On Distance Antimagic Labeling of Some Product Graphs

Anjali Yadav and Minirani S.

***Abstract*—In graph theory, graph labeling is an essential area of study because labeled graphs offer useful mathematical models for coding theory, cryptography, astronomy, radar, database administration and communication networks. Consider a bijection for a graph G of order n, $f\colon V(G) \rightarrow \{1, 2, \ldots, n\}$. The weight of a vertex z of G, expressed as $w(z)$, is defined as the sum of labels assigned to all vertices adjacent to vertex z in G. If the weights are distinct for every unique pair of vertices y, z in $V(G)$, then the labeling f is referred to as distance antimagic. A distance antimagic graph is any graph G that accepts such a labeling. Distance antimagic labeling on various basic graph products are discussed in this paper. We explore results on (a, d)-distance antimagic labeling for the lexicographic product $G \circ H$ and distance antimagic labeling for the cartesian product $G \Box H$, tensor product $G \times H$ and strong product $G \boxtimes H$ in this work, where the graphs G and H are cycle related graphs, paths or complete graphs. Also, computer-aided algorithms are designed to verify that vertex weights are distinct.**

***Index Terms*—Distance magic labeling, distance antimagic labeling, (a, d)-distance antimagic labeling, cartesian product, lexicographic product, tensor product, strong product.**

I. Introduction

GRAPH labeling is the process of mapping a graph's edge or vertex set, under specific criteria, into a set of positive integers. The significance of this subject of study lies in its many applications in coding theory, cryptography, urban planning, networking, telecommunication and crystallography.

Potential applications of graph labeling include solving issues with Mobile Adhoc Networks. A graph model can be used to investigate problems with connectivity, scalability, routing, network modeling, and simulation. Graphs can be represented as matrices and problems can be analysed using algorithms. It is possible to represent node density, mobility, link building, and routing using ideas associated with random graphs. Congestion in Mobile Adhoc Networks can be analysed using a variety of techniques and graph theory principles can be used to model these networks. Another application that has been addressed with the concept of 2-odd labeling in graphs is how to effectively and efficiently design the restricted frequency spectrum of the global mobile communication system with an increasing number of subscribers [1]. Graph labeling also finds application in a transportation network stand where adjacent stations of the same degree are needed to maintain roughly equal capacity and the diversity of connecting highways between stations reaches an extreme value can be represented by the Adjacent Vertex Reducible Total Labeling model [2]. In particular, antimagic labeling in lexicographic order has been used in cryptographic applications that involves encryption and decryption and a plain-text message that has been encrypted produces cipher-text in [3].

This work takes into consideration any finite undirected connected graphs without loops or numerous edges. For a graph $G = (V, E)$, we assume that the order $|V|$ and the size $|E|$ are represented by n and m respectively. Chartrand and Lesniak [4] provide terminologies and notations related to graph theory.

Consider a bijection $f\colon V \rightarrow \{1, 2, ..., n\}$. A graph G of order n is said to have a distance magic labeling f if there is a constant $k^{'}$ such that for every $u \in V$, we have $\sum_{v \in N(u)} f(v) = k^{'}$ where $N(u) = \{v \in V : v \text{ is adjacent to } u\}$ is the open neighbourhood of u. The weight of u is the sum $\sum_{v \in N(u)} f(v)$ and it is represented as$w(u)$.

Another variation of distance magic labeling is distance antimagic labeling, proposed by Kamatchi and Arumugam [5], in which all vertex weights are distinct integers. Also, distance antimagic labeling in which the set of all vertex weights form an arithmetic progression with initial term a and common difference d called (a, d)-distance antimagic labeling is another type arising from distance magic labeling [6]. We refer to Gallian [7] and Arumugam et al. [8] for more insights into recent survey and open problems on labeling.

In [5] Kamatchi and Arumugam posed the problem and asked if join of graphs $G + K_1$, $G + K_2$ and the cartesian product $G \Box K_2$ distance antimagic if G is distance antimagic? By introducing the idea of arbitrary distance antimagic labeling, Handa et al. [9] demonstrated the existence of distance antimagic labeling of the join of two graphs. Further in [10] Cutinho et al. proved that $G = K_n \Box K_n$ is distance antimagic if and only if $n \neq 2$. They also proved that $G = K_3 \Box C_n$ is distance antimagic for odd integer n. Further they posed the problem of distance antimagic labeling of $G = K_3 \Box C_n$ if n is even? Further, Simanjuntak and Tritama [11] proved several results on distance antimagic labeling of graph products.

In this paper, we first discuss distance antimagic labeling of $G \Box K_3$ and then compute distance antimagic labeling of $K_3 \Box C_{2n}^{+}$and$K_3 \Box W_n$. We also discuss the (a, d)-distance antimagic labeling for lexicographic product $K_n \circ K_n \, and \, K_n \circ C_3$ and examine distance antimagic labeling of lexicographic product of star $K_{1,n}$ and bistar $B_{n,n}$ with K_2. Further, in the last section it is proved that tensor product $K_n \times C_3$ and strong product $P_n \boxtimes K_2$ are distance antimagic. It is also shown that vertex weights are distinct using computer-aided algorithms in Python.

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II. Preliminary Definitions

Definition 1 ([6]). *Consider a bijection* $f\colon V \to \{1,2,...,n\}$ *and let the vertex weight for any* $z \in V$ *be* $w(z) = \sum_{y\in N(z)} f(y)$ *where open neighbourhood of* z *is denoted by* $N(z) = \{y \in V : y \text{ is adjacent to } z\}$. *If the set of all vertex weights forms an arithmetic progression with difference* d *and first term* a *and the set of all vertex weights generated is* $\{a, a+d, a+2d, \ldots, a+(n-1)d\}$, *then graph* G *is considered to be (a, d)-distance antimagic.*

Definition 2 ([5]). *The bijection* $f\colon V \to \{1,2,\ldots,n\}$ *for a graph* G *is referred to as distance antimagic labeling if for each pair of different vertices* $y, z \in V(G)$, *we have* $w(y) \neq w(z)$ *where* $w(a) = \sum_{b\in N(a)} f(b)$ *is the weight of vertex* a *and* $N(a)$ *is open neighbourhood of* $a \in V$. G *is referred to as a distance antimagic graph if such a labeling is present.*

Definition 3. *Consider the cycle* $C_{2n} = (u_1, u_2, \ldots, u_{2n}, u_1)$. *M-augmentation of* C_{2n} *represented by* C_{2n}^{+} *is the graph that is created by appending a perfect matching made up of the edges* u_1u_{n+1} *and* u_iu_{2n+2-j} *where* $2 \le j \le n$.

Definition 4. *Consider the cycle* $C_n = (u_1, u_2, \ldots, u_n, u_1)$ *of order* n. *The graph that results by adding vertex* u_{n+1} *to* C_n *and connecting it to all vertices of* C_n *is called a wheel. It is denoted by* W_{n+1}.

Definition 5. *A star graph is the complete bipartite graph* $K_{1,n}$ *of order* $(n+2)$ *containing a central vertex* u *and* $(n+1)$ *adjacent vertices.*

Definition 6. *A bistar graph* $B_{n,n}$ *is the graph of order* $(2n+2)$ *in which the central vertex in two copies of* $K_{1,(n-1)}$ *are joined together.*

Definition 7. *Consider two graphs* G *and* G_0. *The graph represented by* $G\square G_0$ *is the cartesian product of* G *and* G_0. *It has vertex set* $V(G\square G_0) = \{(y,z) : y \in V(G) \text{ and } z \in V(G_0)\}$ *and edge set* $E(G\square G_0) = \{(y,z)(y^{'},z^{'}) : y = y^{'} \text{ and } zz^{'} \in E(G_0) \text{ or } z = z^{'} \text{ and } yy^{'} \in E(G)\}$.

Definition 8. *Consider two graphs* G *and* G_0. *The graph that has vertex set* $V(G \circ G_0) = \{(y,z) : y \in V(G) \text{ and } z \in V(G_0)\}$ *and edge set* $E(G \circ G_0) = \{(y,z)(y^{'},z^{'}) : yy^{'} \in E(G) \text{ or } y = y^{'} \text{ and } zz^{'} \in E(G_0)\}$ *is known as the lexicographic product of* G *and* G_0 *and is represented as* $G \circ G_0$.

Definition 9. *Consider two graphs* G *and* G_0. *The graph that has vertex set* $V(G\times G_0) = \{(y,z) : y \in V(G) \text{ and } z \in V(G_0)\}$ *and edge set* $E(G \times G_0) = \{(y,z)(y^{'},z^{'}) : yy^{'} \in E(G) \text{ and } zz^{'} \in E(G_0)\}$ *is known as the tensor product of* G *and* G_0 *and is represented as* $G \times G_0$.

Definition 10. *Consider two graphs* G *and* G_0. *The graph that has vertex set* $V(G\boxtimes G_0) = \{(y,z) : y \in V(G) \text{ and } z \in V(G_0)\}$ *and edge set* $E(G \boxtimes G_0) = \{(y,z)(y^{'},z^{'}) : yy^{'} \in E(G) \text{ and } zz^{'} \in E(G_0) \text{ or } y = y^{'} \text{ and } zz^{'} \in E(G_0) \text{ or } z = z^{'} \text{ and } yy^{'} \in E(G)\}$ *is known as the strong product of* G *and* G_0 *and is represented as* $G \boxtimes G_0$.

Definition 11. *Consider a graph* G *with* n *vertices.* G *is regarded as a monotonically decreasing graph if there is a bijection* $h\colon V \to \{1,2,...,n\}$ *such that* $w(y) \ge w(z)$ *for all* $h(y) < h(z)$.

III. Distance antimagic labeling of Cartesian Product of graphs

This section discusses distance antimagic labeling of cartesian product $G\square K_3$ and later distance antimagic labeling of K_3 with cycle related graphs are determined.

Theorem 1. *Assume that graph* G *has* n *vertices and is* r*-regular.. Then* $G\square K_3$ *is distance antimagic if* G *is monotonic decreasing and* $r > \sqrt{2n-1}$, $r > \frac{1+\sqrt{1+8n}}{2}$ *or* $r > n + \sqrt{n^2-1}$.

Proof: Consider a bijection $h\colon V(G) \to \{1,2,...,n\}$ where $V(G)$ is the vertex set of G and $h(x_i) = i$ for $1 \le i \le n$. As G is monotonically decreasing, we have $\tau(x_1) \ge \tau(x_2) \ge \cdots \ge \tau(x_n)$ where $\tau(x_i)$ is the weight of vertex x_i in G. Let $V(G\square K_3) = \{x_i\colon 1 \le i \le n\} \cup \{y_i\colon 1 \le i \le n\} \cup \{z_i\colon 1 \le i \le n\}$ and $E(G\square K_3) = E(G) \cup \{(x_i,y_i)\colon 1 \le i \le n\} \cup \{(y_i,z_i)\colon 1 \le i \le n\} \cup \{(x_i,z_i)\colon 1 \le i \le n\} \cup \{(y_i,y_j)\colon 1 \le i < j \le n$ where $(x_i,x_j) \in E(G)\} \cup \{(z_i,z_j)\colon 1 \le i < j \le n$ where $(x_i,x_j) \in E(G)\}$. Also, assume that $w(x_i)$ is the weight of vertices in $V(G\square K_3)$.
Define a bijection $g\colon V(G\square K_3) \to \{1,2,\ldots,3n\}$ such that

$$g(x_i) = h(x_i) = i$$
$$g(y_i) = 2n + 1 - i$$
$$g(z_i) = 2n + i$$

Now, $w(x_i) = \tau(x_i) + g(y_i) + g(z_i) = \tau(x_i) + 4n + 1$
Also,

$$w(y_i) = \sum_{(y_i,x)\in G\square K_3} g(x) = \sum_{(x_i,x_j)\in V(G)} g(y_j) + g(x_i) + g(z_i)$$
$$= 2n + 2i + r(2n+1) - \tau(x_i)$$

$$w(z_i) = \sum_{(z_i,x)\in G\square K_3} g(x) = \sum_{(x_i,x_j)\in V(G)} g(z_j) + g(x_i) + g(y_i)$$
$$= 2n(1+r) + 1 + \tau(x_i)$$

As $\tau(x_i) \ge \tau(x_{i+1}) \implies 4n + 1 + \tau(x_i) \ge 4n + 1 + \tau(x_{i+1}) \implies w(x_i) \ge w(x_{i+1})$. Similarly, $\tau(x_i) \ge \tau(x_{i+1}) \implies 2n+2i+r(2n+1)-\tau(x_i) < 2n+2i+r(2n+1) - \tau(x_{i+1}) \implies w(y_i) < w(y_{i+1})$ for $1 \le i \le (n-1)$.
Also, $\tau(x_i) \ge \tau(x_{i+1}) \implies 2n(1+r) + 1 + \tau(x_i) \ge 2n(1+r) + 1 + \tau(x_{i+1}) \implies w(z_i) \ge w(z_{i+1})$ for $1 \le i \le (n-1)$.
To complete the proof, we have to show that for all $i \neq j$

$$w(x_i) \neq w(y_j)$$
$$w(x_i) \neq w(z_j)$$
$$w(y_i) \neq w(z_j)$$

We first prove $w(x_i) \neq w(y_j)$ for $i \neq j$. Let us assume on the contrary $w(x_i) = w(y_j)$ for some $i \neq j$.
$\implies \tau(x_i) + 4n + 1 = 2n + 2i + r(2n+1) - \tau(x_j)$
$\implies \tau(x_i) + \tau(x_j) = r(2n+1) - 2n + 2i - 1$
Since, $\tau(x_i) + \tau(x_j) \le 2nr - r^2 + r$ so $r(2n+1) - 2n + 2i - 1 \le 2nr - r^2 + r \implies r^2 - 2n + 2i - 1 \le 0 \implies r^2 - 2n + 1 \le 0$ as $1 \le i$.
$\implies r \le \sqrt{2n-1}$ which is a contradiction.

Further to prove $w(x_i) \neq w(z_j)$ for $i \neq j$. Let us assume on the contrary $w(x_i) = w(z_j)$ for some $i \neq j$.
$\implies \tau(x_i) + 4n + 1 = 2n + 1 + 2nr + \tau(x_j) \implies \tau(x_i) - \tau(x_j) = 2nr - 2n$
As $\tau(x_i) - \tau(x_j) \leq \tau(x_i) + \tau(x_j) \leq 2nr - r^2 + r \implies 2nr - 2n \leq 2nr - r^2 + r$
$\implies r^2 - r + 2n \leq 0 \implies r \leq \frac{1+\sqrt{1+8n}}{2}$ which is a contradiction.
Lastly, to prove $w(y_i) \neq w(z_j)$ for $i \neq j$. Let us assume on the contrary $w(y_i) = w(z_j)$ for some $i \neq j$.
$\implies 2n + 2i + 2nr + r - \tau(x_i) = 2n + 2nr + 1 + \tau(x_j)$
$\implies \tau(x_i) + \tau(x_j) = 2i + r - 1$
Since, $\tau(x_i)+\tau(x_j) \leq 2nr-r^2+r$ so $2i+r-1 \leq 2nr-r^2+r$ $\implies r^2 - 2nr + (2i-1) \leq 0 \implies r^2 - 2nr + 1 \leq 0 \implies r \leq n + \sqrt{n^2-1}$ which is a contradiction.
As vertex weights of $G \square K_3$ are distinct, so $G \square K_3$ is distance antimagic.

■

We now provide distance antimagic labeling of cartesian product of graphs where G is not regular or monotonic decreasing.

Theorem 2. *The graph $G = K_3 \square C_{2n}^+$ is distance antimagic for all n except for $n \equiv 0(mod 3)$.*

Proof: Let $G = K_3 \square C_{2n}^+$ and $n \not\equiv 0(mod 3)$, $n > 1$.Let $V(C_{2n}^+) = \{u_1, u_2, \ldots, u_{2n}\}$ and $V(K_3) = \{v_1, v_2, v_3\}$. We denote the vertex (v_i, u_j) in $K_3 \square C_{2n}^+$ by x_{ij} .
Define $g\colon V(G) \to \{1, 2, ..., 6n\}$ as $g(x_{ij}) = 3(j-1) + i$, $1 \leq i \leq 3$ and $1 \leq j \leq 2n$. Clearly, g is a bijection. The vertex weights of graph G are given as

$$w(x_{ij}) = \begin{cases} 9n+8 & :i=1, j=1 \\ 6n-1+9j & :i=1, 2 \leq j \leq n, n+2 \leq j \leq 2n-1 \\ 12n+8 & :i=1, j=n+1 \\ 18n-1 & :i=1, j=2n \\ w(x_{1j})+2 & :i=2, 1 \leq j \leq 2n \\ w(x_{1j})+4 & :i=3, 1 \leq j \leq 2n \end{cases} \tag{1}$$

Let x_{ik} and x_{rs} be two distinct vertices in G and suppose $w(x_{ik}) = w(x_{rs})$ and the following cases be considered
Case 1. $i = r$ and $k \neq s$.If $i = r = 1$, then $w(x_{1k}) = w(x_{1s})$ implies $n \equiv 0(mod 3)$ or $n = 0, 1$ which is a contradiction. If $i = r = 2, 3$ then again we get a contradiction using the case $i = 1$.
Case 2. $i \neq r$ and $k = s$. If $i = 1$ and $r = 2$, then $w(x_{1k}) = w(x_{2k})$ implies $w(x_{1k}) = w(x_{1k}) + 2$ using (1) which is a contradiction. Similar contradictions can be seen if $i = 1, r = 3$ and $i = 2, r = 3$.
Case 3. $i \neq r$ and $k \neq s$. If $k = 1$, $2 \leq s \leq 2n$ and $i \neq r$, then $w(x_{i1}) = w(x_{rs})$ implies the following using (1)
Subcase (i): $w(x_{11}) = w(x_{1s}) + 2$ if $i = 1$ and $r = 2$
Subcase (ii): $w(x_{11}) = w(x_{1s}) + 4$ if $i = 1$ and $r = 3$
Subcase (iii): $w(x_{11}) + 2 = w(x_{1s}) + 4$ if $i = 2$ and $r = 3$
In all subcases we get that n is not an integer for any $2 \leq s \leq 2n$. So, there is a contradiction in all subcases.
Similarly if $2 \leq k \leq n$ or $n+2 \leq k \leq 2n-1$, $s \neq k$ and $i \neq r$, then $w(x_{ik}) = w(x_{rs})$ again implies n is not an integer which is a contradiction using (1).
Also if $k = n+1$ or $k = 2n$, $s \neq k$ and $i \neq r$, then using the same technique as above for subcases we get a contradiction using (1).

Hence $w(x_{ik}) \neq w(x_{rs})$ and so G is distance antimagic when $n \not\equiv 0(mod 3)$.
The algorithm to check that vertex weights of graph G are distinct is given below and is verified using Python programming : ■

Algorithm 1 An algorithm to check vertex weights are distinct.

Require: $n > 1$
Array=np.zeros$(3, 2 \times n)$
Total no of elements=$3 \times 2 \times n$
if $(n \,\%\, 3) \neq 0$ **then**:
for a in range (3) ▷ *loop to iterate over rows of matrix*
for b in range $(2 \times n)$ ▷ *loop to iterate over columns of matrix*
if $b = 0$ **then**:
Array$[a, b] = 9 \times n + 8 + 2 \times a$
else if $((b \geq 1$ and $b \leq (n-1)))$ or $(b \geq (n-1)$ and $b \leq 2 \times n - 2))$ **then**:
Array$[a, b] = 6 \times (n-1) + 9 \times (b+1) + 2 \times a$
else if $b = n$ **then**:
Array$[a, b] = 12 \times n + 8 + 2 \times a$
else if $b = 2 \times n - 1$ **then**:
Array$[a, b] = 18 \times n - 1 + 2 \times a$
end for ▷ *end for inner loop*
end for ▷ *end for outer loop*

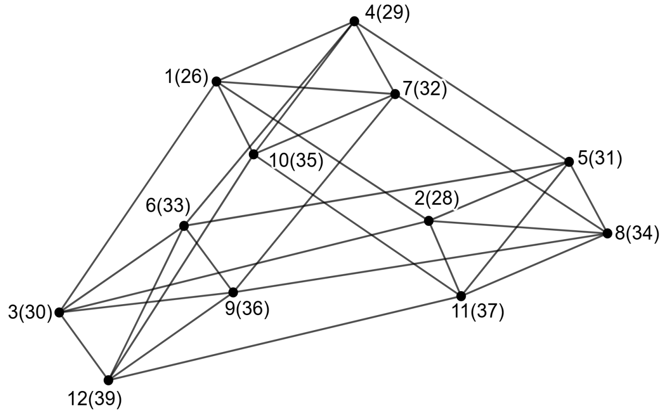

Fig. 1: Distance antimagic labeling of $G = K_3 \square C_4^+$. The vertex weights are given in brackets and labels in usual font.

Theorem 3. *The graph $G = K_3 \square W_n$ is distance antimagic for all n except $n \equiv 0(mod 4)$.*

Proof: Let $G = K_3 \square W_n$ and $n \not\equiv 0(mod 4), n > 2$. Let $V(W_n) = \{u_1, u_2, \ldots, u_n, u_{n+1}\}$ and $V(K_3) = \{v_1, v_2, v_3\}$. We denote the vertex (v_i, u_j) in $K_3 \square W_n$ by x_{ij} .
Define $g\colon V(G) \to \{1, 2, ..., 3n+3\}$ as $g(x_{ij}) = 3(j-1)+i$, $1 \leq i \leq 3$ and $1 \leq j \leq n+1$. It is evident that g is a bijection

and vertex weights are computed as

$$w(x_{ij})=\begin{cases}6n+8 & :i=1,j=1\\ 3n-4+12j & :i=1,2\leq j\leq n-1\\ 12n-4 & :i=1,j=n\\ w(x_{1j})+2 & :i=2,1\leq j\leq n\\ w(x_{1j})+4 & :i=3,1\leq j\leq n\\ w(x_{1n+1})+(n-1)(i-1) & :2\leq i\leq 3,\ j=n+1\end{cases} \tag{2}$$

$$w(x_{ij})=(n+2)(2n+2-k) \quad : i=1,\ j=n+1,\ n=2k+1$$

$$w(x_{ij})=2(k+1)(3n+5) \quad : i=1,\ j=n+1,\ n=4k+2 \tag{3}$$

Let x_{im} and x_{rs} be two distinct vertices in G and suppose $w(x_{im})=w(x_{rs})$ and the following cases be considered.

Case 1. $i=r$ and $m\neq s$. If $i=r=1$ and $m,s\neq(n+1)$, then $w(x_{1m})=w(x_{1s})$ implies $n\equiv 0(mod4)$ or $n=2$ which is a contradiction. In case $m,s=(n+1)$, then $w(x_{1m})=w(x_{1s})$ implies k is not an integer which is a contradiction. If $i=r=2,3$ then again we get a contradiction using the case $i=r=1$.

Case 2. $i\neq r$ and $m=s$. If $i=1$ and $r=2$ and $m,s\neq(n+1)$ then $w(x_{1m})=w(x_{2m})$ implies $w(x_{1m})=w(x_{1m})+2$ using (2) which is a contradiction. If $m,s=(n+1)$ then $w(x_{1m})=w(x_{2m})$ implies $w(x_{1n+1})=w(x_{1n+1})+(n-1)(i-1)$ using (2) implying $n=1$ which is a contradiction. Similar contradictions can be seen if $i=1, r=3$ and $i=2, r=3$.

Case 3. $i\neq r$ and $m\neq s$. If $m=1$ and $2\leq s\leq n$, then $w(x_{i1})=w(x_{rs})$ implies the following using (2)

Subcase (i): $w(x_{11})=w(x_{1s})+2$ if $i=1$ and $r=2$

Subcase (ii): $w(x_{11})=w(x_{1s})+4$ if $i=1$ and $r=3$

Subcase (iii): $w(x_{11})+2=w(x_{1s})+4$ if $i=2$ and $r=3$

In all subcases we get that n is not an integer for any $2\leq s\leq n$. So, there is a contradiction in all subcases.

If $m=1$ and $s=n+1$, then $w(x_{i1})=w(x_{rs})$ implies the following using (2)

Subcase (i): $w(x_{11})=w(x_{2(n+1)})=w(x_{1(n+1)})+(n-1)$ if $i=1$ and $r=2$.

Subcase (ii): $w(x_{11})=w(x_{3(n+1)})=w(x_{1(n+1)})+2(n-1)$ if $i=1$ and $r=3$.

Subcase (iii): $w(x_{21})=w(x_{3(n+1)}) \implies w(x_{11})+2=w(x_{1(n+1)})+2(n-1)$ if $i=2$ and $r=3$.

In all subcases we get that n is not an integer which is a contradiction.

Similarly, if $2\leq m\leq(n-1)$, $s\neq m$ and $i\neq r$, then $w(x_{im})=w(x_{rs})$ implies n is not an integer which is a contradiction.

Also if $m=n$ or $m=(n+1)$, $s\neq m$ and $i\neq r$, then using the same technique as above for subcases we get that n is not an integer which is a contradiction.

Hence $w(x_{im})\neq w(x_{rs})$ and so G is distance antimagic when $n\not\equiv 0(mod4)$.

■

The algorithm to check that vertex weights of graph G are distinct is given below and is verified using Python programming:

Algorithm 2 An algorithm to check vertex weights are distinct.

Require: $n>2$
  if $(n\%2)\neq 0$ **then**:
    $k=n\div 2$
  else if $(n\,\%\,4)\neq 0$ **then**:
    $k=n\div 4$
    Array=np.zeros$(3, 2\times(n+1)$
    Total no of elements=$3\times(n+1)$
    if $(n\,\%\,4)\neq 0$ **then**:
      for a in range (3) ▷ *loop to iterate over rows of matrix*
      for b in range $(n+1)$ ▷ *loop to iterate over columns of matrix*
      if $(a=1$ or $a=2)$ and $(b=n)$ **then**:
        Array$[a,b]$=Array$[0][a]+(n-1)\times a$
      else if $b=0$ **then**:
        Array$[a,b]=6\times n+8+2\times a$
      else if $b\geq 1$ and $b\leq(n-2)$ **then**:
        Array$[a,b]=3\times n-4+12\times(b+1)+2\times i$
      else if $b=n-1$ **then**:
        Array$[a,b]=12\times n-4+2\times a$
      else if $b=n$ and $n\,\%\,2)\neq 0$ **then**:
        Array$[a,b]=(n+2)\times(2\times n+2-k)+2\times a$
      else if $b=n$ and $n\,\%\,4)\neq 0$ **then**:
        Array$[a,b]=(3\times n+5)\times(2\times k+2)+2\times a$
        end for ▷ end for inner loop
        end for ▷ end for outer loop

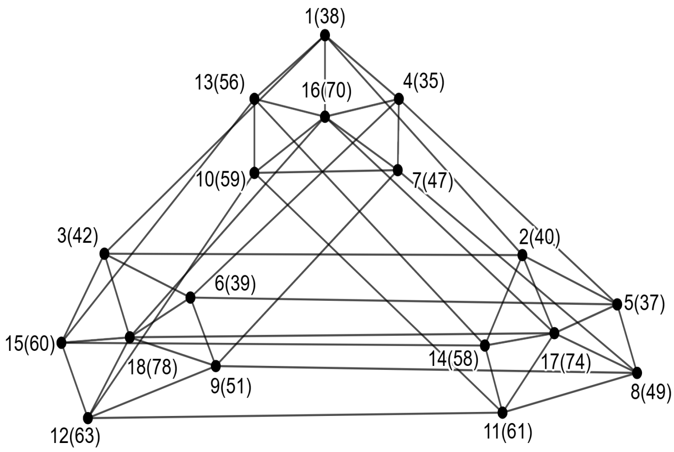

Fig. 2: Distance antimagic labeling of $G=K_3\square W_5$. The vertex weights are given in brackets and labels in usual font.

IV. Distance antimagic labeling of Tensor and Strong Product of graphs

The distance antimagicness of graphs generated by basic graph products: the tensor product and the strong product, is examined in this section.

Theorem 1. *The tensor product $G=K_n\times C_2$ is distance antimagic.*

The tensor product $G=K_n\times C_2$ is an n-crown graph is distance antimagic as shown in [12].

Theorem 2. *The tensor product $G=K_n\times C_3$ is distance antimagic.*

Proof: Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(C_3) = \{v_1, v_2, v_3\}$. We denote the vertex (u_i, v_j) in $G = K_n \times C_3$ by x_{ij} .
Define $g\colon V(G) \to \{1, 2, ..., 3n\}$ as

$$g(x_{ij}) = \begin{cases} i & \text{if } j = 1 \\ n + i & \text{if } j = 2 \\ 2n + i & \text{if } j = 3 \end{cases}$$

where $1 \leq i \leq n$. It is evident that g is a bijection and vertex weights are computed as

$$w(x_{ij}) = \begin{cases} 4n^2 - 2n - 2i & \text{if } j = 1 \\ 3n^2 - n - 2i & \text{if } j = 2 \\ 2n^2 - 2i & \text{if } j = 3 \end{cases}$$

As the vertex weights are monotonically decreasing, so they are all distinct. Therefore, G is distance antimagic graph. ■

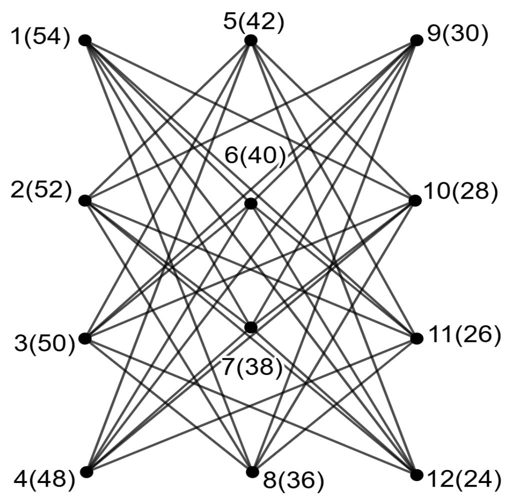

Fig. 3: Distance antimagic labeling of $G = K_4 \times C_3$. The vertex weights are given in brackets and labels in usual font.

Theorem 3. *The strong product $G = P_n \boxtimes K_2$ is distance antimagic.*

Proof: Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_2) = \{v_1, v_2\}$. We denote the vertex (u_i, v_j) in $G = P_n \boxtimes K_2$ by x_{ij} .
Define $g\colon V(G) \to \{1, 2, ..., 2n\}$ as

$$g(x_{i1}) = \begin{cases} i & \text{if } i \text{ is odd} \\ 2n + 1 - i & \text{if } i \text{ is even} \end{cases}$$

$$g(x_{i2}) = \begin{cases} 2n + 1 - i & \text{if } i \text{ is odd} \\ i & \text{if } i \text{ is even} \end{cases}$$

where $1 \leq i \leq n$. It is evident that g is a bijection and vertex weights are computed as

$$w(x_{i1}) = \begin{cases} 4n + 1 & i = 1 \\ 6n + 3 - i & i \text{ is odd, } 3 \leq i \leq n - 1 \\ 4n + 2 + i & i \text{ is even, } 2 \leq i \leq n - 1 \\ 3n + 1 & i = n, \ n \text{ is even} \\ 3n + 2 & i = n, \ n \text{ is odd} \end{cases}$$

$$w(x_{i2}) = \begin{cases} 2n + 2 & i = 1 \\ 4n + 2 + i & i \text{ is odd, } 3 \leq i \leq n - 1 \\ 6n + 3 - i & i \text{ is even, } 2 \leq i \leq n - 1 \\ 3n + 2 & i = n, \ n \text{ is even} \\ 3n + 1 & i = n, \ n \text{ is odd} \end{cases}$$

As the vertex weights are are all distinct, so G is distance antimagic. ■

The algorithm to prove vertex weights are distinct is given below and is verified using Python.

Algorithm 3 An algorithm to check vertex weights are distinct.

Input: n
Output: Weight Matrix W
Matrix of vertices $V_{P_n} \to [1, 2, ..., n]$ and $V_{K2} \to [1, 2]$
$V_{P_n \boxtimes K_2} = [\quad]$
for j in range $(1, 3)$ **do**
  for i in range $(1, len(vertex\_list\_a) + 1)$ **do**
    if $(i\%2 == 1)$ and $(j == 1)$ **then**:
      $V_{P_n \boxtimes K_2}.append(i)$
    else if $(i\%2 == 0)$ and $(j == 1)$ **then**:
      $V_{P_n \boxtimes K_2}.append(2 * n + 1 - i)$
    else if $(i\%2 == 0)$ and $(j == 2)$ **then**:
      $V_{P_n \boxtimes K_2}.append(i)$
    else if $(i\%2 == 1)$ and $(j == 2)$ **then**:
      $V_{P_n \boxtimes K_2}.append(2 * n + 1 - i)$
Generate Weight Matrix W
$def find\_weight\_ab_1() :$
$W = []$
for j in range $(1, 3)$ **do**
  for i in range $(1, n + 1)$ **do**
    if $(i == 1)$ and $(j == 1)$ **then**:
      $W.append(4 * n + 1)$
    else if $(i == 1)$ and $(j == 2)$ **then**:
      $W.append(2 * n + 2)$
    else if $((i\%2 == 1), (j == 1))$ and $((i \geq 3), (i \leq (n - 1))$ **then**:
      $W.append(6 * n + 3 - i)$
    else if $((i\%2 == 1), (j == 2))$ and $((i \geq 3), (i \leq (n - 1))$ **then**:
      $W.append(4 * n + 2 + i)$
    else if $((i\%2 == 0), (j == 1))$ and $((i \geq 2), (i \leq (n - 1))$ **then**:
      $W.append(4 * n + 2 + i)$
    else if $((i\%2 == 0), (j == 2))$ and $((i \geq 2), (i \leq (n - 1))$ **then**:
      $W.append(6 * n + 3 - i)$
    else if $(i == n)$ and $(j == 1)$ and $(n\%2 == 0)$ **then**:
      $W.append(3 * n + 1)$
    else if $(i == n)$ and $(j == 2)$ and $(n\%2 == 0)$ **then**:
      $W.append(3 * n + 2)$
    else if $(i == n)$ and $(j == 1)$ and $(n\%2 == 1)$ **then**:
      $W.append(3 * n + 2)$ =0

else if $(i == n)$ and $(j == 2)$ and $(n\%2 == 1)$ **then**:
$W.append(3 * n + 1)$
return W
if $(len(W) == len(set(W)))$ **then**:
Distinct Elements

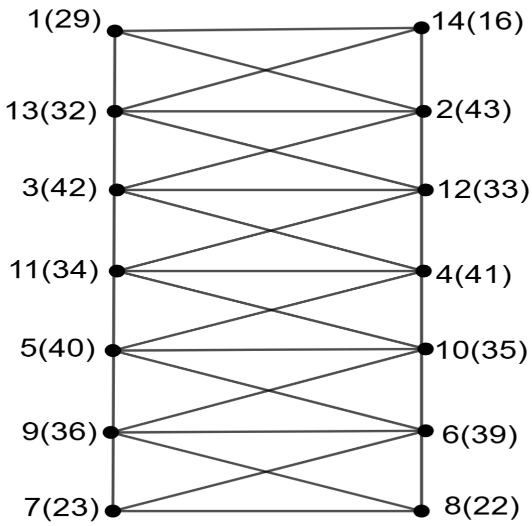

Fig. 4: Distance antimagic labeling of $G = P_7 \boxtimes K_2$.

V. Distance antimagic labeling of Lexicographic Product of graphs

$(a, 1)$-distance antimagic labeling of lexicographic product of complete graphs and cycle is covered in this section. Further, distance antimagic labeling of lexicographic product of star $K_{1,n}$ and bistar $B_{n,n}$ with K_2 is explored.

Theorem 1. *The graph $G = K_n \circ K_n$ is $(a, 1)$-distance antimagic.*

Proof: Let $G = K_n \circ K_n$. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$. We denote the vertex (v_i, v_j) in $K_n \circ K_n$ by x_{ij} . Define $g\colon V(G) \to \{1, 2, \dots, n^2\}$ by $g(x_{ij}) = (j-1)n + i$ for $1 \le i, j \le n$.
So, we get weights of vertices of graph G as

$$w(x_{ij}) = \begin{cases} n^3(t+1) - n^2t + n(1-j) - i & \text{if } n = 2t+1, t \ge 1 \\ n^3t + n^2 - nt + n(1-j) - i & \text{if } n = 2t, t > 1 \end{cases} \tag{4}$$

We will show that the weight of vertices form an arithmetic progression with common difference $d = 1$ using (3) as follows
Case 1. $n = 2t + 1, t \ge 1$
Consider $w(x_{(i-1)j}) - w(x_{ij})$ for $2 \le i \le n$ and $1 \le j \le n$.

$$w(x_{(i-1)j}) - w(x_{ij}) = n^3(t+1) - n^2t + n(1-j) - (i-1) - n^3(t+1) + n^2t - n(1-j) + i = -(i-1) + i = 1$$

Consider $w(x_{n(j-1)}) - w(x_{1j})$ for $2 \le j \le n$.

$$w(x_{n(j-1)}) - w(x_{1j}) = n^3(t+1) - n^2t + n(1-j+1) - n - n^3(t+1) + n^2t - n(1-j) + 1 = n - n + 1 = 1$$

Case 2. $n = 2t, t > 1$
Consider $w(x_{(i-1)j}) - w(x_{ij})$ for $2 \le i \le n$ and $1 \le j \le n$.

$$w(x_{(i-1)j}) - w(x_{ij}) = n^3t + n^2 - nt + n(1-j) - (i-1) - n^3t - n^2 + nt - n(1-j) + i = -(i-1) + i = 1$$

Consider $w(x_{n(j-1)}) - w(x_{1j})$ for $2 \le j \le n$.

$$w(x_{n(j-1)}) - w(x_{1j}) = n^3t + n^2 - nt + n(1-j+1) - n - n^3t - n^2 + nt - n(1-j) + 1 = n - n + 1 = 1$$

We observe that weights of the vertices form an arithmetic progression with first term $a = \frac{(n^2-1)n^2}{2}$ and common difference $d = 1$ in the following sequence: $w(x_{nn})$, $w(x_{(n-1)n})$,..., $w(x_{1n})$, $w(x_{n(n-1)})$, $w(x_{(n-1)(n-1)})$,...,$w(x_{1(n-1)})$,..., $w(x_{n1})$, $w(x_{(n-1)1})$,..., $w(x_{11})$. So, G is $(a, 1)$-distance antimagic. ■

Theorem 2. *The graph $G = K_n \circ C_3$ is $(a, 1)$-distance antimagic for any odd integer $n = 2t + 1$.*

Proof: Let $G = K_n \circ C_3$ where $n = 2t + 1$. Let $V(K_n) = \{u_1, u_2, \dots, u_n\}$ and $V(C_3) = \{v_1, v_2, v_3\}$. We denote the vertex (u_i, v_j) in $K_n \circ C_3$ by x_{ij} . Define $g\colon V(G) \to \{1, 2, \dots, 3n\}$ by $g(x_{ij}) = (j-1)n + i$ for $1 \le i \le n$ and $1 \le j \le 3$. Then for $1 \le i \le n$, we have

$$w(x_{ij}) = \begin{cases} 3n(1+3t) + (n-i) & \text{if } j = 3 \\ 3n(1+3t) + n + (n-i) & \text{if } j = 2 \\ 3n(1+3t) + 2n + (n-i) & \text{if } j = 1 \end{cases} \tag{5}$$

Using (4), we shall demonstrate that the weight of vertices forms an arithmetic progression with common difference $d = 1$ as follows
Consider $w(x_{(i-1)j}) - w(x_{ij})$ for $2 \le i \le n$ and $1 \le j \le 3$.

$$w(x_{(i-1)j}) - w(x_{ij}) = -(i-1) + i = 1$$

Also,

$$w(x_{n2}) - w(x_{13}) = 3n(1+3t) + n + (n-n) - 3n(1+3t) - (n-1) = 1$$

$$w(x_{n1}) - w(x_{12}) = 3n(1+3t) + 2n + (n-n) - 3n(1+3t) - n - (n-1) = 1$$

We note that in the following sequence: $w(x_{n3})$, $w(x_{(n-1)3})$,..., $w(x_{13})$, $w(x_{n2})$, $w(x_{(n-1)2})$,...,$w(x_{12})$, $w(x_{n1})$, $w(x_{(n-1)1})$,..., $w(x_{11})$, the weights of the vertices form an arithmetic progression with initial term $a = 3n(n + t)$ and common difference $d = 1$
Therefore, g is $(a, 1)$-distance antimagic labeling for G. ■

Theorem 3. *The graph $G = K_{1,n} \circ K_2$ is distance antimagic.*

Proof: Let $V(K_{1,n}) = \{u_1, u_2, \dots, u_{n+2}\}$ where u_1 is the central vertex and $V(K_2) = \{v_1, v_2\}$. We denote the vertex (u_i, v_j) in $G = K_{1,n} \circ K_2$ by x_{ij} .
Define $g\colon V(G) \to \{1, 2, ..., 2(n+2)\}$ as

$$g(x_{ij}) = \begin{cases} j & \text{if } i = 1,\ j = 1, 2 \\ 2i - 1 & \text{if } j = 1 \text{ and } 2 \le i \le (n+2) \\ 2i & \text{if } j = 2 \text{ and } 2 \le i \le (n+2) \end{cases}$$

It is evident that g is a bijection and vertex weights are computed as

$$w(x_{ij}) = \begin{cases} (n+2)(2n+5) - j & \text{if } i = 1,\ j = 1, 2 \\ 2i + 3 & \text{if } j = 1,\ 2 \leq i \leq (n+2) \\ 2i + 2 & \text{if } j = 2,\ 2 \leq i \leq (n+2) \end{cases}$$

Clearly, vertex weight are all distinct. Therefore, G is distance antimagic graph. ■

Theorem 4. *The graph $G = B_{n,n} \circ K_2$ is distance antimagic.*

Proof: Let $V(B_{n,n}) = \{u, r, u_1, u_2, \ldots, u_n, r_1, r_2, \ldots, r_n\}$ where u and r are the central vertices and $V(K_2) = \{v_1, v_2\}$. We denote the vertex in $G = K_{1,n} \circ K_2$ by (u_i, v_j) where $u_i \in V(B_{n,n})$ and $v_j \in V(K_2)$.
Define $g\colon V(G) \to \{1, 2, ..., 2(2n+2)\}$ as

$$\begin{aligned} g(u, v_j) &= j && \text{if } 1 \leq j \leq 2 \\ g(r, v_j) &= j + 2 && \text{if } 1 \leq j \leq 2 \\ g(u_i, v_1) &= 2i + 3 && \text{if } 1 \leq i \leq n \\ g(u_i, v_2) &= 2i + 4 && \text{if } 1 \leq i \leq n \\ g(r_i, v_1) &= 2(n+i) + 3 && \text{if } 1 \leq i \leq n \\ g(r_i, v_2) &= 2(n+i) + 4 && \text{if } 1 \leq i \leq n \end{aligned}$$

It is evident that g is a bijection and vertex weights are computed as

$$\begin{aligned} w(u, v_j) &= 2n^2 + 9n + 10 - j && \text{if } 1 \leq j \leq 2 \\ g(r, v_j) &= 6n^2 + 9n + 8 - j && \text{if } 1 \leq j \leq 2 \\ g(u_i, v_1) &= 2i + 7 && \text{if } 1 \leq i \leq n \\ g(u_i, v_2) &= 2i + 6 && \text{if } 1 \leq i \leq n \\ g(r_i, v_1) &= 2(n+i) + 11 && \text{if } 1 \leq i \leq n \\ g(r_i, v_2) &= 2(n+i) + 10 && \text{if } 1 \leq i \leq n \end{aligned}$$

Clearly, vertex weight are all distinct. Therefore, G is distance antimagic graph. ■

VI. Conclusion

Research on graph products enables comprehension of the combinatorial and algebraic aspects of graph theory and improves many academic and practical domains. Graph products are important to graph theory and are used in numerous other areas. In network design, graph products are used to build intricate topologies with certain traits and connections. There are multiple applications for labeling graph products in engineering disciplines. This work presents the results of distance antimagic labeling of the cartesian product, tensor product, strong product and lexicographic product for complete graphs, cycles, paths, star, bistar and graphs connected to cycles. Further research is needed to address the issue of distance antimagic labeling for different additional graph products.

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