Series Solution of Non-linear Initial Value Problems on Certain Physical Systems through MATLAB

Keerthika V and R Prahalatha

Abstract—**The aim of this paper is to find the approximate series solution of IVPs on nonlinear differential equations (NDEs) and on systems of NDEs via an iterative algorithm that can be easily exhibited through MATLAB code. We propose a modified iterative scheme based on the DGJM method of solving functional equations of the form $u = f + N(u)$, and we observe that the method is far better than those existing in the literature. For the sake of illustration, series solutions to a few physical equations proposed by Klien-Gordon, Allen-Cahn and Benjamin-Bona-Mahony are provided along with the MATLAB code used for computation. Moreover, graphs are used to validate the efficacy of the method.**

Index Terms—**Nonlinear differential equations, system of differential equations, series solution, iterative algorithm, decomposition technique.**

I. Introduction

MOST problems in virtually all areas of engineering and science can be expressed as nonlinear or linear, partial or ordinary differential equations. In particular, nonlinear differential equations (NDEs) are widely used in modeling physical phenomena occurring in quantum physics, neural networks, population growth modeling, climate modeling, biotechnology, etc [1], [2]. Typically, there is no universal approach when it comes to choosing the best technique for evaluating NDEs. The problem context, the aggravating conditions, and the desired precision of the solution all have a significant impact on the approach to solving those equations. The complexity adds to the appeal of NDEs makes them a perennial subject of study in many different fields of study. Therefore, solving these types of equations is crucial to modern science, including engineering. Since, traditional approaches are typically complex and difficult to understand, researchers are required to use sophisticated mathematical techniques. As a result, numerous studies have been conducted and a variety of analytical and numerical techniques, including linearization, decomposition, homotopy and perturbation methods, have been developed and applied to estimate the exact solutions of nonlinear equations. The Adomian decomposition method and its modifications [3]–[6], homotopy perturbation analysis [7], variational iteration method [8], [9], optimal decomposition method [10] and Daftardar-Gejji Jafari method [11] are few to be worth mentioned.

Manuscript received April 5, 2024; revised September 29, 2024.

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Our work presents a modified iterative scheme based on the DGJM method of solving functional equations of the form $u = f + N(u)$ [12], that can solve the nonlinear IVPs analytically and efficiently without the need for linearization, perturbation, or decomposition methods. The process entails dividing the equation being studied into nonlinear and/or linear portions. After inverting the linear operator, the equation is transformed into functional equation form along with the initial conditions provided. Finally, the procedure yields a series as a solution, where terms involved are established by an iterative formula.

The article is organized as follows: The iterative scheme of the technique is presented in the first section, along with corresponding examples in the next section. We also proposed a MATLAB algorithm for solving nonlinear differential equations and compared the results of the considered problems along with exact solutions with those obtained using the Adomian decomposition technique. Finally, the section ends with conclusions.

II. Methodology

Let us consider the nonlinear differential equation of the form

$$F(z^*(\tilde{x},\tilde{t})) = H(\tilde{x},\tilde{t}), \tag{1}$$

where the operator F is a combination of nonlinear and linear operators, N and L respectively. Taking L^{-1}, the inverse linear operator (usually means integration) corresponding to L, on both sides of equation (1), we get the solution $z^*(\tilde{x},\tilde{t})$, expressed in the form of a functional equation as:

$$z^*(\tilde{x},\tilde{t}) = h^*(\tilde{x},\tilde{t}) + L^*\left[z^*(\tilde{x},\tilde{t})\right] + N^*\left[z^*(\tilde{x},\tilde{t})\right]. \tag{2}$$

Here $h^*(\tilde{x},\tilde{t})$ represents the homogeneous terms obtained after integration and substitution of the respective initial conditions and

$$L^*\left[z^*(\tilde{x},\tilde{t})\right] = L^{-1}[L[z^*(\tilde{x},\tilde{t})]],$$
$$N^*\left[z^*(\tilde{x},\tilde{t})\right] = L^{-1}[N[z^*(\tilde{x},\tilde{t})]],$$

where $L[z^*(\tilde{x},\tilde{t})]$ represents the linear terms with lower order derivatives and $N[z^*(\tilde{x},\tilde{t})]$ represents the nonlinear terms.

Our goal is to find the approximate value of the equation (2). For instance, suppose that an infinite series $\sum_{r=0}^{\infty} z_r^*(\tilde{x},\tilde{t})$ approximates the solution (2).

Therefore,

$$\begin{aligned} z^*(\tilde{x},\tilde{t}) &= \sum_{r=0}^{\infty} z_r^*(\tilde{x},\tilde{t}) \\ &= h^*(\tilde{x},\tilde{t}) + L^*\left[z^*(\tilde{x},\tilde{t})\right] + N^*\left[z^*(\tilde{x},\tilde{t})\right]. \end{aligned} \tag{3}$$

Equating the terms on both sides for each value of r, we get,

$$
\begin{aligned}
&z_0^*(\tilde{x},\tilde{t}) = h^*(\tilde{x},\tilde{t}),\\
&\sum_{r=0}^{1} z_r^*(\tilde{x},\tilde{t}) = h^*(\tilde{x},\tilde{t}) + L^*\left[z_0^*(\tilde{x},\tilde{t})\right] + N^*\left[z_0^*(\tilde{x},\tilde{t})\right],\\
&\sum_{r=0}^{2} z_r^*(\tilde{x},\tilde{t}) = h^*(\tilde{x},\tilde{t}) + L^*[(z_0^*(\tilde{x},\tilde{t}) + z_1^*(\tilde{x},\tilde{t})]\\
&\qquad + N^*[z_0^*(\tilde{x},\tilde{t}) + z_1^*(\tilde{x},\tilde{t})],\\
&\qquad \vdots\\
&\sum_{r=0}^{k} z_r^*(\tilde{x},\tilde{t}) = h^*(\tilde{x},\tilde{t})\\
&\qquad + L^*[(z_0^*(\tilde{x},\tilde{t}) + z_1^*(\tilde{x},\tilde{t}) + \dots z_{k-1}^*(\tilde{x},\tilde{t})]\\
&\qquad + N^*[z_0^*(\tilde{x},\tilde{t}) + z_1^*(\tilde{x},\tilde{t}) + \dots z_{k-1}^*(\tilde{x},\tilde{t})],\\
&\qquad \vdots
\end{aligned}
$$

If the nonlinear term N^* is a contraction, then this infinite series will converge uniformly to the solution of the given differential equation (1).
To solve the functional equation (3), we propose a modified iterative method based on DGJM as follows:
Let v_k denotes the k^{th} approximation of the solution $z^*(\tilde{x},\tilde{t})$. That is,

$$v_k = \sum_{r=0}^{k} z_r^*(\tilde{x},\tilde{t}) \tag{4}$$

which comprises of $(k+1)$ terms. Thus, by combining equations (3) and (4), we get the simplest form of iterative solution as follows:

$$
\begin{aligned}
v_0 &= z_0^*(\tilde{x},\tilde{t}) = h^*(\tilde{x},\tilde{t}),\\
v_k &= \sum_{r=0}^{k} z_r^*(\tilde{x},\tilde{t})\\
&= h^*(\tilde{x},\tilde{t}) + L^*[v_{k-1}] + N^*[v_{k-1}], \forall k \geq 1.
\end{aligned} \tag{5}
$$

As, $k \to \infty$, $v_k \to z^*(\tilde{x},\tilde{t})$, which is the required solution of equation (1).
In contrast to the solution methods discussed in the literature for nonlinear equations, the method given in (5) uses values obtained in the immediately preceding step and converges more quickly to the exact solution (if exists) of the differential equation (1) within a few iterations. Moreover, these new iterative equations can be easily programmed using mathematical software such as MATLAB, Mathematica, etc. Computation becomes easy and thus reduces the time spent on manual calculations.

Theorem 1. *Condition for convergence:*
If N^* and L^* are the nonlinear and linear operators, respectively, defined on the Banach space $\mathcal{B}$ and satisfy the Lipschitz condition, whose derivatives are bounded by the values K_N and K_L respectively, then the sequence of iterated values $\{v_k\}$ in (5) converges uniformly to the solution function $z^*(\tilde{x},\tilde{t}), as\ k \to \infty$, whenever $0 < K = (K_L + K_N) < 1$.

Proof: By mean value theorem on Banach spaces $\mathcal{B}$ [13],

$$
\begin{aligned}
&\|v_{n+1} - v_n\|\\
&= \|(L^*[v_n] - L^*[v_{n-1}]) + (N^*[v_n] - N^*[v_{n-1}])\|\\
&\leq (K_L + K_N)\|v_n - v_{n-1}\|, \ \forall n = 0, 1, \cdots, k-1
\end{aligned}
$$

Let $K = (K_L + K_N)$ with $0 < K < 1$.
Then,

$$
\begin{aligned}
\|v_{n+1} - v_n\| &\leq K\|v_n - v_{n-1}\| \leq K^n\|v_1 - v_0\|\\
\implies \|v_k\| &= \left\|v_0 + \sum_{n=0}^{k-1}(v_{n+1} - v_n)\right\|\\
&\leq \|v_0\| + \sum_{n=0}^{k-1} K^n\|v_1 - v_0\|
\end{aligned}
$$

By Weistrass M-test, the series $\sum_{n=0}^{\infty} K^n\|v_1 - v_0\|$ converges and hence, the sequence $\{v_k\}$ in (5) converges uniformly to the solution $z^*(\tilde{x},\tilde{t}), as\ k \to \infty$.
Note: The condition mentioned in theorem 1 is only sufficient for convergence of the method. (For detailed study of convergence, one can refer to the articles [12], [14]–[16]) ■

A. Algorithm

In this section, we provide an algorithm for solving NDEs using the iterative method in MATLAB, which efficiently uses symbolic tools and string functions to calculate all desired nonlinear components. The algorithm is as follows:

Step 1. Write down the solution of the given NDE in functional equation form, $z^* = h^* + N^*(z^*)$.
Step 2. Create a built-in function for computing the value of the nonlinear component $N^*(z^*)$ with a suitable integration operator.
Step 3. Assign the initial value h^* to the solution function z^*.
Step 4. Use "for loop" *for i=1:k* to proceed with the iterative scheme $z = h^* + N^*(z^*)$, by calling the function N^* with the input argument z^*, for each value of i. Here k denotes the order of approximation of the infinite series solution z^*.
Step 5. The output obtained from each iteration (i) of step 4 gives the required approximated solution with $(i+1)$ terms. Display the corresponding output using the commands "fprintf('v(%d)',i)" and "disp(z)".

The suggested approach to compute the nonlinear term is incredibly efficient, clear, and simple. Since the method transformed the nonlinear terms into a function file in MATLAB, it may be readily incorporated into any code that deals with the solution of NDEs, and it can be used anywhere in the program to compute any desired estimated component of interesting nonlinearity. The effective utilization of MATLAB's embedded symbolic toolboxes and string functions significantly reduces the length of computational coding. As can be seen, the code is brief and straightforward, which reduces the calculation time and volume.

B. Algorithm for the System of Nonlinear Differential Equations

Suppose that, we have a system of m NDEs of the form

$$F_i(z^*(\tilde{x},\tilde{t})) = H_i(\tilde{x},\tilde{t}), \quad 1 \leq i \leq m. \tag{6}$$

To find the set of approximate solution $\{z_i^*\}$ for this system (6), the iterative scheme (5) will be modified as:

$$\begin{aligned} v_{i0} &= z_{i0}^*(\tilde{x},\tilde{t}) = h_i^*(\tilde{x},\tilde{t}), \\ v_{ik} &= \sum_{r=0}^{k} z_{ir}^*(\tilde{x},\tilde{t}) \\ &= h_i^*(\tilde{x},\tilde{t}) + L_i^*[v_{ik-1}] + N_i^*[v_{ik-1}], \forall k \geq 1, \end{aligned} \tag{7}$$

where $h_i^*(x,t)$, N_i^* and L_i^* denotes the homogeneous term, nonlinear and linear differential operators, respectively, obtained from the functional equation form,

$$\begin{aligned} z_i^*(\tilde{x},\tilde{t}) = \sum_{r=0}^{\infty} z_{ir}^*(\tilde{x},\tilde{t}) = h_i^*(\tilde{x},\tilde{t}) + L_i^*[z_1^*, z_2^*, \cdots, z_m^*] \\ + N_i^*[z_1^*, z_2^*, \cdots, z_m^*], \end{aligned} \tag{8}$$

corresponding to the i^{th} differential equation of the given system (6). Hence, the proposed approach can be easily applied to nonlinear systems involving functions with one or more independent variables, and we can modify the algorithm by simply creating the respective function for the nonlinear operator as mentioned in step 2 and by calling the function with one or more input arguments as per the requirement, based on the structure of the differential system considered.

III. Applications

This section covers examples that demonstrate the process of solving nonlinear ordinary and partial differential equations (NODEs and NPDEs) using the algorithm mentioned in Section II.

A. Nonlinear Ordinary Differential Equations

Example 1. *Riccati differential equation:*
Consider the homogenous Riccati differential equation [17], $y' + y - y^2 = 0$ *with initial condition* $y(0) = 2$.

Let us convert this IVP to functional equation form,

$$y(x) = 2 + \int_0^t y^2 \ dt - \int_0^t y \ dt. \tag{9}$$

Taking $h^*(x) = 2$, $N^*[y(x)] = \int_0^t y^2 \ dt$ and $L^*[y(x)] = -\int_0^t y(t) \ dt$ and substituting in (5), we get,

$$\begin{aligned} v_0 &= y_0(x) = 2, \\ v_1 &= y_0(x) + y_1(x) = 2 + L^*[v_0] + N^*[v_0] \\ &= 2 + 2x, \\ v_2 &= y_0(x) + y_1(x) + y_2(x) = 2 + L^*[v_1] + N^*[v_1] \\ &= 2 + 2x + 3x^2 + \frac{4x^3}{3}, \\ v_3 &= 2 + 2x + 3x^2 + \frac{13x^3}{3} + 4x^4 + \frac{43x^5}{15} + \frac{4x^6}{3} + \frac{16x^7}{63}, \\ &\vdots \end{aligned}$$

while proceeding the process, we get the series that exactly corresponds to the Taylor series expansion of the function $\frac{-2}{e^x-2}$. The curves representing the estimated solution $v_5 = \sum_{j=0}^{5} y_j(x)$ and the exact solution $y(x) = \frac{-2}{e^x-2}$ is plotted in figure 1.

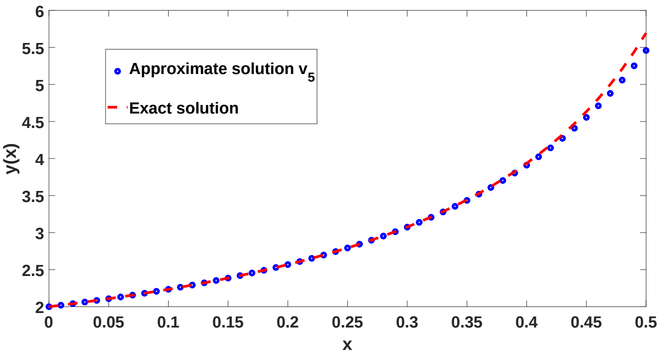

Fig. 1. Exact and the estimated solution of Riccati differential equation (9) for $x \in (0{,}0.5)$.

Example 2. *Lane-Emden equation:*
Consider the Lane-Emden equation,

$$y'' + \frac{2}{x}y' + (8e^y + 4e^{\frac{y}{2}}) = 0, \tag{10}$$

with initial conditions, $y(0) = y'(0) = 0$.

To overcome the difficulties caused by the singularity, let us rewrite the equation (10) as

$$x^{-2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -\left(8e^y + 4e^{\frac{y}{2}}\right).$$

Let us denote the linear operator, $L := x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right)$, then by applying the corresponding inverse operator, $L^{-1} := x^{-2}\int_0^t \left(x^2 \int_0^t (*)dt\right) dt$ on both sides of (10), we get the functional equation form of solution as

$$y(x) = h^*(x) + N^*[y(x)],$$

where,

$$\begin{aligned} h^*(x) &= 0, \\ N^*[y(x)] &= -x^{-2}\int_0^t x^2 \int_0^t \left(8e^y + 4e^{\frac{y}{2}}\right) dtdt. \end{aligned}$$

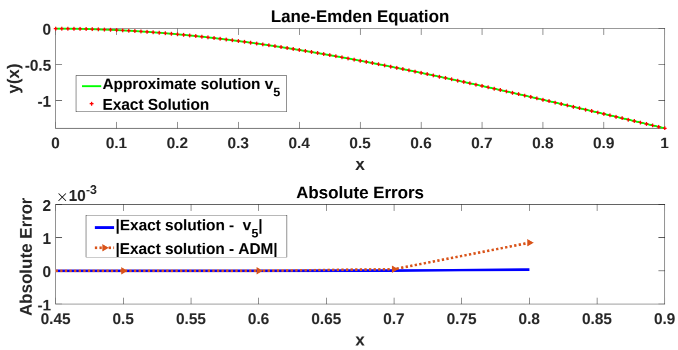

Fig. 2. (a)Exact and the iterated solution v_5 of Lane-Emden equation (10) for $x \in (0{,}1)$. (b) Absolute error of solution v_5 and the one obtained by using ADM explained by Hosseini *et al.* in the article [18].

We see that the nonlinear term contains a trigonometric form of the solution function. So, to avoid prolongation of iterative process, we replace exponential function by its series form of expansion (as taken in the article [19]) with few numbers

TABLE I
THE ABSOLUTE ERROR OF SOLUTIONS OBTAINED USING ITERATIVE METHOD AND THOSE VALUES OBTAINED BY USING ADM EXPLAINED BY HOSSEINI *et al.* IN THE ARTICLE [18]

x	Estimated v_5	Exact solution	Solution by ADM	Exact - v_5	Exact - ADM
0.1	-0.01990066	-0.01990066	-0.01990066	0.00000000	0.00000000
0.2	-0.07844143	-0.07844143	-0.07844143	0.00000000	0.00000000
0.3	-0.17235539	-0.17235539	-0.17235539	0.00000000	0.00000000
0.4	-0.29684001	-0.29684001	-0.29684001	0.00000000	0.00000000
0.5	-0.44628701	-0.4462871	-0.44628707	0.00000009	0.00000003
0.6	-0.61496836	-0.6149694	-0.61496760	0.00000104	0.00000180
0.7	-0.79754497	-0.79755224	-0.79750315	0.00000727	0.00004909
0.8	-0.98935562	-0.98939248	-0.98854610	0.00003686	0.00084638

of terms (say j terms). For this problem, let us take the value of $j = 5$. Hence, the nonlinear term will be

$$N^*[y(x)] = -x^{-2}\int_0^t x^2 \int_0^t \left(8\sum_{j=0}^{5}\frac{y^j}{j!} + 4\sum_{j=0}^{5}\frac{(y/2)^j}{j!}\right) dtdt.$$

Hence, by using the iterative procedure (5), we get,

$$\begin{aligned}
v_0 &= y_0(x) = 0,\\
v_1 &= y_0(x) + y_1(x) = N^*[v_0]\\
&= -2x^2 + x^4 - \frac{3x^6}{7} + \frac{17x^8}{108} - \frac{x^{10}}{20} + \frac{x^{12}}{72},\\
v_2 &= y_0(x) + y_1(x) + y_2(x) = N^*[v_1]\\
&= -2x^2 + x^4 - \frac{2x^6}{3} + \frac{353x^8}{756} - \frac{1247x^{10}}{3780} + \frac{253x^{12}}{1092}\\
&- \frac{437137x^{14}}{2778300} + \frac{181957x^{16}}{1799280} - \frac{1103789x^{18}}{18098640}\\
&+ \frac{144598339x^{20}}{4200789600} - \frac{3683556971x^{22}}{202438051200}\\
&+ \frac{231935153x^{24}}{25719120000} - \frac{534607386479x^{26}}{127394825725440}\\
&+ \frac{25495822731x^{28}}{122797481587200} - \frac{1276867135939x^{30}}{1687709229696000}\\
&+ \frac{1231131080579x^{32}}{4192051957632000} - \frac{167018930215267x^{34}}{1554870180648960000}\\
&+ \frac{193233191054773x^{36}}{5221227934347264000} - \frac{139430104011049x^{38}}{11618408106160128000}\\
&+ \frac{248746818701x^{40}}{68026866647040000} - \frac{1140675477038039x^{42}}{1092224438158049280000}\\
&+ \frac{2471631549683x^{44}}{8870039734026240000} - \frac{1722217118149x^{46}}{24905228448006144000}\\
&+ \frac{3979105027x^{48}}{250869810659328000} - \frac{4079967859x^{50}}{1223950287974400000}\\
&+ \frac{11381281x^{52}}{17946361036800000} - \frac{58304909x^{54}}{543063657185280000}\\
&+ \frac{136201x^{56}}{8646759913881600} - \frac{18161x^{58}}{9534654916853760}\\
&+ \frac{13x^{60}}{75539184353280} - \frac{13x^{62}}{1451095347953664},\\
&\vdots
\end{aligned}$$

The exact solution of (10) is $y(x) = -2ln(1 + x^2)$. We see that, even in the second iterative step, we get the closest approximation to the exact solution and the results closely match with those obtained in [20]–[22]. The comparison between the iterative solution $v_2(x)$, the exact solution $y(x)$ and the solution obtained by using Adomian decomposition method explained by Hosseini *et al.* in the article [18] is given in table I and the corresponding error is depicted in figure 2.

B. Nonlinear Partial Differential Equations

Example 3. *Klien-Gordon equation:*

Extending a linear wave equation with additional linear and/or nonlinear variables to reflect relativistic energy results in the Klein-Gordon equation [23], which is used in numerous applications based on quantum field theory. Let us find the approximate solution of the following Klien-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 - u^2, \tag{11}$$

with $u(x, 0) = 0$ and $\frac{\partial u(x,0)}{\partial t} = e^x$.
Here, the linear operator is $L := \frac{\partial^2}{\partial t^2}$, and by applying the corresponding inverse operator, $L^{-1} := \iint_0^t *dt\ dt$, on both sides of (11), we get the functional equation form of the solution,
$u(x, t) = h^*(x, t) + L^*[u(x, t)] + N^*[u(x, t)]$,
where,

$$\begin{aligned}
h^*(x, t) &= te^x,\\
L^*[u(x, t)] &= \int_0^t \int_0^t \frac{\partial^2 u}{\partial x^2}\, dtdt,\\
N^*[u(x, t)] &= \int_0^t \int_0^t \left(\left(\frac{\partial u}{\partial x}\right)^2 - u^2\right) dtdt.
\end{aligned}$$

Hence, by using the iterative procedure (5), we get,

$$\begin{aligned}
v_0 &= u_0(x, t) = te^x,\\
v_1 &= u_0(x, t) + u_1(x, t)\\
&= te^x + L^*[v_0] + N^*[v_0]\\
&= e^x t + \frac{e^x t^3}{6}\\
&= e^x\left(t + \frac{t^3}{3!}\right),\\
v_2 &= u_0(x, t) + u_1(x, t) + u_2(x, t)\\
&= te^x + L^*[v_1] + N^*[v_1]\\
&= e^x t + \frac{e^x t^3}{6} + \frac{e^x t^5}{120}\\
&= e^x\left(t + \frac{t^3}{3!} + \frac{t^5}{5!}\right),
\end{aligned}$$

$$\begin{aligned} v_3 &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \\ &= te^x + L^*[v_2] + N^*[v_2] \\ &= e^x t + \frac{e^x t^3}{6} + \frac{e^x t^5}{120} + \frac{e^x t^7}{5040} \\ &= e^x \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} \right), \\ &\vdots \end{aligned}$$

We see that by proceeding the process, the approximate series of solution v_k converges to $e^x \sinh(t)$, which is the exact solution of (11). The obtained solution structure is compared with the exact solution in figure 3.

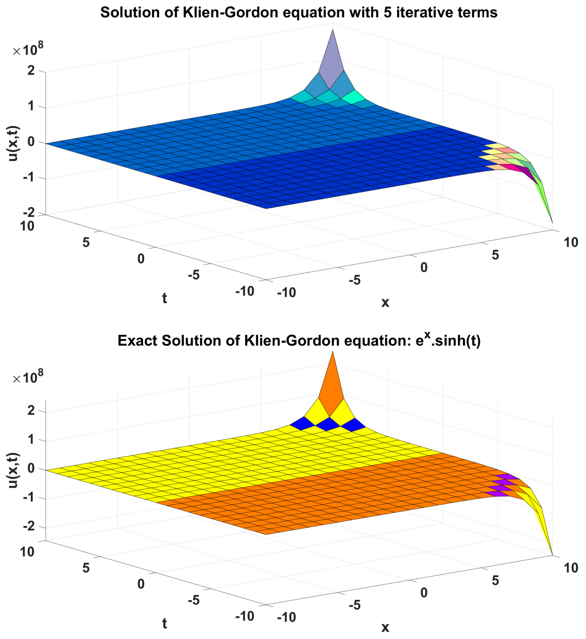

Fig. 3. Graphical simulation of the estimated solution v_4 and the exact solution of Klien-Gordon equation (11).

Example 4. *Allen-Cahn equation:*
Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u^3 + u, \tag{12}$$

with $u(x,0) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}$.

Integrating both sides of (12) with respect to t partially and substituting the given conditions, we get,

$$u(x,t) = \frac{1}{1+e^{\frac{x}{\sqrt{2}}}} + \int_0^t (u_{xx} + u)\, dt - \int_0^t u^3\, dt. \tag{13}$$

Comparing equation (13) with equation (2), we get,

$$\begin{aligned} h^*(x,t) &= \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}, \\ L^*[u(x,t)] &= \int_0^t (u_{xx} + u)\, dt, \\ N^*[u(x,t)] &= -\int_0^t u^3\, dt. \end{aligned}$$

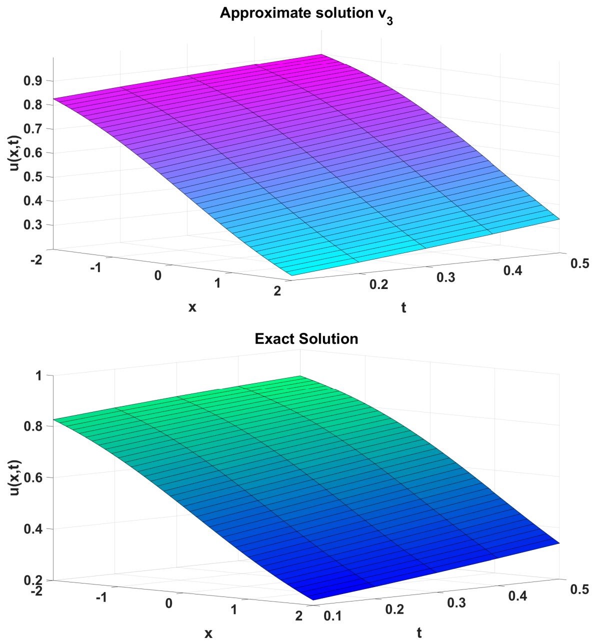

Fig. 4. Graphical simulation of the approximate solution v_3 and the exact solution of equation (12).

Hence, by using the iterative procedure (5), we obtain,

$$\begin{aligned} v_0 &= \frac{1}{1+e^{\frac{x}{\sqrt{2}}}}, \\ v_1 &= h^*(x,t) + L^*[v_0] + N^*[v_0] \\ &= \frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)} + \frac{3te^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}, \\ v_2 &= h^*(x,t) + L^*[v_1] + N^*[v_1] \\ &= \frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)} + \frac{\left[\begin{array}{c} 3te^{\left(\frac{x}{\sqrt{2}}\right)}\left(96e^{\left(\sqrt{2}x\right)} - 12t + 16e^{\left(2\sqrt{2}x\right)}\right. \\ +64e^{\left(\frac{3x}{\sqrt{2}}\right)} + 12te^{\left(2\sqrt{2}x\right)} + 24te^{\left(\frac{3x}{\sqrt{2}}\right)} \\ \left.-24t^2e^{\left(\sqrt{2}x\right)} - 9t^3e^{\left(\sqrt{2}x\right)} + 16\right) \\ -\left(3te^{\left(\sqrt{2}x\right)}\right)\left(24t + 24t^2 - 64\right) \end{array}\right]}{32\left(1+e^{\left(\frac{x}{\sqrt{2}}\right)}\right)^6}, \\ &\vdots \end{aligned}$$

The obtained iterated value resembles the Taylor series expansion of the function $\frac{1}{\left(1+e^{\left(\frac{x}{\sqrt{2}}-\frac{3t}{2}\right)}\right)}$ given by Shehata *et al.* in [24], which is the exact solution of equation (12). The approximate solution and exact solution are plotted in figure 4. The comparison of the respective solutions obtained at time $t = 0$ and $t = 0.5$ is also portrayed in the plot 5.

Example 5. *Regularized long-Wave equation:*
Next we analyse the adaptability of our method in solving Benjamin-Bona-Mahony (BBM) equation [24]–[26].

$$\begin{aligned} u_t - u_{xxt} + u_x + u.u_x &= 0, \\ u(x,0) &= sech^2\left(\frac{x}{4}\right). \end{aligned} \tag{14}$$

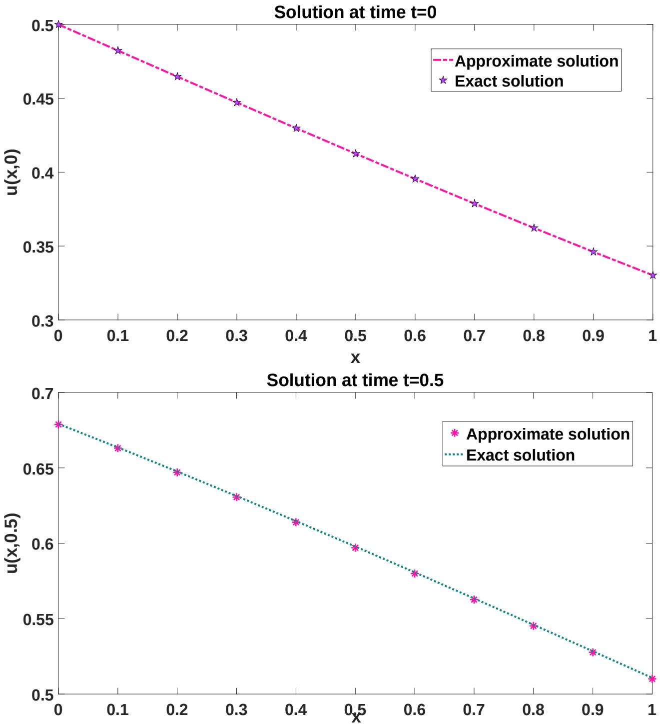

Fig. 5. Solutions obtained at time $t = 0$ and $t = 0.5$ of the Allen-Cahn equation (12).

The functional form of equation (14) is:

$$u = h^*(x,t) + L^*\left[u(x,t)\right] + N^*\left[u(x,t)\right]$$

$$= sech^2\left(\frac{x}{4}\right) + \int_0^t \left(u_{xxt} - u_x\right)\ dt - \int_0^t u.u_x\ dt.$$

Hence, by using (4) and (5), we get,

$$v_0 = sech^2\left(\frac{x}{4}\right),$$

$$v_1 = h^*(x,t) + L^*[v_0] + N^*[v_0]$$

$$= \frac{\left(t\sinh\left(\frac{x}{4}\right) + 2\cosh\left(\frac{x}{4}\right)^3 + t\cosh\left(\frac{x}{4}\right)^2\sinh\left(\frac{x}{4}\right)\right)}{2\cosh\left(\frac{x}{4}\right)^5},$$

$$v_2 = h^*(x,t) + L^*[v_1] + N^*[v_1]$$

$$= \frac{\left[\begin{array}{c} 48\cosh\left(\frac{x}{4}\right)^9 - 5t^3\sinh\left(\frac{x}{4}\right) - 21t^2\cosh\left(\frac{x}{4}\right)^3 \\ -12t^2\cosh\left(\frac{x}{4}\right)^5 + 15t^2\cosh\left(\frac{x}{4}\right)^7 + 6t^2\cosh\left(\frac{x}{4}\right)^9 \\ -4t^3\cosh\left(\frac{x}{4}\right)^2\sinh\left(\frac{x}{4}\right) + 3t^3\cosh\left(\frac{x}{4}\right)^4\sinh\left(\frac{x}{4}\right) \\ +2t^3\cosh\left(\frac{x}{4}\right)^6\sinh\left(\frac{x}{4}\right) - 45t\cosh\left(\frac{x}{4}\right)^4\sinh\left(\frac{x}{4}\right) \\ +30t\cosh\left(\frac{x}{4}\right)^6\sinh\left(\frac{x}{4}\right) + 30t\cosh\left(\frac{x}{4}\right)^8\sinh\left(\frac{x}{4}\right)\Big) \end{array}\right]}{48\cosh\left(\frac{x}{4}\right)^{11}},$$

$$v_3 = h^*(x,t) + L^*[v_2] + N^*[v_2]$$

$$= \frac{\left[\begin{array}{c} \Big(1290240\cosh\left(\frac{x}{4}\right)^{21} - 5500t^7\sinh\left(\frac{x}{4}\right) - 46550t^6\cosh\left(\frac{x}{4}\right)^3 \\ -496125t^4\cosh\left(\frac{x}{4}\right)^7 - 13020t^6\cosh\left(\frac{x}{4}\right)^5 + 613200t^4\cosh\left(\frac{x}{4}\right)^9 \\ +85260t^6\cosh\left(\frac{x}{4}\right)^7 + 3900960t^2\cosh\left(\frac{x}{4}\right)^{13} + 668745t^4\cosh\left(\frac{x}{4}\right)^{11} \\ +21560t^6\cosh\left(\frac{x}{4}\right)^9 - 2268000t^2\cosh\left(\frac{x}{4}\right)^{15} - 662130t^4\cosh\left(\frac{x}{4}\right)^{13} \\ -46270t^6\cosh\left(\frac{x}{4}\right)^{11} - 2540160t^2\cosh\left(\frac{x}{4}\right)^{17} - 276990t^4\cosh\left(\frac{x}{4}\right)^{15} \\ -10780t^6\cosh\left(\frac{x}{4}\right)^{13} + 846720t^2\cosh\left(\frac{x}{4}\right)^{19} + 134400t^4\cosh\left(\frac{x}{4}\right)^{17} \\ +7000t^6\cosh\left(\frac{x}{4}\right)^{15} + 241920t^2\cosh\left(\frac{x}{4}\right)^{21} + 38640t^4\cosh\left(\frac{x}{4}\right)^{19} \\ +1680t^6\cosh\left(\frac{x}{4}\right)^{17} - 113400t^5\cosh\left(\frac{x}{4}\right)^4\sinh\left(\frac{x}{4}\right) - 3000t^7\cosh\left(\frac{x}{4}\right)^2\sinh\left(\frac{x}{4}\right) \\ -661500t^3\cosh\left(\frac{x}{4}\right)^8\sinh\left(\frac{x}{4}\right) + 186144t^5\cosh\left(\frac{x}{4}\right)^6\sinh\left(\frac{x}{4}\right) \\ +9720t^7\cosh\left(\frac{x}{4}\right)^4\sinh\left(\frac{x}{4}\right) + 2431800t^3\cosh\left(\frac{x}{4}\right)^{10}\sinh\left(\frac{x}{4}\right) \\ +269304t^5\cosh\left(\frac{x}{4}\right)^8\sinh\left(\frac{x}{4}\right) + 4800t^7\cosh\left(\frac{x}{4}\right)^6\sinh\left(\frac{x}{4}\right) \\ -915600t^3\cosh\left(\frac{x}{4}\right)^{12}\sinh\left(\frac{x}{4}\right) - 177408t^5\cosh\left(\frac{x}{4}\right)^{10}\sinh\left(\frac{x}{4}\right) \\ -5180t^7\cosh\left(\frac{x}{4}\right)^8\sinh\left(\frac{x}{4}\right) - 1975680t^3\cosh\left(\frac{x}{4}\right)^{14}\sinh\left(\frac{x}{4}\right) \\ -161280t^5\cosh\left(\frac{x}{4}\right)^{12}\sinh\left(\frac{x}{4}\right) - 2280t^7\cosh\left(\frac{x}{4}\right)^{10}\sinh\left(\frac{x}{4}\right) \\ +246960t^3\cosh\left(\frac{x}{4}\right)^{16}\sinh\left(\frac{x}{4}\right) + 29232t^5\cosh\left(\frac{x}{4}\right)^{14}\sinh\left(\frac{x}{4}\right) \\ +800t^7\cosh\left(\frac{x}{4}\right)^{12}\sinh\left(\frac{x}{4}\right) + 325920t^3\cosh\left(\frac{x}{4}\right)^{18}\sinh\left(\frac{x}{4}\right) \\ +25200t^5\cosh\left(\frac{x}{4}\right)^{16}\sinh\left(\frac{x}{4}\right) + 320t^7\cosh\left(\frac{x}{4}\right)^{14}\sinh\left(\frac{x}{4}\right) \\ +26880t^3\cosh\left(\frac{x}{4}\right)^{20}\sinh\left(\frac{x}{4}\right) + 2016t^5\cosh\left(\frac{x}{4}\right)^{18}\sinh\left(\frac{x}{4}\right) \\ +4233600t\cosh\left(\frac{x}{4}\right)^{14}\sinh\left(\frac{x}{4}\right) - 4233600t\cosh\left(\frac{x}{4}\right)^{16}\sinh\left(\frac{x}{4}\right) \\ +846720t\cosh\left(\frac{x}{4}\right)^{18}\sinh\left(\frac{x}{4}\right) + 846720t\cosh\left(\frac{x}{4}\right)^{20}\sinh\left(\frac{x}{4}\right)\Big) \end{array}\right]}{1290240\cosh\left(\frac{x}{4}\right)^{23}},$$

$$\vdots$$

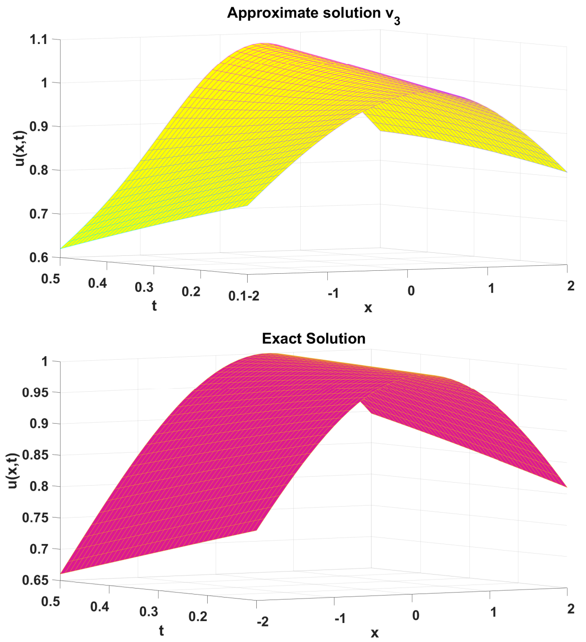

Fig. 6. Approximate solution v_3 and the exact solution $sech^2\left(\frac{x}{4}-\frac{t}{3}\right)$ of the BBM equation (14).

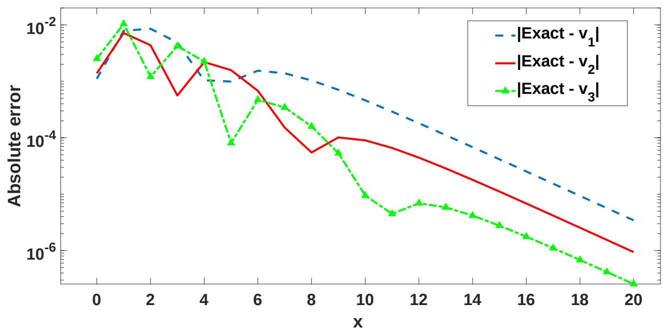

Fig. 7. Absolute error of solutions obtained at time $t = 0.1$ of the BBM equation (14).

The estimated solution and the exact solution of (14) are plotted in figure 6 and the respective absolute error corresponding to each iterated values are pictured in figure 7.

C. System of Nonlinear Differential Equations

In this section, we use the iterative method of approach to solve the system of nonlinear ordinary as well as partial differential equations.

Example 6. *Consider the system of ordinary differential equations discussed in [3]*

$$\begin{aligned} y_1'''(t) &= y_2(t), \\ y_2'(t) &= -y_2(t) - y_1^2(t) + t^4. \end{aligned} \tag{15}$$

At initial time, the function and its derivatives has following values:
$y_1(0) = 0, y_1'(0) = 0, y_1''(0) = 2, y_2(0) = 0.$

Now, let us find the solution of this system (15) using our iterative method (7).
Take $L_1(.)$ as the differential operator $\frac{d^3}{dt^3}$ and $L_2(.)$ as the differential operator $\frac{d}{dt}$. Then by applying the corresponding inverse operators with respect to t on both sides of the system (15) and using the initial values of functions, we get,

$$\begin{aligned} y_1(t) &= h_1^*(t) + L_1^*[y_1, y_2] \\ &= t^2 + \iiint_0^t y_2(t)\ dtdtdt, \\ y_2(t) &= h_2^*(t) + L_2^*[y_1, y_2] + N_2^*[y_1, y_2] \\ &= \int_0^t \left(t^4 - y_2(t)\right)\ dt - \int_0^t y_1^2(t)\ dt. \end{aligned}$$

Let $v_{ik} = \sum_{r=0}^{k} y_{ir}(t), 1 \le i \le 2$ denotes the k^{th} term approximation to the solution y_i, then by using (7), we get the following series of solution:

$$\left.\begin{aligned} v_{10} &= y_{10} = h_1^*(t) &= t^2 \\ v_{20} &= y_{20} = h_2^*(t) &= 0 \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} v_{11} &= y_{10} + y_{11} \\ &= h_1^* + L_1^*[v_{10}, v_{20}] \\ &= t^2 \\ v_{21} &= y_{20} + y_{21} \\ &= h_2^* + L_2^*[v_{10}, v_{20}] + N_2^*[v_{10}, v_{20}] \\ &= 0 \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} v_{12} &= y_{10} + y_{11} + y_{12} &= t^2 \\ v_{22} &= y_{20} + y_{21} + y_{22} &= 0 \end{aligned}\right\} \tag{18}$$

Thus $\forall k$,

$$\left.\begin{aligned} v_{1k} &= \textstyle\sum_{r=0}^{k} y_{1r}(t) &= t^2 \\ v_{2k} &= \textstyle\sum_{r=0}^{k} y_{1r}(t) &= 0 \end{aligned}\right\} \tag{19}$$

Hence, the solution of the given system of equations (15) is $(y_1(t), y_2(t)) = (t^2, 0)$.

Example 7. *Let us take the nonlinear system of coupled partial differential equations dealt by Bissanga et al. in [27]*

$$\begin{aligned} \frac{\partial P(x,y,t)}{\partial t} - \frac{\partial Q(x,y,t)}{\partial x}.\frac{\partial R(x,y,t)}{\partial y} &= 1, \\ \frac{\partial Q(x,y,t)}{\partial t} - \frac{\partial R(x,y,t)}{\partial x}.\frac{\partial P(x,y,t)}{\partial y} &= 5, \\ \frac{\partial R(x,y,t)}{\partial t} - \frac{\partial P(x,y,t)}{\partial x}.\frac{\partial Q(x,y,t)}{\partial y} &= 5, \end{aligned} \tag{20}$$

with initial conditions,

$$\begin{aligned} P(x,y,0) &= x + 2y, \\ Q(x,y,0) &= x - 2y, \\ R(x,y,0) &= -x + 2y. \end{aligned}$$

Let the operator $L(.)$ denote the partial derivative on t of order 1. Then, applying the corresponding inverse operator $\int_0^t dt$ to the system (20) along with the initial conditions we get,

$$\begin{aligned} P(x,y,t) &= x + 2y + t + \int_0^t \frac{\partial Q(x,y,t)}{\partial x}.\frac{\partial R(x,y,t)}{\partial y}\ dt, \\ Q(x,y,t) &= x - 2y + 5t + \int_0^t \frac{\partial R(x,y,t)}{\partial x}.\frac{\partial P(x,y,t)}{\partial y}\ dt, \\ R(x,y,t) &= -x + 2y + 5t + \int_0^t \frac{\partial P(x,y,t)}{\partial x}.\frac{\partial Q(x,y,t)}{\partial y}\ dt. \end{aligned}$$

Let the nonlinear operators be

$$N_1^*[P,Q,R]=\int_0^t \frac{\partial Q(x,y,t)}{\partial x}.\frac{\partial R(x,y,t)}{\partial y}\,dt,$$
$$N_2^*[P,Q,R]=\int_0^t \frac{\partial R(x,y,t)}{\partial x}.\frac{\partial P(x,y,t)}{\partial y}\,dt,$$
$$N_3^*[P,Q,R]=\int_0^t \frac{\partial P(x,y,t)}{\partial x}.\frac{\partial Q(x,y,t)}{\partial y}\,dt.$$

Implementing the iterative procedure (7) on the functions (P,Q,R), we obtain the following set of solutions:

$$\left.\begin{array}{lllll} v_{10} &=& P_0=h_1^*(x,y,t) &=& x+2y+t\\ v_{20} &=& Q_0=h_2^*(x,y,t) &=& x-2y+5t\\ v_{30} &=& R_0=h_3^*(x,y,t) &=& -x+2y+5t \end{array}\right\} \quad (21)$$

$$\left.\begin{array}{lll} v_{11} &=& P_0+P_1=h_1^*(x,y,t)+N_1^*[v_{10,20,30}]\\ &=& x+2y+3t\\ v_{21} &=& Q_0+Q_1=h_2^*(x,y,t)+N_2^*[v_{10,20,30}]\\ &=& x-2y+3t\\ v_{31} &=& R_0+R_1=h_3^*(x,y,t)+N_3^*[v_{10,20,30}]\\ &=& -x+2y+3t \end{array}\right\} \quad (22)$$

$$\left.\begin{array}{lll} v_{12} &=& P_0+P_1+P_2\\ &=& h_1^*(x,y,t)+N_1^*[v_{11,21,31}]\\ &=& x+2y+3t\\ v_{22} &=& Q_0+Q_1+Q_2\\ &=& h_2^*(x,y,t)+N_2^*[v_{11,21,31}]\\ &=& x-2y+3t\\ v_{32} &=& R_0+R_1+R_2\\ &=& h_3^*(x,y,t)+N_3^*[v_{11,21,31}]\\ &=& -x+2y+3t \end{array}\right\} \quad (23)$$

We see that we reach the exact solution of the system (20) within two iteration steps. Hence, the solution of the system is $(P,Q,R)=(x+2y+3t,x-2y+3t,-x+2y+3t)$.

Example 8. *Consider the nonlinear system of partial differential equations*

$$\frac{\partial u(x,t)}{\partial t}+2w(x,t)\frac{\partial u(x,t)}{\partial x}-u(x,t)=2,$$
$$\frac{\partial w(x,t)}{\partial t}-3u(x,t)\frac{\partial w(x,t)}{\partial x}+w(x,t)=3, \quad (24)$$

with initial conditions,

$$u(x,0)=e^x,$$
$$w(x,0)=e^{-x}.$$

Let the operator $L(.)$ denote the partial derivative with respect to t of order 1. Then, by applying the corresponding inverse operator $\int_0^t dt$ on to the system (24) along with initial conditions, we get,

$$u(x,t)=e^x+\int_0^t\left[2+u(x,t)-2w(x,t)\frac{\partial u(x,t)}{\partial x}\right]dt,$$
$$w(x,t)=e^{-x}+\int_0^t\left[3+3u(x,t)\frac{\partial w(x,t)}{\partial x}-w(x,t)\right]dt.$$

Using iterative scheme (7), we obtain the following results:

$$\left.\begin{array}{lllll} v_{10} &=& u_0=h_1^*(x,t) &=& e^x\\ v_{20} &=& w_0=h_2^*(x,t) &=& e^{-x} \end{array}\right\} \quad (25)$$

$$\left.\begin{array}{lllll} v_{11} &=& u_0+u_1 &=& e^x(1+t)\\ v_{21} &=& w_0+w_1 &=& e^{-x}(1-t) \end{array}\right\} \quad (26)$$

$$\left.\begin{array}{lll} v_{12} &=& u_0+u_1+u_2\\ &=& e^x\left(1+t+\frac{t^2}{2}\right)+\frac{2t^3}{3}\\ v_{22} &=& w_0+w_1+w_2\\ &=& e^{-x}\left(1-t+\frac{t^2}{2}\right)+t^3 \end{array}\right\} \quad (27)$$

$$\left.\begin{array}{lll} v_{13} &=& u_0+u_1+u_2+u_3\\ &=& e^x\left(1+t+\frac{t^2}{2}+\frac{t^3}{6}-\frac{t^4}{2}-\frac{2t^5}{5}\right)\\ && +\frac{t^4}{6}-\frac{t^5}{10}\\ v_{23} &=& w_0+w_1+w_2+w_3\\ &=& e^{-x}\left(1-t+\frac{t^2}{2}-\frac{t^3}{6}-\frac{t^4}{2}+\frac{2t^5}{5}\right)\\ && -\frac{t^4}{6}-\frac{3t^5}{20} \end{array}\right\} \quad (28)$$

Proceeding the iteration, we get the series expansion that resembles the Taylor series expansion of the function $u(x,t)=e^{x+t}, w(x,t)=e^{-(x+t)}$, which is the exact solution of the system given in (24).

IV. Discussion of Results

The proposed iterative scheme has been successfully applied to ordinary and partial differential equations with nonlinearities. The method was also used to solve the system of nonlinear differential equations. The equations are chosen based on the fact that the exact solution is already known, as it can be easily compared with the results obtained to analyze the accuracy. The following observations are captured:

The graphical simulation of the exact and approximate solution surfaces depicted in figures 3, 4 and 6 has a similar structure, which indicates the compatibility of the suggested technique with almost all types of non-linearities. Furthermore, we observe from the figures 1 and 5, we see that the two solution curves overlap each other, which shows that the results obtained with our method are in good agreement with the exact solution. For the simple system of differential equations, we observe that the solution converges directly to the exact solution within a few iteration steps. Moreover, in problem 2, we implement the suggested technique for solving the nonlinear differential equation with singularity. The numerical values obtained are listed in table I against the values obtained for the same problem using ADM, along with the respective absolute error. We have also visualized these values in figure 2. From the above illustrations, we observe that an increase in the number of iterations generally results in an improvement in the accuracy of the stated iterative algorithm outputs. However, this is valid only if the values of independent variables lie within the method's convergence region. The iterated solution does not converge to the computational results for values of the mentioned variables above the radius of convergence.

Although these equations are processed by many researchers using various analytical, approximate and numerical methods, our scheme of iteration is computationally simple and versatile, as it can be applied to any kind of function equation. Since the method is adaptable to any mathematical software, computation becomes easy and the required solution can be obtained within optimal execution time.

V. Conclusion

Once the basic concepts behind the method's general theory have been established, the algorithm is employed in order to solve specific nonlinear differential equations and systems of equations symbolically using the function "$N^*(z^*)$" and their results are presented. Graphical simulations illustrate the validity and potential of the method, and by comparing the results of problem discussed in Section III with the exact solution, it is evident that the current method is simply effective and reliable in finding the solution to NDEs. Moreover, it is worth mentioning that the obtained results are compatible with the exact solution.

APPENDIX

MATLAB code for approximate solutions to a few NDEs discussed in Section III is provided in this section:
Here are some terms used in MATLAB program for calculations:

```
Input:
    x,t := Independent variables
    N(Input arguments):= nonlinear
    function
    h:=initial condition
    y(x),u(x,t),p(x,y,t):= solution
    functions
    i:= number of iterative steps
    required.
Output:
    Series solutions of the given
    nonlinear differential equations.
```

1. MATLAB program for example 2:
Function:

```
function [y] = lnem(y,ey,ey2)
syms x
y=int(x^-2*int(-x^2*(8*ey+4*ey2),x,0,x),x,0,x);
end
```

Command:

```
syms x k;
h=0;
y=h;
ey=0;
ey2=0;
for i=1:5
ey=symsum(y^k/factorial(k),k,0,5);
ey2= symsum((y/2)^k/factorial(k),k,0,5);
y=h+lnem(y,ey,ey2);
disp(y)
end
```

2. MATLAB program for example 3:
Function:

```
function [u] = kg(u)
syms x t ux uxx
ux=diff(u,x);
uxx=diff(ux,x);
u=int(int(uxx-ux^2+u^2,t,0,t),t,0,t);
end
```

Command:

```
syms t x;
h=t*exp(x);
u=h;
for i=1:5
u=h+kg(u);
fprintf('u(%d)=',i)
disp(expand(u))
end
```

3. MATLAB program for example 7:
Function:

```
function [p1,q1,r1] = s2(p,q,r)
syms t y x px qx rx py qy ry;
px=diff(p,x);
qx=diff(q,x);
rx=diff(r,x);
py=diff(p,y);
qy=diff(q,y);
ry=diff(r,y);
p1=int(ry*qx,t,0,t);
q1=int(rx*py,t,0,t);
r1=int(px*qy,t,0,t);
end
```

Command:

```
syms t x y;
h1=x+2*y+t;
h2=x-2*y+5*t;
h3=-x+2*y+5*t;
p=h1;
q=h2;
r=h3;
for i=1:2
[p1,q1,r1]=s2(p,q,r);
p=simplify(h1+p1);
q=simplify(h2+q1);
r=simplify(h3+r1);
fprintf('p(%d)=',i)
disp(p)
fprintf('q(%d)=',i)
disp(q)
fprintf('r(%d)=',i)
disp(r)
end
```

4. MATLAB program for example 8:
Function:

```
function [u1,w1] = s1(u,w)
syms t x ux wx;
ux=diff(u,x);
wx=diff(w,x);
u1=int(2-2*ux*w+u,t,0,t);
w1=int(3+3*wx*u-w,t,0,t);
end
```

Command:

```
syms t x;
h1=exp(x);
h2=exp(-x);
u=h1;
w=h2;
for i=1:4
[u1,w1]= s1(u,w);
u=expand(h1+u1);
w=expand(h2+w1);
fprintf('u(%d)=',i)
disp(u)
fprintf('w(%d)=',i)
disp(w)
end
```

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