, coefficients of ϵ, are

$$\mathcal{O}(\epsilon): y_1''(t) + y_1(t) = -\cos t, \quad y_1(0) = 0, \ y_1'(0) = 0. \tag{8}$$

The solution of Equation (8) is

$$y_1(t) = \frac{-t}{2} \sin t.$$

Thus, the first two-term of the approximative solution of is as follows:

$$y(t) = \cos t - \epsilon \frac{t}{2} \sin t.$$

As seen in Figures 1, 2 and Tables I, II, the solution obtained by the regular perturbation method exhibits excellent agreement with the exact solution if we choose ϵ to be small (if ϵ is smaller, the solution will be better). Since the absolute error is low, the regular perturbation approach is a very good way to solve this equation.

TABLE I
NUMERICAL RESULTS OBTAINED BY REGULAR PERTURBATION METHOD OF EXAMPLE (2) WHEN $\epsilon = 0.01$.

t	approximate	exact	absolute error
0	1	1	0
$\frac{\pi}{2}$	-0.00785398	-0.00783436	1.96175E-05
π	-1	-0.9998770	1.22755E-04
$\frac{3\pi}{2}$	0.0235619	0.0235012	6.07760E-05
2π	1	0.9995090	4.90988E-04
$\frac{5\pi}{2}$	-0.0392699	-0.0391622	1.07704E-04
3π	-0.0392699	-0.9988950	1.10461E-03
$\frac{7\pi}{2}$	0.0549779	0.0548136	1.64247E-04
4π	1	0.998037	1.96347E-03

Example 3: Consider the second order nonlinear differential equation of the form

$$m\frac{d^2y}{d\tau^2} + a\Big(\frac{dy}{d\tau}\Big)^2 + ky(\tau) = 0, \ y(0) = x_0, \ \frac{dy}{d\tau}(0) = 0,$$

here the constant a satisfy $0 < a \ll 1$.
We utilize the two new variables, t (time) and $y(t)$ (distance), as well as ϵ for any small parameter to reformulate the problem in dimensionless form. We utilize the two new variables, t (time) and $y(t)$ (distance), as well as ϵ for any small parameter to reformulate the problem in dimensionless

TABLE II
NUMERICAL RESULTS BY REGULAR PERTURBATION METHOD OF EXAMPLE (2) WHEN $\epsilon = 0.1$.

t	approximate	exact	absolute error
0	1	1	0
$\frac{\pi}{2}$	-0.0785398	-0.0765937	1.94615E-03
π	-1	-0.988267	1.17332E-02
$\frac{3\pi}{2}$	0.235619	0.227984	7.63581E-03
2π	1	0.953343	4.66574E-02
$\frac{5\pi}{2}$	-0.392699	-0.374024	1.86754E-02
3π	-1	-0.896047	1.03953E-01
$\frac{7\pi}{2}$	0.549779	0.511287	3.8492E-02
4π	1	0.817724	1.82276E-01

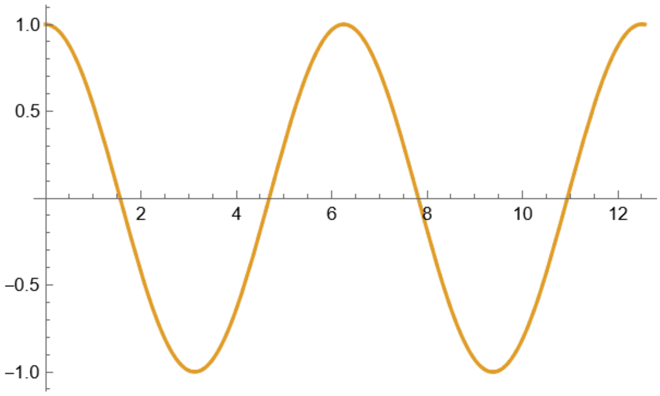

Fig. 1. The exact solution (dashed line) and approximation solution (orange line) when $\epsilon = 0.01$ for example (2)

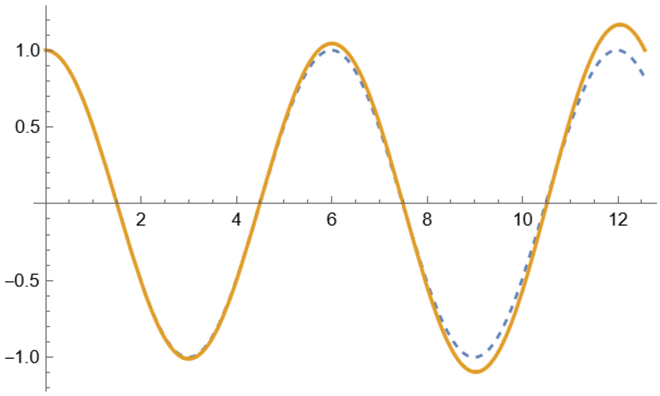

Fig. 2. The exact solution (dashed line) and approximation solution (orange line) when $\epsilon = 0.1$ for example 2

form, m is mass, $a = \ddot{x}(t)$ is acceleration with units $\frac{L}{T^2}$, and F is the force, L is Length and T is time. The constants appeared in the given differential equation are of dimensions: $[k] = MT^{-2}$, $[a] = ML^{-1}$, $[x_0] = L$, $[m] = M$, and we scale the new variable's as: $u(t) = \frac{y(\tau)}{x_0}$, and $t = \tau/\sqrt{\frac{m}{k}}$. The given ODE reduces to

$$m\frac{kx_0}{m}\frac{d^2u}{dt^2} + a(x_0^2\frac{k}{m})\Big(\frac{du}{dt}\Big)^2 + kx_0u(t) = 0,$$

simplify, we obtain

$$\frac{d^2u}{dt^2} + \epsilon\Big(\frac{du}{dt}\Big)^2 + u(t) = 0, \quad u(0) = 1, \quad \frac{du}{dt}(0) = 0,$$

where the small parameter, $\epsilon = \frac{ax_0}{m} \ll 1$. Next, we substitute $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$ into the above differential equation, to obtain

$u_0''(t) + \epsilon u_1''(t) + \ldots + \epsilon\Big(u_0'^2(t) + 2u_0'(t)u_1'(t)\epsilon + \ldots\Big) + u_0(t) + \epsilon u_1(t) + \ldots = 0.$

Equation coefficients of equal powers of ϵ, we get

$$u_0(0) + u_1(0)\epsilon + \ldots = 1$$
$$u_0'(0) + u_1'(0)\epsilon + \ldots = 0.$$

The ODE that represents the leading order is

$$u_0''(t) + u_0(t) = 0, \;\; u'(0) = 0, \; u(0) = 0,$$

where the solution is

$$u_0(t) = \cos t.$$

Finally, we solve initial value problem for $u_1(t)$ to find

$$u_1''(t) + u_1(t) = -\sin^2 t. \tag{9}$$

Solving the Equation (9), we get

$$u_1(t) = -\frac{1}{2} - \frac{1}{6}\cos 2t + \frac{2}{3}\cos t.$$

The approximate solution is

$$u(t) = u_0(t) + u_1(t) = \cos t + (-\frac{1}{2} - \frac{1}{6}\cos 2t + \frac{2}{3}\cos t)\epsilon.$$

Example 4: The nonlinear differential equation of second order that follows is of the following form:

$$\frac{d^2u}{dt^2} + u(t) = a + \epsilon u^2(t), \quad u(0) = b, \quad \frac{du}{dt}(0) = 0. \tag{10}$$

To solve the above ODE, we substitute $u(t) = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots$, we get,

$\Big(u_0''(t) + \epsilon u_1''(t) + \epsilon^2 u_2''(t) + \ldots\Big)^2 + \Big(u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots\Big) = a + \epsilon(u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + \ldots)^2,$

The initial conditions, also reduces to

$$u_0(0) + \epsilon u_1(0) + \ldots = b$$
$$u_0'(0) + \epsilon u_1'(0) + \ldots = 0.$$

Leading terms of order :$\mathcal{O}(1)$ i.e : $\epsilon = 0$

$$\frac{d^2u_0}{dt^2} + u_0(t) = a.$$

Where the solution is given by

$$u_0(t) = (b - a)\cos t + a.$$

For more terms, we need to find $\mathcal{O}(\epsilon)$, the initial value problem for $u_1(t)$ is given by

$$\frac{d^2u_1}{dt^2} + u_1(t) = u_0^2(t), \;\; u_1(0) = 0, \;\; u_1'(0) = 0.$$

Then the solution of $u_1(t)$ is
$u_1(t) = (a^2\cos t + a(b-a)\frac{\cos 2t}{2} + (b-a)^2\frac{\cos^3 t}{3})\cos t + (a^2\sin t + a(b-a)(t + \frac{\sin 2t}{2}) + (b-a)^2(t - \frac{\sin^3 t}{3}))\sin t + -(a^2 + \frac{a(b-a)}{2} + \frac{(b-a)^2}{3})\cos t.$

Then the approximate solution of $u(t)$ is
$u(t) = ((b-a)\cos t + a) + ((a^2\cos t + a(b-a)\frac{\cos 2t}{2} + (b-a)^2\frac{\cos^3 t}{3})\cos t + (a^2\sin t + a(b-a)(t + \frac{\sin 2t}{2}) + (b-a)^2(t - \frac{\sin^3 t}{3}))\sin t - (a^2 + \frac{a(b-a)}{2} + \frac{(b-a)^2}{3})\cos t)\epsilon.$

V. Perturbation Methods for Strongly Nonlinear Equations

We use two techniques in this section to explore and illustrate the science of oscillation in solutions to nonlinear differential equations: the singular perturbation method and the Lindstedt-Poincaré method. Several methods are available for obtaining a near-exact approximate solution to a variety of nonlinear differential equations. For a wide range of nonlinear differential equations, the strategy behind these techniques works well. The primary procedures for the two approaches that will be applied in this paper will be discussed.

A. Lindstedt-Poincaré Method

A perturbation theory technique known as the Lindstedt-Poincaré (LP) method was developed by Henri Poincare and Anders Lindstedt. Approximate periodic solutions to ordinary differential equations can be found approximately around 1890, if regular perturbation approaches are unsuccessful. For second-order nonlinear differential equations, we apply the Lindstedt-Poincaré method in this section. We will start with a brief discussion of the traditional LP method. To make the excitation and nonlinearity appear simultaneously in the perturbation scheme, we must arrange them in a certain order. A new variable is introduced, $\tau = \omega t$, so that τ appears in the governing equation. The standard LP method states that u and ω are typically expanded in powers of ϵ as,

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2^2 + ...$$

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + \epsilon^2 u_2^2(\tau) + ...$$

Upon substituting the two aforementioned equations into the newly created differential equation and setting the coefficients of powers of ϵ to zero, multiple readily solvable differential equations for the unknowns $u_0, u_1, ...$ arise, that can be solved sequentially. Finally the solution is expressed.

Example 5: Consider the second-order nonlinear differential equation

$$y''(t)+y(t) = \epsilon y(t)(1-y'^2(t)),\ y(0) = 1,\ y'(0) = 0. \quad (11)$$

The approximate solution by regular perturbation method is

$$y(t) = \cos t + \Big(\frac{3}{8}t\sin t - \frac{1}{32}\cos 3t + \frac{1}{32}\cos t\Big)\epsilon.$$

The failure of the regular perturbation method stems from the need for a periodic approximate solution, which includes a secular term $t\sin t$. The transformation $\tau = \omega t$ converts Equation (11) to and ODE in which we can use the Lindstedt-Poincaré technique.

$$\omega^2 u''(\tau) + u(\tau) - \epsilon u(\tau) + \epsilon\omega^2 u(\tau)u'^2(\tau) = 0, \quad (12)$$

subject to the initial conditions $u(0) = 1,\ u'(0) = 0$. Where $y(t) = u(\tau)$, then with

$$\omega = 1 + \epsilon\omega_1 + ... \quad (13)$$

and

$$u(\tau) = u_0(\tau) + \epsilon u_1(\tau) + ... \,. \quad (14)$$

Through inserting Equations (13) and (14) into Equation (12), yields

$$\Big(1+\epsilon\omega_1+...\Big)^2\Big(u_0''(\tau)+\epsilon u_1''(\tau)+...\Big)+\Big(u_0(\tau)+\epsilon u_1(\tau)+ ...\Big) - \epsilon\Big(u_0(\tau) + \epsilon u_1(\tau) + ...\Big) + \epsilon\Big(1 + \epsilon\omega_1 + ...\Big)^2\Big(u_0(\tau) + \epsilon u_1(\tau) + ...\Big)\Big(u_0'(\tau) + \epsilon u_1'(\tau) + ...\Big)^2 = 0,$$ with conditions

$$u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + ... = 1$$

$$u_0'(0) + \epsilon u_1'(0) + \epsilon^2 u_2'(0) + ... = 0.$$

Equating the coefficients of various powers of ϵ to 0, as

$$\mathcal{O}(1):\ u_0''(\tau) + u_0(\tau) = 0,\ \ u_0(0) = 1,\ u_0'(0) = 0$$

and for $\mathcal{O}(\epsilon)$, we have

$$2\omega_1 u_0''(\tau) + u_1''(\tau) + u_1(\tau) - u_0(\tau) + u_0(\tau)(u_0'(\tau))^2 = 0,$$

with conditions $u_1(0) = 0,\ u_1'(0) = 0$. The solutions after avoiding the occurrence of secular terms are given as: The solution of the Leading order equation is

$$u_0(\tau) = \cos(\tau).$$

While the solution of $\mathcal{O}(\epsilon)$ equation is

$$u_1(\tau) = \frac{1}{32}(\cos\tau - \cos 3\tau).$$

Finally, a two-term approximation of the periodic solution of Equation (11), when we replace $\tau = (1 - \frac{3\epsilon}{8})t$ and $y(t) = u(\tau)$ is given by

$$y(t) = \cos(1 - \frac{3\epsilon}{8})t + \frac{\epsilon}{32}(\cos(1 - \frac{3\epsilon}{8})t - \cos 3(1 - \frac{3\epsilon}{8})t).$$

But the exact solution of Equation (11) by MATHEMATICA software is

$$y(t) = \cos(\sqrt{\frac{(1-\epsilon)}{(1+\epsilon)}})t. \quad (15)$$

The Lindstedt-Poincaré method yields an excellent agreement with the one obtained by MATHEMATICA software (DSolve tool) if we choose ϵ to be small (the solution would be better if ϵ is smaller). It has been demonstrated by Nayfeh and Mook [38] that this solution is fully consistent with the harmonic balance method solution when ϵ has a small value. This solution contains some glaring mistakes if ϵ is large. Which is demonstrated by the small absolute error in Figures 3, 4 and Table III, IV (the absolute error is small).

TABLE III
Results for obtained approximation solution using Lindstedt-Poincaré method of example (5) when $\epsilon = 0.01$.

t	approximate	exact	absolute error
0	1	1	0
0.1	0.995104	0.995103	1.000 E-06
0.2	0.980338	0.980461	1.220E-04
0.3	0.955851	0.956214	3.630E-04
0.4	0.921883	0.922604	7.210E-04
0.5	0.878772	0.879957	1.185E-03
0.6	0.826944	0.828692	1.748E-03
0.7	0.766915	0.769311	2.396E-03
0.8	0.699277	0.702395	3.120E-03
0.9	0.624704	0.628600	3.896E-03
1	0.543933	0.548648	4.715E-03

TABLE IV
RESULTS FOR OBTAINED APPROXIMATION SOLUTION USING LINDSTEDT-POINCARÉ METHOD OF EXAMPLE (5) WHEN $\epsilon = 0.05$.

t	approximate	exact	absolute error
0	1	1	0
0.1	0.995249	0.99548	2.31E-04
0.2	0.981038	0.981959	9.21E-04
0.3	0.957487	0.959561	2.074E-03
0.4	0.924801	0.928488	3.687E-03
0.5	0.883262	0.889021	5.759E-03
0.6	0.833238	0.841516	8.278E-03
0.7	0.775175	0.786403	1.1228E-02
0.8	0.709597	0.72418	1.4583E-02
0.9	0.637108	0.65541	1.8302E-02
1	0.558385	0.580715	2.233E-02

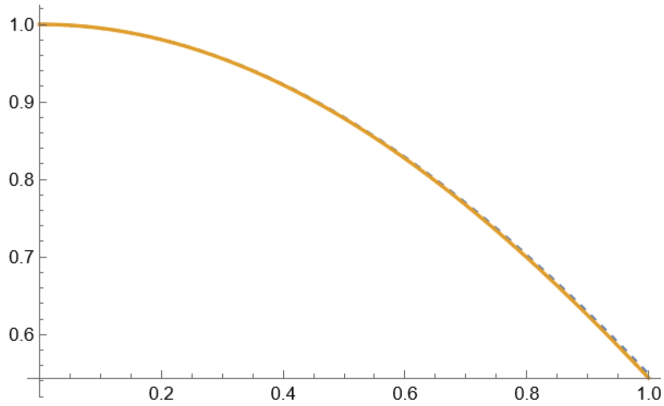

Fig. 3. The exact solution using Lindstedt-Poincaré method (dashed line) and the approximate solution (Orange line) of example (5) when $\epsilon = 0.01$.

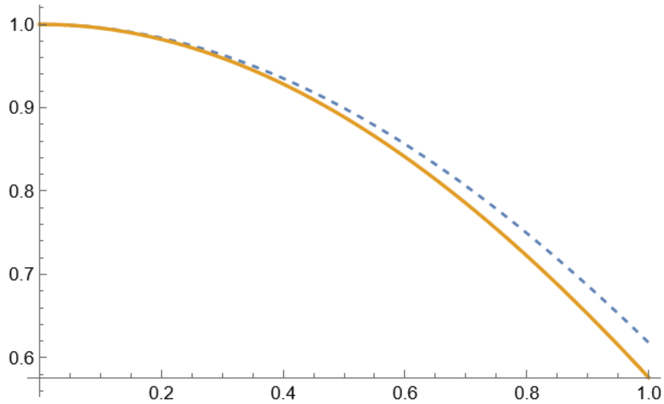

Fig. 4. The exact solution using Lindstedt-Poincaré method (dashed line) and the approximate solution (Orange line) of example (5) when $\epsilon = 0.1$.

B. Singular Perturbation Method

In the 1940s, Wolfgang R. Wasow and Kurt Otto Friedrichs created the singular perturbation method [39]. The study of problems involving a parameter for which the solution varies from the limit of the general solution at a limiting value of the parameter is its main focus. The general solution of the problem converges to the limit solution in regular perturbation problems as the parameter gets closer to the limit value. A problem with a small parameter that cannot be approximated by setting the parameter value to zero is known as a singular perturbation problem. More specifically, an asymptotic expansion cannot consistently approximate the solution. Consider the following solution to the second order nonlinear differential equation as an initial illustration of the application of the singular perturbation method.

Example 6: Consider the second order nonlinear differential equation of the form

$$\epsilon y''(t)+(t+1)y'(t)+y(t)=0, \;\; y(0)=0, \; y(1)=1. \quad (16)$$

By means of regular perturbation method, we use the perturbation series

$$y(t)=y_0(t)+\epsilon y_1(t)+\dots \,. \quad (17)$$

Inserting Equation (17) into Equation (16) yields

$$\epsilon\Big(y_0''(t) + \epsilon y_1''(t)+\dots\Big)+(t+1)\Big(y_0'(t)+\epsilon y_1'(t)+\dots\Big) + \Big(y_0(t)+\epsilon y_1(t)+\dots\Big)=0,$$

with conditions,

$$y_0(0)+\epsilon y_1(0)+\dots=0$$

$$y_0(1)+\epsilon y_1(1)+\dots=1.$$

Equating the coefficients of various powers of ϵ to 0, we get

$$\mathcal{O}(1):(1+t)y_0'(t)+y_0(t)=0, \;\; y_0(0)=0, \; y_0(1)=1$$

and

$$\mathcal{O}(\epsilon)\;:y_0''(t)+(1+t)y_1'(t)+y_1(t)=0, \;\; y_1(0)=0, \; y_1(1)=0.$$

The solution of the Leading order $\mathcal{O}(1)$ is

$$y_0(t)=\frac{c}{t+1}.$$

If $y_0(0)=0$, then $y_0(t)=0$ is the solution; if $y_0(1)=1$, then $y_0(t)=\frac{2}{t+1}$ is the solution. The regular perturbation method fails at the first step because $y_0(t)$ cannot satisfy both end conditions. Therefore, we will use the singular perturbation method. Using the singular perturbation method, we set $\epsilon=0$ in Equation (16) to find the outer solution. This yields

$$(1+t)y'(t)+y(t)=0, \;\; y(1)=1. \quad (18)$$

Solving Equation (18), we get the outer solution to be $y_{outer}(t)=2/(t+1)$. To find the inner solution, the transformation $\tau=\frac{t}{\delta(\epsilon)}$ convert Equation (16), into a new form that yields

$$\epsilon\frac{1}{\delta^2(\epsilon)}Y''(\tau)+(1+\tau\delta(\epsilon))\frac{1}{\delta(\epsilon)}Y'(\tau)+Y(\tau)=0, \quad (19)$$

where $y(t)=Y(\tau)$. We are able to determine the value $\delta(\epsilon)=\epsilon$ by applying the dominate balance criteria. Taking $\epsilon=0$ in Equation (19) as a result, we obtain

$$Y''(\tau)+Y'(\tau)=0, \;\; Y(0)=0. \quad (20)$$

Solving Equation (20) to obtain inner approximation

$$Y_{inner}(\tau)=A(1-e^{-\tau}).$$

By matching the outer and inner solutions, A can be found with the following value:

$$\lim_{t\to 0}\frac{2}{1+t}=\lim_{\tau\to\infty}A(1-e^{-\tau}).$$

indicating that $A=2$, and thus the inner equation is

$$Y_{inner}(\tau)=2(1-e^{-\tau}).$$

Finally, we need to construct the approximate solution for $0\le t\le 1$ that is given by

$$y(t)=y_{outer}(t)+y_{inner}(t)-\lim_{t\to 0}y_{outer}(t).$$

That simplifies to be

$$y(t)=\frac{2}{1+t}-2e^{-t/\epsilon}. \tag{21}$$

As can be seen in Figures 5, 6 and Tables V, VI (the absolute error is small), the solution obtained by the singular perturbation method displays an excellent agreement with the one obtained by MATHEMATICA software (DSolve tool) if we choose ϵ to be small (if ϵ is smaller, the solution would be better).

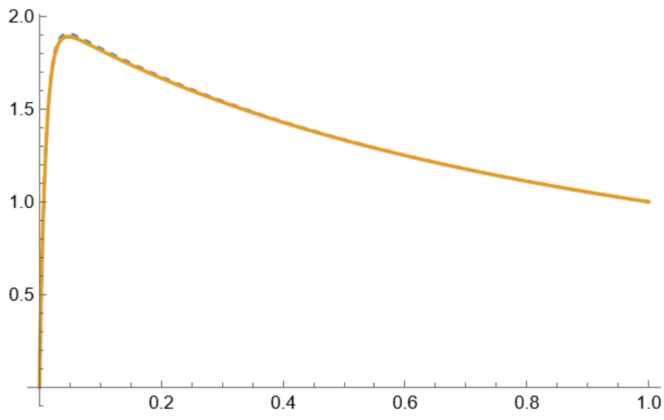

Fig. 5. The exact solution (dashed line) and the approximate solution by singular perturbation (Orange line) for example (6) when $\epsilon=0.01$

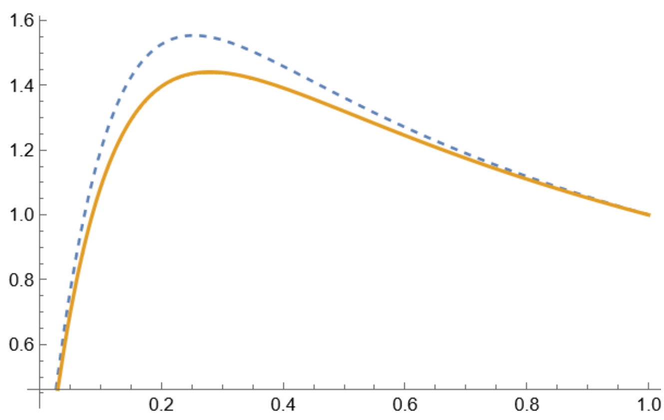

Fig. 6. The exact solution (dashed line) and the approximate solution by singular perturbation (Orange line) for example (6) when $\epsilon=0.1$

TABLE V
NUMERICAL RESULTS FOR EXAMPLE (6) WHEN $\epsilon=0.05$.

t	approximate	exact	absolute error
0	0	0	0
0.1	1.81809	1.82893	1.084E-02
0.2	1.66667	1.67427	7.600E-03
0.3	1.538467	1.54384	5.373E-03
0.4	1.42857	1.43237	3.810E-03
0.5	1.33333	1.33597	2.64E-03
0.6	1.25	1.25179	1.79E-03
0.7	1.17647	1.17762	1.15E-03
0.8	1.11111	1.11177	6.6E-04
0.9	1.05263	1.05292	2.9E-04
1	1	1	0

TABLE VI
NUMERICAL RESULTS FOR EXAMPLE (6) WHEN $\epsilon=0.01$.

t	approximate	exact	absolute error
0	0	0	0
0.1	1.54751	1.62464	7.713E-02
0.2	1.63004	1.68407	5.403E-02
0.3	1.5335	1.56629	3.279E-02
0.4	1.4279	1.44926	2.136E-02
0.5	1.33324	1.34767	1.443E-02
0.6	1.24999	1.25963	9.64E-03
0.7	1.17647	1.18261	6.14E-03
0.8	1.11111	1.11463	3.52E-03
0.9	1.05263	1.05416	1.53E-03
1	1	1	0

As an additional example of using the singular perturbation method, we provide the solution to the second order nonlinear differential equation below.

Example 7: Consider the nonlinear differential equation of second order of the form:

$$\epsilon y''(t)+2y'(t)+e^{y(t)}=0,\quad y(0)=0,\ y(1)=0. \tag{22}$$

The perturbation series is employed through the regular perturbation method

$$y(t)=y_0(t)+\epsilon y_1(t)+\dots\ . \tag{23}$$

Inserting Equation (23) into Equation (22), yields

$$\epsilon(y_0''(t)+\epsilon y_1''(t)+\dots)+2(y_0'(t)+\epsilon y_1'(t)+\dots)$$

$$+e^{y_0(t)+\epsilon y_1(t)+\cdots}=0,$$

subject to the conditions

$$y_0(0)+\epsilon y_1(0)+\dots=0$$

$$y_0(1)+\epsilon y_1(1)+\dots=0.$$

Comparing the coefficients of various power ϵ and equate them to 0, we obtain

$$\mathcal{O}(1):2y_0'(t)+e^{y_0(t)}=0,\quad y_0(0)=0,\ y_0(1)=0$$

and

$$\mathcal{O}(\epsilon)\ :y_0''(t)+2y_1'(t)+y_1(t)e^{y_0(t)}=0,\quad y_1(0)=0,\ y_1(1)=0.$$

Solving the $y_0(t)$ equation when $y_0(0)=0$, the solution is $y_0(t)\ =\ -\ln(t/2+1)$, and if we solve it with the condition $y_0(1)=0$, the solution is $y_0(t)=-\ln\left(\frac{t+1}{2}\right)$.

The regular perturbation method fails at the first step, as $y_0(t)$ is unable to satisfy both end conditions, as demonstrated above. Thus, the singular perturbation method will be employed. In order to use the singular perturbation method, we first attempt to find the outer solution. By setting $\epsilon = 0$ in Equation (22) and applying the singular perturbation method, this yields

$$2y'(t) + e^{y(t)} = 0, \quad y(1) = 0. \tag{24}$$

Solving Equation (24) yields to

$$y_{outer}(t) = -\ln(\frac{t+1}{2}).$$

To find inner solution, the transformation $\tau = \frac{t}{\delta(\epsilon)}$ convert Equation (22) to the form

$$\epsilon\frac{1}{\delta^2(\epsilon)}Y''(\tau) + \frac{2}{\delta(\epsilon)}Y'(\tau) + e^{Y(\tau)} = 0 \tag{25}$$

where $y(t) = Y(\tau)$. Using the dominate balance idea, we can find value of $\delta(\epsilon) = \epsilon$. Then set $\epsilon = 0$ in Equation (25), we obtain

$$Y''(\tau) + 2Y'(\tau) = 0, \quad Y(0) = 0. \tag{26}$$

Solving Equation (26), we get

$$Y_{inner}(\tau) = A(1 - e^{-2\tau}). \tag{27}$$

To find the value of the constant A, we do a matching between the outer and inner solutions

$$\lim_{t\to 0} -\ln(\frac{t+1}{2}) = \lim_{\tau\to\infty} A(1 - e^{-2\tau}),$$

to end up with a value $A = \ln 2$, so the inner equation is

$$Y_{inner}(\tau) = \ln 2(1 - e^{-2\tau}).$$

Finally, we need to construct the approximate solution for $0 \leq t \leq 1$, and re-write the solution in the form

$$y(t) = y_{outer}(t) + y_{inner}(t) - \lim_{t\to 0} y_{outer}(t)$$

$$y(t) = -\ln(\frac{t+1}{2}) - \ln 2\, e^{-2t/\epsilon}.$$

Because of the intricacy of the equations' form, we won't write the exact solution here; suffice it to say, it can be found using the MATHEMATICA software. The singular perturbation method's result and the one produced by the MATHEMATICA software (DSolve tool) agree on every detail. The singular perturbation is an effective method for solving such nonlinear differential equations if we choose ϵ to be small (if ϵ is smaller, the solution would be better). The absolute error is small in Figures 7, 8 and Tables VII, VII.

TABLE VII
NUMERICAL RESULTS FOR EXAMPLE (7) WHEN $\epsilon = 0.01$.

t	approximate	exact	absolute error
0	0	0	0
0.1	0.597837	0.570086	2.77506E-02
0.2	0.510826	0.493325	1.75008E-02
0.3	0.430783	0.420316	1.04669E-02
0.4	0.356675	0.350877	5.79826E-03
0.5	0.287682	0.284832	2.84985E-03
0.6	0.223144	0.222017	1.12686E-03
0.7	0.162519	0.162272	2.46724E-04
0.8	0.105361	0.105449	8.81202E-05
0.9	0.0512933	0.0514032	1.09880E-04
1	-9.592E-88	-1.161E-14	1.16100E-14

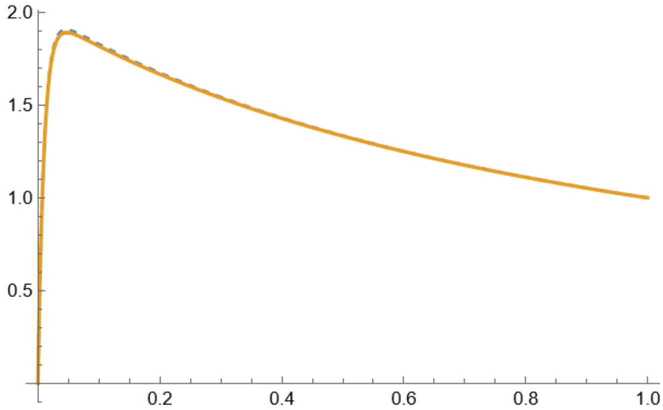

Fig. 7. The exact solution (dashed line) and the approximate solution (Orange line) for example (7) when $\epsilon = 0.01$

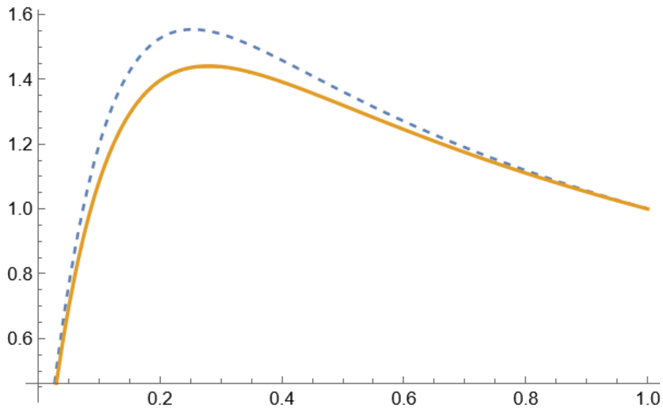

Fig. 8. The exact solution (dashed line) and the approximate solution (Orange line) for example (7) when $\epsilon = 0.1$

TABLE VIII
NUMERICAL RESULTS FOR EXAMPLE (7) WHEN $\epsilon = 0.05$.

t	approximate	exact	absolute error
0	0	0	0
0.1	0.585142	0.564685	2.04457E-02
0.2	0.510593	0.499251	1.1342E-02
0.3	0.430779	0.425446	5.3330E-03
0.4	0.356675	0.355061	1.6170E-03
0.5	0.287682	0.288148	4.6600E-04
0.6	0.223144	0.224539	1.3950E-03
0.7	0.162519	0.164071	1.5520E-03
0.8	0.105361	0.106589	1.45967E-03
0.9	0.0512933	0.0519453	6.5200E-04
1	-2.945E-18	2.0392E-15	2.0421E-15

VI. SOLUTIONS TO WEAKLY NONLINEAR DIFFERENTIAL EQUATIONS

One popular perturbation method for finding analytic solutions to weakly nonlinear oscillators is the Lindstedt-

Poincaré method. Its main idea is to express the frequency unknown in terms of powers of small parameters, where the coefficients of the unknown are known to prevent secular solutions.

A. The Lindstedt-Poincaré Method

An extension of the perturbation method, the Lindstedt-Poincaré method finds periodic solutions to differential equations by removing secular terms. The perturbation series solution will be comparable to the simple approach. We substitute another polynomial series based on the perturbation parameter for the oscillation frequency in order to remove the secular terms. Once the secular terms are removed, the coefficients of the polynomial are obtained. The differential equation needs to have the potential for periodic response in order to be solved using the Lindstedt-Poincaré method. It is a technique for figuring out the oscillating systems' steady-state solution as a result.

$$y''(t) + \omega_0 y(t) = \epsilon f(t, y(t), y'(t), \epsilon), \quad 0 < \epsilon \ll 1.$$

The idea of Lindstedt-Poincaré method is to assume a new time $\tau = \omega t = (\omega_0 + \epsilon\omega_1 + ...)t$, and $y(t) = u(\tau)$, then the new variable has the form

$$u(\tau) = u_0(\tau) + \epsilon u_1 + \epsilon^2 u_2(\tau) + ... \,.$$

Example 8: The Lindstedt-Poincaré method uses the perturbation method to approximate the periodic solution of the Duffing equation

$$y''(t) + y(t) + \epsilon y^3(t) = 0, \quad y(0) = 1, \;\; y'(0) = 0. \quad (28)$$

The approximate solution by regular perturbation method is given by

$$y(t) = \cos t + \left(\frac{-3}{8} t \sin t + \frac{1}{32}\cos 3t - \frac{1}{32}\cos t\right)\epsilon.$$

The reason the regular perturbation method fails is that, although we require a periodic approximate solution, this solution includes a secular term $t \sin t$.That is why the Lindstedt-Poincaré Method is required. The perturbation series' time scale is distorted by the Lindstedt-Poincaré Method as

$$u(\tau) = u_0(\tau) + \epsilon u_1 + \epsilon^2 u_2(\tau) + ...$$

then define $\tau = \omega t$, where $y(t) = u(\tau)$. So in the new time variable, the initial value problem becomes

$$\omega^2 u''(\tau) + u(\tau) + \epsilon u^3(\tau) = 0, \quad u(0) = 1, \; u'(0) = 0. \quad (29)$$

Derivation of the time variable τ is represented by the prime. What happens when the initial value problem is solved by replacing the perturbation series with the distorted time scale:

$$(1+\epsilon\omega_1+...)^2(u_0''(\tau)+\epsilon u_1''(\tau)+...)+(u_0(\tau)+\epsilon u_1(\tau)+ ...) + \epsilon(u_0(\tau) + \epsilon u_1(\tau) + ...)^3 = 0,$$

$$u_0(0) + \epsilon u_1(0) + ... = 1$$

$$u_0'(0) + \epsilon u_1'(0) + ... = 0.$$

When the powers of ϵ are equationed to 0, a sequence of linear second-order initial value problems of the following form results.

$$\mathcal{O}(1): \; u_0''(\tau) + u_0(\tau) = 0, \;\; u_0(0) = 1, \;\; u_0'(0) = 0$$

and

$$\mathcal{O}(\epsilon): \; u_1''(\tau) + u_1(\tau) = -2\omega_1 u_0''(\tau) - u_0^3(\tau),$$

$$u_1(0) = 0, \; u_1'(0) = 0.$$

The solution of leading order and $\mathcal{O}(\epsilon)$ and avoiding the occurrence of secular terms are given

$$u_0(\tau) = \cos\tau, \;\; u_1(\tau) = \frac{1}{32}(\cos 3\tau - \cos\tau).$$

So, we obtain a two-term approximation of the periodic solution given as

$$u(\tau) = \cos\tau + \frac{\epsilon}{32}(\cos 3\tau - \cos\tau). \quad (30)$$

For the distorted time $\tau = \omega t = t + \frac{3}{8}\epsilon t + ...$.
Finally, the approximation is valid on $(0, \infty)$ for sufficiently small ϵ and given by

$$y(t) = \cos(1+\frac{3}{8}\epsilon)t + \frac{\epsilon}{32}\Big(\cos 3(1+\frac{3}{8}\epsilon)t - \cos(1+\frac{3}{8}\epsilon)t\Big).$$

Example 9: Consider the nonlinear differential equation of second order

$$y''(t) + y(t) = \epsilon y(t) y'^2(t), \quad y(0) = 1, \quad y'(0) = 0, \quad 0 < \epsilon \ll 1.$$

The approximate solution by regular perturbation method is given by

$$y(t) = \cos t + (\frac{1}{8} t \sin t + \frac{1}{32}\cos 3t - \frac{1}{32}\cos t)\epsilon.$$

Because a periodic approximate solution is required and this solution includes a secular term $t \sin t$, the regular perturbation method fails. The Lindstedt-Poincaré Method must therefore be applied. Second-order nonlinear differential equations are subjected to the Lindstedt-Poincaré method. In a perturbation series, the Poincaré-Lindstedt Method warps the time scale.

$$u(\tau) = u_0(\tau) + \epsilon u_1 + \epsilon^2 u_2(\tau) + ... \,.$$

Then define $\tau = \omega t$, where $y(t) = u(\tau)$. So in the new time variable, the initial value problem becomes

$$\omega^2 u''(\tau) + u(\tau) - \epsilon\omega^2 u(\tau) u'^2(\tau) = 0, \quad u(0) = 1, u'(0) = 0.$$

The prime here represents differentiation with respect to the time variable τ. Substituting the perturbation series with the distorted time scale into the initial value problem yields

$$(1 + \epsilon\omega_1 + ...)^2(u_0''(\tau) + \epsilon u_1''(\tau) + ...) + (u_0(\tau) + \epsilon u_1(\tau) + ...) - \epsilon(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + ...)^2(u_0'(\tau) + \epsilon u_1'(\tau) + ...)^2(u_0(\tau) + \epsilon u_1(\tau) + ...) = 0,$$

$$u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + ... = 1$$

$$u_0'(0) + \epsilon u_1'(0) + \epsilon^2 u_2'(0) + ... = 0.$$

Equating the coefficients of powers of ϵ to zero, produces a sequence of linear second order initial value problems

$$\mathcal{O}(1):\ u_0''(\tau)+u_0(\tau)=0,\ \ u_0(0)=1,\ \ u_0'(0)=0$$

and,

$$\mathcal{O}(\epsilon):\ 2\omega_1 u_0''(\tau)+u_1''(\tau)+u_1(\tau)-u_0(\tau)u_0'^2(\tau)=0,$$

subject to the conditions $u_1(0)=0,\ u_1'(0)=0.$

The solution of the $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ second order initial value problem and avoiding the occurrence of secular terms are given

$$u_0(\tau)=\cos\tau.$$

$$u_1(\tau)=\frac{1}{32}(\cos 3\tau-\cos\tau).$$

Therefore, a two-term approximation of the periodic solution is given

$$u(\tau)=\cos\tau+\frac{\epsilon}{32}(\cos 3\tau-\cos\tau). \tag{31}$$

For the distorted time $\tau=\omega t=(1-\frac{1}{8}\epsilon+....)t$. The approximation is valid on $(0,\infty)$ for sufficiently small ϵ

$$y(t)=\cos(1-\frac{1}{8}\epsilon)t+\frac{\epsilon}{32}\Big(\cos 3(1-\frac{1}{8}\epsilon)t-\cos(1-\frac{1}{8}\epsilon)t\Big).$$

B. Singular Perturbation Theory

In mathematics, a singular perturbation problem is a small parameter problem that cannot be approximated by setting the parameter value to zero. Specifically, an asymptotic expansion cannot be used to uniformly approximate the solution. If a single asymptotic expansion can be used to approximate the solution of a perturbed problem over its whole domain (both in space and time) it is said to have a regular perturbation. Typically, a regularly perturbed problem can be found with an acceptable approximation by simply replacing all instances of the small parameter ϵ in the problem statement with zero. When ϵ decreases, the approximation that arises from taking only the first term of the expansion approaches the true solution gradually. Singular perturbation theory is a topic that researchers, physicists, and mathematicians are actively exploring. There are a lot of ways to deal with problems in this area. The most basic techniques for solving spatial problems are the matched asymptotic expansions method and the WKB approximation; for solving time problems, the Poincaré-Lindstedt method, the method of multiple scales, and periodic averaging are employed. Numerical techniques are also widely used to solve singular perturbation problems. We may apply Lindstedt-Poincaré method to initial value problem of the general form

$$F(t,y(t),y'(t),y''(t),\epsilon)=0,\ \ y(0)=a,\ y'(0)=b.$$

Also apply singular perturbation methods to boundary value problem of the form

$$F(t,y(t),y'(t),y''(t),\epsilon)=0,\ y(A)=a,\ y(B)=b.$$

Example 10: Consider the second order nonlinear differential equation of the form

$$\epsilon y''(t)+(t+1)y'(t)=1,\ \ y(0)=0,\ y(1)=1+\ln 2. \tag{32}$$

By means regular perturbation method, we use the a perturbation series

$$y(t)=y_0(t)+\epsilon y_1(t)+...\ . \tag{33}$$

Substitution of Equation (33) into Equation (32) to obtain

$$\epsilon(y_0''(t)+\epsilon y_1''(t)+...)+(t+1)(y_0'(t)+\epsilon y_1'(t)+...)=1,$$
$$y_0(0)+\epsilon y_1(0)+...=0$$
$$y_0(1)+\epsilon y_1(1)+...=1+\ln 2.$$

Equating the coefficients of various powers of ϵ to 0, then we obtain

$$\mathcal{O}(1):\ (t+1)y_0'(t)=1,\ \ y_0(0)=0,\ \ y_0(1)=1+\ln 2.$$

So, solving the above differential equation being simple separable differential equation yields to the solution:

$$y_0(t)=\ln(t+1)+C.$$

There is only one constant and two conditions in the solution above. If we apply the condition $y_0(0)=0$, then $y_0(t)=\ln(t+1)$. On the other hand, the solution reduces to $y_0(1)=\ln(1+1)+1$, which is confusing if we use the other condition $y_0(1)=1+\ln 2$. Regular perturbation is therefore undesirable because $y_0(t)$ is unable to satisfy both end conditions. As long as we avoid $t=0$, the exact solution (numerically) matches to $y_0(t)=\ln(1+t)+C$ with initial condition $y_0(1)=1+\ln 2$.

To find the outer solution, we take $\epsilon=0$ in Equation (32) to get

$$(1+t)y'(t)=1,\ y(1)=1+\ln 2.$$

The outer solution is

$$y_{outer}(t)=\ln(1+t)+1.$$

To find inner approximate solution, re-scale the t-variable by $\xi=\frac{t}{\delta(\epsilon)}$, so that

$$y(t)=Y(\xi). \tag{34}$$

Substitute Equation (34) in Equation (32), we obtain the new ordinary differential equation of the form

$$\frac{\epsilon}{\delta^2(\epsilon)}Y''(\xi)+\frac{1}{\delta(\epsilon)}Y'(\xi)+\xi Y'(\xi)=1.$$

With the help of dominant balancing criteria, we can find value of $\delta(\epsilon)=\epsilon$, so

$$Y''(\xi)+Y'(\xi)+\epsilon\xi Y'(\xi)=\epsilon,\quad Y(0)=0. \tag{35}$$

Then set $\epsilon=0$ in Equation (35), we get

$$Y''(\xi)+Y'(\xi)=0, Y(0)=0.$$

The solution of the above ODE is

$$y_{inner}(t)=C_1(1-e^{\frac{-t}{\epsilon}}). \tag{36}$$

Apply the matching process to find value of C_1, by matching outer and inner solutions as

$$\lim_{t\to 0}(\ln(t+1)+1)=\lim_{\xi\to\infty}C_1(1-e^{-\xi}).$$

Then the value of $C_1 = 1$, so the inner equation is

$$Y_{inner}(\tau) = (1 - e^{-\tau}).$$

We write down the approximate solution over $0 \leq t \leq 1$

$$y(t) = y_{outer}(t) + y_{inner}(t) - \lim_{t \to 0} y_{outer}(t),$$

then the approximate solution

$$y(t) = 1 + \ln(t+1) - e^{-t/\epsilon}.$$

VII. Conclusions

Here, we investigated the perturbation method, which is used to find approximate or more accurate solutions to linear and nonlinear differential equations when exact solutions are unavailable. We investigated some problems and solved them using three perturbation methods, yielding results that were both convenient and satisfactory. Various perturbation methods were considered, including regular, singular, and Lindstedt-Poincaré. Most perturbation methods assume the existence of a small parameter in the equation.

Solving differential equations using the regular perturbation method, a perturbation series was used to provide a general approximate solution. These may or may not converge, but they provide a useful approximation to the original problem in a truncated form with only two or three terms. However, the regular perturbation method cannot be used in all cases, so there is another method for solving nonlinear differential equations.

The Lindstedt-Poincaré method was applied to find an approximate periodic solution to a nonlinear ordinary differential equation in the presence of an initial value condition. This was achieved by converting the original equation into a different equation and then applying valid perturbation series for periodic solutions to second-order nonlinear differential equations. The singular perturbation method is used to find an approximate solution to a nonlinear ordinary differential equation with a boundary value condition. The outer and inner solutions are found using the perturbation series, and the approximate solution is obtained by combining them. For every method, the approximate solutions converge to the exact solution very quickly improved approximation solutions. We contrasted exact solutions obtained with MATHEMATICS software with approximate solutions obtained through perturbation methods. The numerical results demonstrate how straightforward and understandable the perturbation method used in these problems is, and how close the approximate solution is to the exact solution, suggesting that this method yields a reliable approximation of the exact solution. We intend to solve systems of nonlinear differential equations using perturbation methods in the future and compare the outcomes with other existence approaches.

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