Surveying the Current State of Uncertain Optimization Models and Methodologies

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Abstract—**Uncertain parameters are pervasive across various domains, necessitating sophisticated approaches for their effective management within optimization problems. This review extensively examines state-of-the-art models and algorithms to tackle uncertain optimization challenges. We delve into a broad spectrum of contemporary research hotspots, including stochastic programming, fuzzy optimization, interval optimization, and polymorphic uncertain optimization. The paper meticulously analyzes essential solution methods such as the expectation method, chance-constrained optimization, sample average approximation, robust optimization, distributionally robust optimization, data-driven method, α-cut set method, and fuzzy expectation method. Each approach is scrutinized for its applicability, strengths, and limitations, offering a nuanced perspective on its practical value. By presenting these insights, we establish a robust theoretical foundation and provide valuable guidance for future research in uncertain optimization. This research highlights significant advances and identifies promising future research directions, making a substantial contribution to the evolving optimization field under uncertainty.**

Index Terms—**Stochastic programming, robust optimization, distributionally robust optimization, interval optimization, fuzzy optimization, data-driven method**

I. Introduction

In intricate environments, accurately estimating parameters, and predicting the probability distribution pose significant challenges due to the dynamic evolution of circumstances, information asymmetry, and external interferences. Uncertainty is an objective and inherent phenomenon. Product demand in supply chain problems, taxi driver earnings per unit time in transportation management, stock prices in financial markets, machine failure randomness and workshop scheduling in industry, wind and photovoltaic power generation in renewable energy systems, and aquifer and hydraulic properties in environmental problems are all examples of uncertain variables. Randomness, imprecision, or ambiguity can all induce uncertainty. Ignoring uncertainty in decision-making processes may result in suboptimal solutions [1]. This has significantly propelled research on complex optimization models and methodologies designed to address uncertain scenarios, stimulating scholars to analyze and solve such uncertain optimization problems with a rigorous scientific approach. Numerous uncertain optimization models have been discovered and applied to solve problems in various domains such as supply chain, investment portfolio management, transportation, aerospace engineering, machine learning, energy environment, and bioengineering. Therefore, advancing research in uncertain optimization is of paramount importance. The mathematical representation of an optimization model incorporating uncertain parameters is presented as follows:

$$\min_{x\in X} \varphi(x;\xi). \tag{1}$$

Given the inherent uncertainty of the underlying parameters, the optimization model's objective function may pose complex challenges. This complexity may stem from non-smoothness, nonlinearity, non-differentiability, non-convexity, and noise, presenting significant hurdles to the optimization process. Establishing effective optimization models and finding stable and reliable solutions is at the forefront of academic challenges. Traditional deterministic models are insufficient for addressing these issues. Even in the case of smooth nonlinear optimization problems, where the unknown decision variables of the model are related to complex multi-dimensional integrals, solving the objective function remains extremely challenging. For non-smooth optimization problems, conventional optimization techniques that rely on gradients and other standards are rendered inapplicable due to the absence of derivative information. Moreover, existing software packages such as MATLAB, CPLEX, and Lingo are not directly suited to solving these intricate models. All these factors contribute to the complexity of research.

Despite the multitude of methods developed to tackle complex optimization problems with uncertainty, the efficiency of their performance remains suboptimal due to the inherent limitations of these strategies. Stochastic programming methods heavily depend on the cumulative distribution function (CDF) or probability density function (PDF), which satisfy and require sufficient independent measurement sample data. Fuzzy optimization methods suffer from a high degree of subjectivity in their representation of uncertainty. Traditional heuristic algorithms are plagued by significant computational costs and present challenges in verifying convergence and avoiding local optima dilemmas. These challenges underscore the ongoing difficulties in optimizing performance in uncertainty.

A thorough overview of uncertain optimization models and their feasible methods is presented in this paper. Initially,

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various uncertain optimization models such as stochastic programming, fuzzy optimization, interval optimization, polymorphic uncertain optimization, and their corresponding solution methods are systematically summarized and categorized. These models and methods can effectively solve the problems of uncertain data input and uncertain parameters in objective functions or constraints. Subsequently, these models and methods are analyzed and compared in detail to provide valuable insights for researchers in related fields.

The subsequent sections of the paper are structured as follows. The next section focuses on the recent research hotspots in the stochastic programming model and multiple methodologies for handling known and unknown random parameter distributions. In Section III, the fuzzy optimization model and methodologies are reviewed. Section IV presents the interval optimization model. Section V is devoted to the development of polymorphic uncertain optimization. Finally, followed by a summary and comparative analysis, we highlight potential future research directions and draw conclusions in Section VI. Refer to Fig. 1 for the research framework diagram.

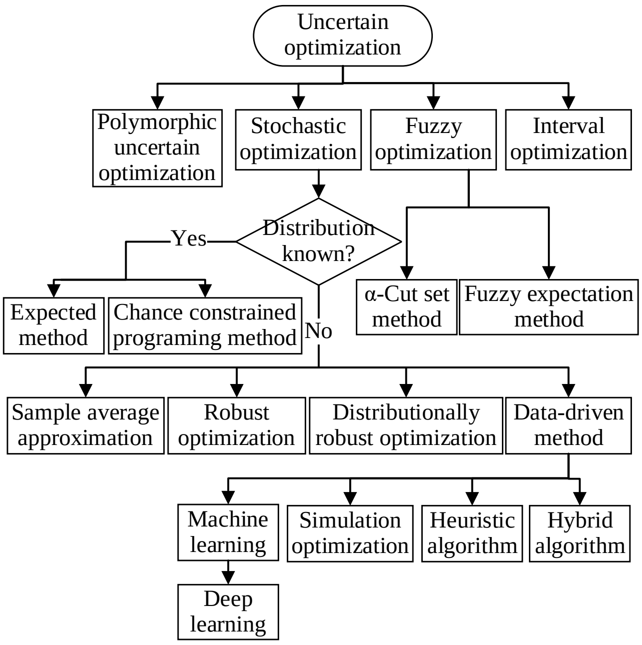


Fig. 1. Research framework.

II. Stochastic Programming

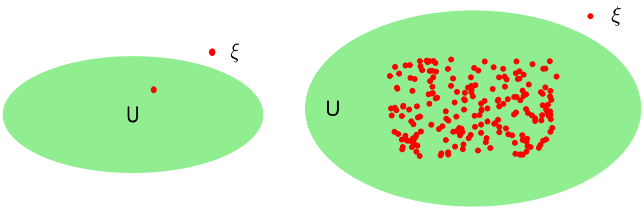
Stochastic programming (SP) is the most prevalent method for addressing optimization problems under uncertainty. Stochastic programming models involve optimization with random parameters and are approached using probability statistics theory, stochastic processes, and stochastic analysis. Advances in solving stochastic programming problems have been substantial, establishing SP as one of the most effective methods for decision-making in uncertain environments.

The mathematical form of a stochastic programming model with the random parameter ξ is expressed as follows:

$$\min_{x \in X} \varphi(x;\xi), \tag{2}$$

where R^n denotes the n-dimensional Euclidean space. $x=(x_1,\cdots,x_n)\in X$ represents the n-dimensional decision variable. Given a real-valued function $\varphi := X\times\Omega \to R$, the goal is to find the global minimum solution x^* that satisfies $\varphi(x^*)\le\varphi(x), \forall x\in R^n$.

In stochastic programming, when the support set of the random parameter is U, deterministic optimization can be regarded as a case where the uncertainty set contains only one specific element [2], as illustrated in Fig. 2.



(a) Deterministic Optimization (b) Stochastic Programming

Fig. 2. The distinction and interconnection between deterministic optimization and stochastic programming.

For stochastic programming models, an initial step involves analyzing the distributional characteristics of the random parameters before making optimization decisions. Uncertainty can be classified into two distinct categories: pure risk and strict uncertainty. Pure risk occurs when the probabilities of all potential states of the random parameters are fully known. Conversely, strict uncertainty occurs when the probability distribution is not uniquely determinable. Classical stochastic programming models typically operate under the assumption that the probability distribution of the uncertain parameters is known.

Research in stochastic programming mainly differs in the assumptions regarding the exact form of the probability distribution. Currently, the methodologies for solving stochastic programming encompass several broad approaches. These include the expectation method, chance-constrained programming, sample average approximation method, and distributionally robust optimization. Emerging techniques such as machine learning, heuristic algorithms, simulation optimization, and hybrid algorithms are also being explored.

A. *The known probability distribution*

1. Expectation method

If the probability distribution of uncertain parameters is known, statistical techniques can handle the randomness inherent in the objective function or constraint conditions [3]. Among these techniques, the expectation method (EM) stands as a prevalent approach for transforming stochastic programming (2) into an equivalent deterministic optimization problem:

$$\min_{x\in X} E_P\left[\varphi(x;\xi)\right]. \tag{3}$$

The problem (3) represents a large class of stochastic programming problems predicated on the following critical assumptions [4].

Hypothesis 1: The distribution of random parameters is assumed to be known, although this may not hold in most cases.

Hypothesis 2: The objective function and constraints must be known or observable. This implicit assumption is essential for constructing an optimization model.

Hypothesis 3: Any function that depends on random parameters should be represented through a probability distribution. Risk measures, such as expected value, probability, or conditional value-at-risk (CVaR), can facilitate reformulating the objective function. Select the probability distribution based on the decision maker's risk preference. For instance, conventional expectation methods are considered risk-neutral, whereas objective functions incorporating conditional value-at-risk reflect a risk-averse optimization approach.

In recent years, numerous novel expectation methods have been developed to solve uncertain optimization problems. Wan et al. [5] have introduced a new variance expectation synthesis approach to tackle the multi-objective constrained stochastic programming model. The deterministic equivalence class of the optimization problem is defined, and an interactive method grounded in the decision-maker's preferences is proposed. Deng et al. [6] further transformed the stochastic programming model into a deterministic mathematical model with complementarity constraints (MPCC). By employing partial smoothing technology [7], MPCC was transformed into a sequence of standard smoothing optimization subproblems for solution, and a gradient-based algorithm was proposed to solve the original model. Deng's subsequent research [8] proposed a novel stochastic programming decision model based on the random net income of urban taxi services. The newly established expected profit maximization and risk minimization models were further verified to be risk-neutral through the expectation method. Wang employs the expected value method to tackle a complex stochastic multi-item lot-sizing optimization problem pertinent to make-to-order manufacturing, with the ultimate aim of wealth maximization [9]. Akande presents a comparative analysis and comprehensive overview of stochastic and deterministic models employed in solving scheduling problems [10].

In the problem (3), when the random parameter is a multi-dimensional random vector, the random parameter is referred to as a joint probability density. However, Faes et al. [11] assert that constructing the necessary prior estimate of the joint probability density with uncertain parameter values is subjective and requires strict statistical uniformity assumption of the random field.

2. Chance-constrained programming method

Another effective and convenient method for handling optimization problems characterized by uncertain parameters is the chance-constrained programming method (CCPM). This method is designed to manage risk in decision-making scenarios under uncertainty. The origins of chance-constrained programming can be traced back to the 1950s when Charnes et al. first proposed [12]. Over the decades, it has evolved into an essential tool in diverse decision-making environments.

Despite its utility, CCPM is not without limitations. In practice, the probability distributions of random parameters are often unknown, which can introduce biases into the solutions obtained by this method. CCPM allows the decision to ensure that the probability or possibility of satisfying the constraints is not less than a predefined confidence level. The optimal solution satisfies the uncertain constraints with at least a certain probability β. For instance, decision-makers in the financial industry may want to ensure that their investment portfolio achieves a target return with at least a certain probability β.

Typically, the problem (2) can be reformulated into a typical chance-constrained problem as follows:

$$\begin{aligned} &\min_{x\in X}\ \bar{\varphi}(x;\xi) \\ &s.t.\ \begin{cases} P\left(\varphi(x;\xi)\le\bar{\varphi}\right)\ge\beta, i\in I, \\ P\left(g_i(x;\xi)\le 0, i\in I\right)\ge\alpha, \end{cases} \end{aligned} \tag{4}$$

where β and α represent the pre-specified confidence levels for the objective function and constraint conditions, respectively. Since its introduction, the chance-constrained programming method has developed rapidly and has been effectively applied to solve numerous practical problems. However, several significant challenges persist:

1) Prior knowledge requirement: CCPM necessitates prior knowledge of the probability density and the inverse of the random variables involved.

2) Non-convexity of Chance Constraints: The chance constraints frequently exhibit nonconvex characteristics, complicating the optimization process.

3) Complexity of calculation: Calculating the probability associated with chance constraints is challenging due to the involvement of complex, high-dimensional integral problems.

Several approaches have been developed to address these challenges. Charnes et al. [13] and Calafiore et al. [14] introduced a nonlinear but convex chance-constrained scheme for exceptional cases to address the first challenge. Nemirovski et al. [15] and Chen et al. [16] provided conservative convex approximations. Approximation methods such as stochastic simulation are often adopted in engineering to address the second challenge. For the third challenge, Nemirovski et al. [17] and Luedtke et al. [18] proposed scenario approximation methods, simplifying the computational process and ensuring that the optimal solution satisfies the chance constraints with high probability. Notably, Luedtke et al. [19]-[20] have successfully addressed chance-constrained integer optimization problems.

The limitations inherent in the optimization methods for random parameters with known probability distributions are manifested in several key aspects:

1) Unknown probability distributions: If the probability distribution of the random parameters is unknown, a large amount of data or statistical moment estimation is required to fit the distribution that satisfies [11]. This process can be data-intensive and computationally demanding.

2) Inference of distribution type: In scenarios where distribution types are inferred based on sample experiments, data from analogous events, or accumulated experience, there is inherent subjectivity in determining the joint probability density of uncertain parameters. This inference relies heavily on prior estimation and requires a strict assumption of statistical uniformity of the random field.

3) Quantifying all possible joint possibilities: Accurately and objectively quantifying all possible joint possibilities of parameter values necessitates sufficient independent measurement data. In cases where such data is lacking, this fitting will significantly affect the reliability and accuracy of the quantification results [21].

B. The unknown probability distribution

Assuming a known probability distribution for uncertain parameters is unrealistic in practical scenarios. Firstly, the precise probability distribution of uncertain parameters is frequently unknown. Secondly, even if an approximate probability distribution is used, it may not be reliable. When the probability distribution of uncertain parameters is unknown, accurately calculating the expectation in problem (3) becomes challenging. The amount of data or statistical moment estimation is required to fit the probability distribution it satisfies. In practice, parametric and nonparametric methods can be used. This situation necessitates substantial data or statistical moment estimation to fit an appropriate probability distribution.

In practice, two main approaches can be employed: parametric and nonparametric methods. The parametric approach presupposes that the actual distribution belongs to a specific family of parametric distributions. Data can also be utilized to infer the prior distribution of unknown parameters or parameters [22]. In contrast, nonparametric methods do not need to know a predefined form for the distribution. Typically, specific probability distributions are obtained by storing, observing, and analyzing historical data, collecting expert opinions, and calculating simulation results.

1. Sample average approximation

Historical data can be treated as samples extracted from the underlying probability distribution. Akcay et al. [23] fitted the samples to a probability distribution within a parameter family containing many common distributions. However, distribution fitting methods can lead to substantial errors. The sample average approximation method (SAA) often offers superior accuracy compared to traditional distribution fitting approaches. The standard SAA methodology is to consider a stochastic approximation family of functions $\{\varphi_N(\cdot)\}$, each defined as $\varphi_N(\cdot) := \frac{1}{N}\sum_{j=1}^{N}\varphi(x;\xi^j)$. $\xi^1, \cdots, \xi^N$ are samples drawn from the distribution of the random variable ξ. Given a class of estimators $\{\varphi_N(\cdot)\}$, an approximation problem (3) can be constructed:

$$\min_{x\in X} \varphi_N(x;\xi) = \frac{1}{N}\sum_{j=1}^{N}\varphi(x;\xi^j). \tag{5}$$

SAA methods are frequently employed to address two distinct categories of stochastic programming problems:

The first category of problems remains computationally challenging even when the distribution is known, such as two-stage discrete problems and those involving complex utility function expectations. The SAA method approximates the complex, known objective function through sampling and solves the corresponding equivalent problem. For two-stage stochastic integer optimization, Shapiro et al. [24] have proven that the optimal solution of the SAA method converges to the actual optimal value with a probability of 1. The accuracy of the SAA method is contingent upon the variability of the objective function and the size of the feasible domain, and they hold the view that the SAA method is overly conservative.

The second category involves problems where the distribution is known, and the objective function is straightforward to estimate, such as the newsvendor problem. The newsvendor problem is one of the most fundamental and widely utilized inventory models. Retsef and Levi et al. [25]-[26] analyzed the application of the SAA method to data-driven supplier newsvendor problems and assessed the accuracy of the SAA method in solving such problems.

Shapiro [27] has shown that if the feasible set is tight and the loss function is uniformly continuous on the feasible set, the optimal value and solution of the SAA problem converge almost surely to the actual problem. When the loss function satisfies the Lipschitz continuous condition, the SAA method can guarantee good performance under limited samples [28]. However, in cases where the sample size is small and acquiring additional samples is costly, the optimal solution of the SAA method often exhibits poor out-of-sample performance.

2. Robust optimization

An alternative method to the chance-constrained programming method is robust optimization (RO). RO operates under the assumption that the exact distribution is unknown. However, the known probability distribution belongs to a specified uncertainty set, which may be characterized by a specific structure, such as an ellipsoid or polytope. Refer to Fig. 3 for details.

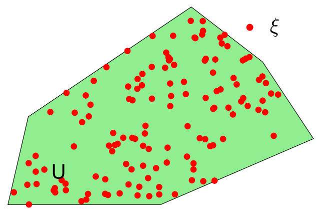


Fig. 3. The uncertainty set of RO.

Robust optimization is performed for the worst-case scenario within the uncertainty set, inevitably leading to overly conservative and suboptimal decisions. The formal RO methodology is defined as follows:

$$\min_{x\in X}\sup_{\xi\in U}\varphi(x;\xi), \tag{6}$$

where U denotes the support set of random parameters. Robust optimization provides several key advantages when tackling problems with uncertainty, noise, or interference.

1) Self-adaptation capability. Robust optimization is inherently adaptable, allowing real-time adjustments based on current environmental conditions and input data, which maintain excellent performance.

2) Fault tolerance. Robust optimization seeks the optimal solution to worst-case scenarios, including all conceivable situations. As a result, robust optimization possesses a measure of fault tolerance and can continue to provide superior decisions even when faced with unexpected failures

or inconsistencies.

3) Diversity. Robust optimization considers all potential strategies when seeking optimal decisions to increase the adaptability of decisions to parameter changes.

4) Global performance optimization. Robust optimization focuses on the superiority of maintaining global performance under various conditions.

RO is a suitable approach in the following scenarios.

1) The random parameters in the optimization problem need to be estimated accurately. However, estimations inherently carry risk.

2) Any possible realization of the uncertain parameters in the optimization model must satisfy the constraint functions.

3) The objective function or the optimal solution is highly sensitive to perturbations in the optimization model parameters. RO offers a robust solution. By focusing on worst-case scenarios, RO mitigates the impact of such sensitivities and stabilizes the decision-making process.

4) The decision maker cannot bear the enormous risks of low-probability events.

Stochastic programming and robust optimization are two fundamentally different methods. Despite their successes, each method also has distinct limitations:

1) Utilization of data and conservatism: Apart from calibrating the uncertainty set, the RO method does not fully utilize the rich data available to the decision maker [29]. If RO ignores valuable probability information, its solution may be overly conservative. Conversely, modelling for stochastic programming requires too much information about the probability distribution, which may be unavailable. If the distribution used in the model is inaccurate, the stochastic programming problem may produce suboptimal solutions.

2) Performance in out-of-sample: Solutions derived from robust or stochastic programming models may perform poorly in out-of-sample tests, or simply increasing the sample size may not eliminate inherent biases.

3) Computational complexity: The RO may be difficult to compute, mainly when dealing with large-scale data. Similarly, stochastic programming often involves complex, high-dimensional integrals that are generally challenging to solve.

3. Distributionally robust optimization

To consider the distribution of random parameters more accurately, the distributionally robust optimization (DRO) proposed by Wiesemann et al. [30] and Delage et al. [31] has extensively promoted the development of stochastic programming techniques. As an essential extension of robust and stochastic programming, DRO provides a powerful modelling framework that addresses the above limitations of robust optimization and stochastic programming. It is not limited to the true probability distribution that may be difficult to obtain in applications and allows for some ambiguity in the probability distribution. For instance, rather than presuming a specific Gaussian distribution with fixed mean and variance, DRO considers all Gaussian distributions with the same mean but varying variances or even all distributions with a mean close to 0 and variance close to 1. It forms an ambiguity set of distributions. All potential probability distributions that can be considered are elements of the set. The significant distinction between distributionally robust optimization and stochastic programming lies in their respective sets. In DRO, the ambiguity set consists of a collection of distributions, reflecting the inherent uncertainty about the exact probability distribution. In contrast, stochastic programming relies on an uncertainty set. This difference is illustrated in Fig. 4.

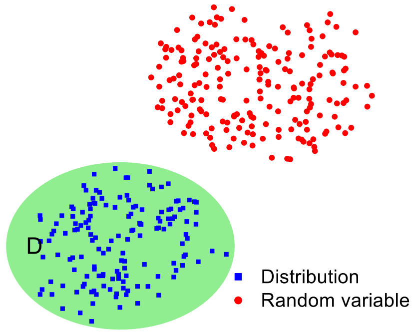


Fig. 4. The ambiguity set of DRO.

Distributionally robust optimization and stochastic programming adopt distinct methodologies and strategies in the optimization process. A stochastic programming model with a known probability distribution can be viewed as a particular case where the ambiguity set of distributions comprises only a single element Q_0 . Choosing the probability distribution becomes crucial when the ambiguity set of distributions $\mathcal{P}$ contains multiple elements. The general approach addresses the worst-case scenario across all distributions within the ambiguity set, which minimizes the average cost of the loss function. The DRO formulation of the distributionally robust stochastic programming problem (2) is:

$$\min_{x\in X} \sup_{f\in\mathcal{P}} E_f\left[\varphi(x;\xi)\right]. \tag{7}$$

This technique considers the ambiguity set of distributions that encompasses all possible probability distributions for the parameters rather than relying on a predefined probability distribution. DDRO hedges against the uncertainty of the probability distribution by leveraging the ambiguity distribution set. The worst distribution in the ambiguity set is optimized, corresponding to minimizing the expected cost in the worst-case scenario. The DRO method is proficient in effectively addressing both parameter uncertainty and uncertainty related to the probability distribution.

The significant challenges associated with distributionally robust optimization algorithms lie in constructing the ambiguity set of the distribution and determining the optimal solution. Although these two problems involve distinct techniques, they can be addressed concurrently. The structure of the ambiguity set heavily influences the tractability of DRO solutions. The optimal solution can often be derived using the Lagrange dual method. A DRO that lacks computing tractability- encompassing both practical solvability and approximability- is essentially useless.

The ambiguity set of distributions is the most crucial element of any distributionally robust optimization model. To ensure robustness and efficacy, an ambiguity set must meet several criteria. Firstly, a good ambiguity set of distributions should encompass the true probability distribution with high confidence. Secondly, the ambiguity set of distributions should be narrow enough to exclude pathological distributions that could lead to overly conservative decisions. Finally, the ambiguity set of distributions should be easily parameterized from available data. Ideally, this set should facilitate reforming the distributionally robust optimization problem into a structured mathematical programming problem that can be solved using standard optimization software.

Broadly, the ambiguity set of distributions can be divided into those based on moment information and those grounded in statistical distance.

1) Ambiguity set based on moment information

Ambiguity sets defined by moments encompass a family of distributions whose moments satisfy certain conditions. The earliest model of this concept can be traced back to Scarf's stochastic programming model proposed in 1958 for the newsvendor problem. Scarf investigated robust ordering quantities within an ambiguity set of distributions [32]. Dupacova et al. subsequently validated this pioneering model [33] in 1960. The distribution robust newsvendor model with moment constraints remains tractable. Saghafian et al. combined empirical data with moment information to study the newsvendor problem. Empirical moment information often defines a distribution ambiguity set using a moment information-based approach [34]. For example, the mean and covariance can be fixed to specific values, or all distributions satisfying these moment conditions can define a set of distributions $\mathcal{P}=\left\{f\left(\xi\right):E\left[\xi\right]\leq\mu,E\left[\xi\xi^{T}\right]\leq\Sigma,\ldots\right\}$. For more complex families of distributions, higher-order moments information may be used. Complex ambiguity sets based on cross-moment information were proposed by Delage E et al. [31] as follows:

$$\mathcal{P}\left(\Omega,\mu_0,\Sigma_0,r_1,r_2\right) =\left\{f\in M\left|\begin{array}{c}\mathbb{P}\left(\xi\in\Omega\right)=1\\ \left(E\left[\xi\right]-\mu_0\right)^T\Sigma_0^{-1}\left(E\left[\xi\right]-\mu_0\right)\leq r_1\\ E\left[\left(\xi-\mu_0\right)\left(\xi-\mu_0\right)^T\right]\preceq r_2\Sigma_0\end{array}\right.\right\},$$

where M is the set of all probability measures in measurable space. $\Omega\in R^m$ represents a closed convex set. Delage E et al. introduced a DRO model based on support moments, where the first moment of the random parameter is confined within an ellipsoid, and the second moment is at the intersection of two semi-positive cones. Then, the problem (2) is transformed into the following form based on the distribution ambiguity set $\mathcal{P}$:

$$\min_{x\in X}\sup_{f\in\mathcal{P}}E_f\left[\varphi\left(x;\xi\right)\right]$$
$$s.t.\begin{cases}E\left[1\right]=1,\\ E\left[\left(\xi-\mu_0\right)\left(\xi-\mu_0\right)^T\right]\preceq r_2\Sigma_0,\\ \begin{bmatrix}\Sigma_0 & E\left[\xi\right]-\mu_0\\ \left(E\left[\xi\right]-\mu_0\right)^T & r_1\end{bmatrix}\succeq 0,\\ f\left(\xi\right)\geq 0,\ \ \forall\xi\in\Omega.\end{cases}\tag{8}$$

With the development of moment-based distribution ambiguity sets, Bertsimas [35], Goh and Sim [36], and Wiesemann [30] have proposed various extensions to address different objective functions and consider the probability distribution shapes. These extensions include formulating new distribution ambiguity sets that account for symmetric and asymmetric and unimodal and multimodal distributions.

Moment-based ambiguity sets contain probability distributions with common characteristics regarding moment information rather than specific probability distributions. There is a lack of a straightforward way to define an empirical distribution in pure moment-based ambiguity sets. An empirical probability distribution can be derived from sampling the true distribution. Further, study a family of distributions in ambiguity set closely to the particular empirical distribution. The correct ambiguity set should at least statistically contain the true unknown probability distribution. Consequently, a pivotal challenge in studying ambiguity sets is accurately measuring the closeness between probability distributions, a classic problem in statistics.

In many cases, the optimization problem obtained can be reformulated into quadratic or semidefinite programming problems. However, considering all distributions that only use first and second moments information may lead to overly conservative results [37]. Only the moment information of the data is used, while other important information that may exist is ignored. It may affect the performance and results of the model. Furthermore, obtaining moment information in practical problems is also tricky. Technical limitations or incomplete data collection methods, historical data is insufficient, and it is challenging to estimate moments. Even with precise moment information, constructing an ambiguity set based only on the moment information can forfeit valuable prior information about the shape of the distribution, potentially leading to suboptimal solutions for decision-making.

2) Ambiguity set based on statistical distance

Another method for quantifying the statistical distance between two probability distributions is considering probability measures and constructing a set of possible distributions centered around an underlying probability distribution. The candidate distributions in the set are close to the underlying distribution in the sense of statistical distance. The underlying probability distribution is usually an empirical or Gaussian distribution with fixed mean and variance. The empirical distribution is generally determined by data-driven methods such as sampling, expert opinions, and simulations. The conservatism of the optimization problem is controlled by adjusting the radius of the ambiguity set. Specifically, as the radius decreases, the ambiguity set correspondingly contracts to incorporate solely the underlying probability distribution. In this case, the distribution robustness problem is simplified to a stochastic programming problem with no ambiguity. One significant advantage of this method is that it can effectively incorporate observed or sampled data directly into the optimization problem. This approach is a data-driven method because of its direct and extensive use of actual data[38].

Research on distributionally robust optimization primarily focuses on the choice of measure for assessing the statistical distance between probability distributions. Commonly used measures include the Prokhorov measure [39] and the Kantorovich measure [40]. Kuhn [41] and some scholars study specific divergences (ϕ), such as relative entropy [42] χ^2-distance [43]. The choice of divergence depends on the risk preferences of the modelers. ϕ-divergence has convexity and can use the first and second moments and other numerical characteristics. Ben-Tal et al. [44] have explored ambiguity set models based on ϕ-divergence and tested their computational viability. However, Gao noted that the ϕ-divergence-based distribution ambiguity set may sometimes fail to include the distributions that decision-makers want to include, and ϕ-divergence does not consider the proximity between two points in the support, resulting in overly conservative or pessimistic results [45]. Blanchet et al. [46] have employed the Wasserstein distance. The Wasserstein distance accurately evaluates the closeness between two points and accurately measures the distance between two distributions[45][47]. Hanasusanto et al. [48] comprehensively review various ambiguity set types. Table I summarizes commonly used probability measures for constructing ambiguity sets [49].

TABLE I
PROBABILITY MEASURES COMMONLY USED FOR CONSTRUCTING AMBIGUITY SET

Divergence	$D_\phi(P,Q)$
Kullback-Leibler (ϕ_{KL})	$\Sigma_i q_i \log\left(\frac{p_i}{q_i}\right)$
Burg Entropy (ϕ_B)	$\Sigma_i q_i \log\left(\frac{q_i}{p_i}\right)$
Hellinger divergence (ϕ_H)	$\Sigma_i\left(\sqrt{p_i}-\sqrt{q_i}\right)^2$
χ-divergence of order 2(ϕ_χ)	$\Sigma_i \frac{(p_i-q_i)^2}{q_i}$
Variation distance (ϕ_V)	$\Sigma_i \lvert p_i-q_i\rvert$
Wasserstein distance (ϕ_W)	$\sum_{i=1}^{m}\sum_{j=1}^{n}\lVert p_i-q_j\rVert^p \pi_{ij}$

The two vectors $P=\{p_1,\cdots,p_m\}^T$ and $Q=\{q_1,\cdots,q_m\}^T$ represent probabilities on these m scenarios. Two discrete probability distributions (P, Q) are defined on the same Ω, except for the Wasserstein distance. These two probability distributions are respectively supported on $\{\xi_1,\cdots,\xi_m\}$ and $\{\zeta_1,\cdots,\zeta_n\}$ for the Wasserstein distance. Notably, most ϕ-divergences are not metrics, though they help characterize the divergence between distributions. Only the Variation distance is metric, which satisfies the properties of a metric.

Based on these probability measures, an ambiguity set can be constructed as

$$\mathcal{P}(Q_0,r)=\left\{P\in R_+^m : P(\xi\in\Omega)=1, D_\phi(P,Q_0)\le r\right\},$$

where Q_0 represents the underlying distribution. The inner maximization problem in (7) is given below.

$$\max_P \ \sum_{i=1}^m p_i\varphi(x;\xi_i)$$
$$s.t.\begin{cases}\sum_{i=1}^m p_i=1,\ p_i\ge 0, \forall i=1,2,\cdots,m,\\ D_\phi(P,Q_0)\le r.\end{cases}$$

The critical step is to dualize the above problem. Its dual problem is given by

$$\min_{\lambda_1,\lambda_2\ge 0}\left\{\lambda_1+r\lambda_2+\max_{P\ge 0}\left[\sum_{i=1}^m p_i\left(\varphi(x;\xi_i)-\lambda_1\right)-\lambda_2 D_\phi(P,Q_0)\right]\right\}.$$

Through Lagrangian duality theory, the equivalent reformulation of the DRO problem based on various ambiguity sets of probability measures takes the form of Table II.

TABLE II
THE EQUIVALENT REFORMULATION OF THE DRO PROBLEM BASED ON DIFFERENT PROBABILITY MEASURES AMBIGUITY SET

Probability measures	The equivalent reformulation
Kullback-Leibler	$\min_{x\in X,\lambda_1,\lambda_2\ge 0}\left\{\lambda_1+(r-1)\lambda_2+\lambda_2\sum_{i=1}^m q_i\exp\left(\frac{\varphi(x;\xi_i)-\lambda_1}{\lambda_2}\right)\right\}$
Hellinger divergence	$\min_{x\in X,\lambda_1,\lambda_2\ge 0}\left\{\lambda_1+r\lambda_2+\lambda_2\sum_{i=1}^m q_i\frac{\varphi(x;\xi_i)-\lambda_1}{\lambda_1+\lambda_2-\varphi(x;\xi_i)}\right\}$ $s.t.\ \varphi(x;\xi_i)-\lambda_1\le\lambda_2, \forall i=1,2,\cdots,m$
Variation distance	$\min_{x\in X,\lambda_1,\lambda_2\ge 0}\left\{\lambda_1+r\lambda_2+\lambda_2\sum_{i=1}^m q_i\max\left\{\varphi(x;\xi_i)-\lambda_1,-\lambda_2\right\}\right\}$
Wasserstein distance	$\min_{x\in X,\lambda\ge 0}\left\{r^p\lambda-\sum_{j=1}^n q_i\min_{i=1,\cdots,m}\left\{\lambda\lVert\xi_i-\zeta_j\rVert^p-\varphi(x;\xi_i)\right\}\right\}$

Kuhn [41] proved that the DRO method based on Wasserstein distance is precious, offering finite sample guarantees, asymptotic consistency, and computational tractability. Wasserstein distance has received much attention due to its many desirable properties. It has seen increasing application in generative adversarial networks (GANs), autoencoders, and machine learning regularization.

Although ambiguity sets based on moment information have better tractability, those based on Wasserstein distance provide strong out-of-sample performance guarantees and allow decision-makers to control the conservatism of the model. Therefore, recent DRO research has increasingly focused on methodologies grounded in statistical distance.

4. Data-driven method

With today's explosive growth in data, various industries are collecting vast quantities of data regularly. Suppliers track order patterns throughout the supply chain, the World Health Organization compiles comprehensive infection data, GPS tracks drivers' movements, and brokerages record historical stock prices. These massive amounts of data have catalyzed a shift from traditional reasoning and assumptions to data-centric approaches. Data-driven algorithms (DDA) are at the forefront and are emerging and becoming increasingly prevalent and significant in modern computing.

For instance, data-driven robust optimization algorithms can make the most of data. The algorithm can provide detailed descriptions of random parameters using historical data and statistical inference. Data-driven distributionally robust optimization methods can fully consider uncertain factors related to decisions and use collected data to estimate distribution sets more accurately, transcending limited to specific distributions. Constructing new optimization techniques tailored to data-driven environments can obtain reliable solutions.

As researchers grow more interested in leveraging data and advanced technologies to solve practical problems, data-driven algorithms evolve and improve.

1) Machine Learning

Several researchers have employed machine learning techniques to tackle vendor-managed inventory (VMI) problems, addressing demand uncertainty and managing inventory costs effectively. For instance, Bertsimas et al. [50] applied machine learning techniques to control the inventory for video entertainment products in VMI frameworks. Ferreira [51] utilized machine learning techniques to optimize pricing decisions for online clothing retailers by predicting future demand for new products. Y. Shi et al. [52] employed data-driven methodologies to manage inventory allocation for overseas warehouses. Their approach randomly partitioned the complete data set into training and validation sets according to a specific ratio. It used machine learning techniques to analyze product attributes from historical sales data. After fitting the model to the training data set, the effectiveness of model performance was evaluated based on the out-of-sample of the validation data set.

Deep learning (DL) represents an emerging and prominent area of study within machine learning (ML). It is well known that problems can be solved analytically if the probability distribution is known. However, the probability distribution is frequently unavailable or unknown. Typically, decision-makers only have a series of historical data. Oroojlooyjadid et al. [53] proposed a deep learning method that utilizes prior knowledge of the demand's probability distribution to address this issue. The product order quantities are optimized based on the characteristics of demand data. Numerical experiments on real-world data show that for highly volatile demand, the deep learning method significantly outperforms other conventional data-driven methods.

2) Simulation optimization

In mathematical programming problems under uncertainty, simulation optimization can simulate the objective function and constraint conditions of the optimization problem through a "black box" when they cannot be expressed in analytical form. These simulations can operate under random inputs to the simulation system and the corresponding random outputs based on the objective function and constraints. The simulation system has adjustable design variables to optimize performance [54]. Amaran et al. developed a simulation optimization library that includes various simulation optimization examples and applications [55].

3) Heuristic algorithm

Heuristic algorithms are essential tools for dealing with complex optimization models. The core idea is to iteratively update various feasible points and systematically search the solution space to find the global optimal solution. Practical metaheuristic algorithms include particle swarm, local search, tabu search, simulated annealing, evolutionary algorithms, scatter search, ant colony optimization, bee colony optimization, and artificial immune systems. Among them, particle swarm optimization (PSO) is a population-based metaheuristic method inspired by swarm intelligence. Each particle moves in the solution space, and its position represents a point visited in the decision variable space and the value of the objective function at that point. Another attribute of the PSO algorithm is the particle's velocity, which represents its movement direction and step size. PSO iteratively updates particle positions by collectively integrating information from individual particles and the swarm. The algorithm intends to efficiently search the complex solution space from an adaptive particle system to determine the global optimum solution [56]. Hughes et al. [57] improved the PSO algorithm by proposing a robust metaheuristic solution for black-box problems under uncertainty. It can be applied to more general-scale optimization problems with less information on the objective function and can be achieved under expected evaluation times. Xing further enhanced the PSO comprehensively and deeply by introducing the mutation operator in the genetic algorithm (GA) and the Metropolis criterion in the simulated annealing algorithm (SI) [58]. Bertsimas et al. [59-61] proposed an improved robust local search adaptive method based on a downhill direction (d.d.). This method is essentially a robust local search.

Given a starting point in the decision variable space, internal maximization is performed in the uncertain neighborhood, unwanted high-cost points (HCP) are identified from this search, and the best direction away from all these HCP points is determined through quadratic programming. Iterate to a new point in this descent direction and repeat the process until no further descent direction can be identified. Hughes et al. [62] proposed the largest empty hypersphere metaheuristic algorithm (LEH). This global optimization method moves the search to a feasible location within the farthest distance from all previously visited "bad" points. Using the idea of identifying high-cost points globally with the d.d. method, LEH uses the historical set of evaluated points and the high-cost set. The latter is a subset of the historical set containing all points with objective function values greater than the threshold, set to the current estimated robust global minimum value. Beyer and Sendhoff et al. [63] proposed a genetic algorithm (GAs/RS3) based on a robust solution search plan to solve issues such as parameter drift within the running time, model sensitivity, and other aspects.

Heuristic optimization algorithms are characterized by their independence from initial conditions. In addition, these algorithms do not rely heavily on the structure of the solution space and do not require differentiability or continuity in the solution domain, making them suitable for solving problems with unknown solution space or discrete variables. Heuristic methods start from any initial solution and explore the optimal solution in the entire solution space based on a specific mechanism and probability. Furthermore, heuristic optimization algorithms are easy to implement.

However, heuristic optimization algorithms also present certain disadvantages. While they can provide approximate solutions, the inherent randomness of their iterative search process often incurs significant computational costs as a trade-off. The convergence speed tends to be relatively slow, and verifying their convergence can be challenging, especially for high-dimensional and multimodal problems, where the algorithm may face convergence issues or get stuck in local optima.

4) Hybrid algorithm

In recent years, hybrid random optimization algorithms based on data-driven techniques have garnered significant attention among practitioners and researchers. Jiang et al. [64] developed a precise method to solve stochastic programming problems through data-driven chance constraints (DCC) with ϕ-divergence. Two types of confidence sets are constructed for stochastic distributions based on nonparametric statistical estimation of moments and probability density from historical datasets. Their approach effectively solved stochastic programming with DCC under moment-based and density-based confidence sets. In addition, they derived a quantitative relationship between DCC and the sample size of historical data, thereby demonstrating the intrinsic value of data. Similarly, Lee et al. [65] studied the data-driven distribution of robust newsvendor problems under Wasserstein distance. They derived a closed-form solution and characterized the worst-case distribution.

Additionally, McLachlan et al. clustered multivariate data observed from random samples using finite mixture models and fitted them employing the expectation-maximization (EM) method with maximum likelihood [66]. However, this method has limitations in addressing variance characteristics concentrated in the uncertainty set. Pierpaolo D U et al. [67] proposed several uncertainty clustering methods based on distinct theoretical methods for modelling uncertainties to address these limitations.

Despite the growing prominence of data-driven methods, Elishakoff et al. [21] argue that their practical application remains challenging. This difficulty arises from all joint probabilities of parameters should be objectively quantified. In scenarios with insufficient independent measurement data, this fitting process can significantly impact the accuracy of quantification results. Furthermore, these methods usually require substantial computational costs, especially when the failure probability is tiny.

Next, a motivating example of a linear programming problem by Shang et al. [68] is used to illustrate these models:

$$\begin{aligned} &\max z = \xi_1 x_1 + \xi_2 x_2 \\ &s.t.\ \ 5x_1 + 2x_2 \le 80, \\ &\qquad 6x_1 + 8x_2 \le 200, \\ &\qquad x_1, x_2 \ge 0, \end{aligned}$$

where ξ_1 and ξ_2 represent uncertain parameters. Data samples are generated randomly based on their nominal values [8, 2]. The comparative analysis of running time and optimal values across varying sample sizes in the motivating case are depicted in Fig. 5 and Fig. 6. Specifically, in the motivating example, when the means of the random variables are known to be [8,2], the solutions obtained from the deterministic optimization model and EM align perfectly.

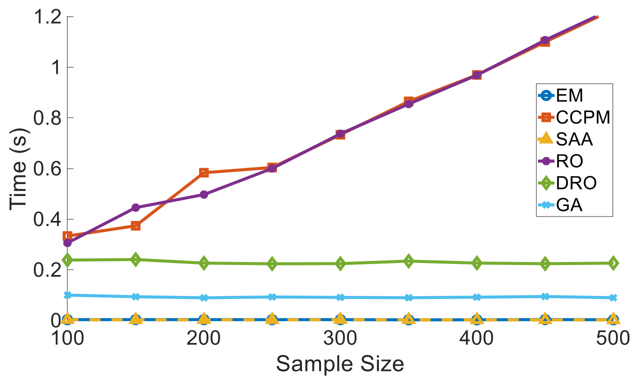


Fig. 5. Comparisons of running time across varying sample sizes.

Figure 5 illustrates that the CCPM and RO methods exhibit substantial increases in running time with growing sample sizes, indicating a significant impact of algorithmic complexity on their performance when dealing with large-scale data. In contrast, the EM, SAA, DRO, and GA methods display minimal variation in running times across different sample sizes, reflecting high efficiency and potential scalability. Despite this, GA, as an intelligent algorithm, seeks local optimal solutions, which results in unstable optimal values. Furthermore, the SAA and EM methods show the lowest running times, indicating good performance in handling large-scale problems. SAA and DRO are often preferable for highly uncertain datasets due to their design, which incorporates risk considerations into the optimization process.

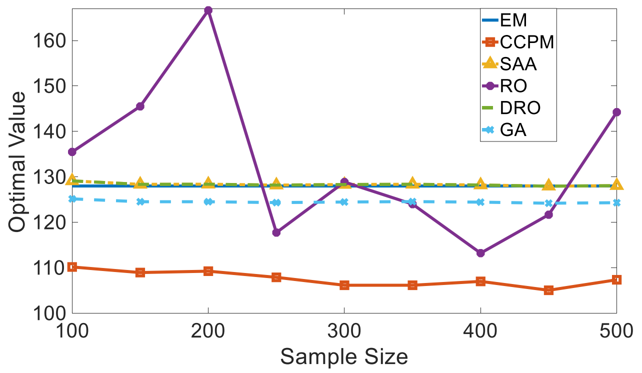


Fig. 6. Comparisons of optimal values across varying sample sizes.

As illustrated in Figure 6, when the means of the random variables are known, EM demonstrates stability, exhibiting minimal variation with changes in sample size. This consistency indicates that EM might be robust to fluctuations in the amount of data. Conversely, CCPM exhibits a noticeable fluctuation in optimal values with varying sample sizes. This variability indicates that the CCPM may be sensitive to sample size, with performance possibly improving with larger samples but also some instability. RO shows significant variability in the optimal values, reflecting a high dependence on the sample size and data characteristics. The method may perform inconsistently across different sample sizes, which could reflect sensitivity to the specific data or sample characteristics. DRO and GA present a consistent and decreasing trend in optimal values similar to the SAA. It indicates improved performance with larger sample sizes and more excellent stability as sample size increases. However, the optimal values obtained by GA are relatively small and do not reach the global optimum, as evidenced by the comparison results in Table III.

The performance of these methods is further evaluated using the optimal values performance loss ratios. These ratios are computed by comparing the loss values of the optimal values obtained through each method to those from the deterministic model. Specifically, the performance loss ratio quantifies the extent to which the optimal values from given methods deviate from the optimal values provided by the deterministic model.

TABLE III
COMPARES RESULTS ACROSS VARYING MODELS

Model	Optimal value	Loss ratio
EM	128.0000	0.0000
CCPM	110.1825	-0.1392
SAA	129.1200	0.0088
RO	135.4909	0.0585
DRO	129.1200	0.0088
GA	125.1658	-0.0221

The comparison of linear programming methods reveals that both CCPM and RO experience increased running times with larger samples, indicating high algorithmic complexity. Conversely, EM, SAA, DRO, and GA maintain efficiency and stability across varying sample sizes. Due to their risk management capabilities, SAA and DRO are particularly well-suited for uncertain datasets. EM remains stable when the means of the random variables are known, suggesting robustness. CCPM displays variability with sample size, while RO demonstrates significant fluctuations in optimal values, reflecting sensitivity to data characteristics. The optimum value of the GA is relatively small and does not achieve the global optimum. Overall, SAA and DRO offer reliable performance and scalability with larger sample sizes, making them advantageous choices for managing uncertainty and handling large datasets effectively.

III. Fuzzy Optimization Model

When decision tasks are too complex for quantitative description or faced with some non-precise information lacking clear boundaries, such as information related to human language or behaviors, fuzzy optimization (FO) provides practical techniques and methods.

The fuzzy optimization problem involves an optimization model with either the objective function or constraints containing fuzzy variables. This approach is particularly suitable for dynamic, uncertain, or complex systems. Since LA Zadeh [69] proposed the concept of “fuzzy sets” in 1965, numerous new methods and mathematical theories have been developed to address imprecision, fuzziness, and uncertainty.

These advancements are often extensions of fuzzy set theory, while others attempt to mathematically model imprecision and uncertainty in different frameworks (Kerre and Burgin [70][71]). Such studies generally extend fuzzy set theory as standard probability theory often falls short in sufficiently handling complex problems involving randomness, imprecision, fuzziness, and partial incompleteness. Fuzzy theory draws on hesitant sets and inference logic (conditional event function) and appropriately complements traditional probability theory to better handle the complexities inherent in statistical inference problems. Prominent examples include Type-2 fuzzy set theory [72], Intuitionistic fuzzy set theory [73], Rough set theory [74], Shadowed set theory [75], Credibility set theory [73], and theory of evidence [76]. Pioneering works by Bellman, Kalaba, and Zadeh et al. [77] laid the groundwork for clustering based on fuzzy set theory. Their research results were prototypes of clustering algorithms. In addition, numerous studies have employed fuzzy theory to solve uncertainty in practical applications [78]-[80]. The fuzzy linear optimization model is as follows:

$$\min_{x\in X}\varphi\left(x;\tilde{\xi}\right)=\tilde{\xi}_i x_i$$
$$s.t.\begin{cases}\sum_{i=1}^{n}\tilde{a}_{ij}x_i\le\tilde{b}_j,\ j=1,2,\cdots,m',\\ x_i\ge 0, i=1,2,\cdots,n',\end{cases}\tag{9}$$

where x represents the decision variable. $\tilde{\xi}$, $\tilde{a}_{ij}$, $\tilde{b}_j$ denote the fuzzy variable. The α-Cut set, and fuzzy expectation method are commonly employed to solve fuzzy optimization models.

1) α-Cut Set Method

The basic principle of handling fuzzy optimization is to treat the fuzzy objective function and fuzzy constraints as fuzzy subsets within the solution space. These fuzzy sets are represented by their respective membership functions. The intersection of the fuzzy objective function and fuzzy constraints is analyzed to find an optimal solution. Maximizing the membership function of the intersection is the optimal solution to the fuzzy optimization problem.

Let the fuzzy subset $\tilde{A}_\alpha=\left\{x\middle|\mu_{\tilde{A}}(x)\ge\alpha\right\}$ be the α-cut set of the fuzzy set *A*, where *α* is the confidence level. The fuzzy linear optimization model (9) can be converted into an interval optimization model using the concept of cut set:

$$\min_{x\in X}\varphi\left(x;\tilde{\xi}\right)=\left[\xi_i^-,\xi_i^+\right]x_i$$
$$s.t.\begin{cases}\sum_{i=1}^{n}\left[a_{ij}^-,a_{ij}^+\right]x_i\le\left[b_i^-,b_i^+\right],\ j=1,2,\cdots,m',\\ x_i\ge 0, i=1,2,\cdots,n'.\end{cases}$$

2) Fuzzy Expectation Method

For any confidence level $\beta\in[0,1]$, the fuzzy linear optimization problem (9) can be reformulated as the following interval optimization model:

$$\min_{x\in X}E\left[\varphi\left(x;\tilde{\xi}\right)\right]$$
$$s.t.\begin{cases}C_k b_i-\sum_{i=1}^{n}\tilde{a}_{ij}x_i\ge 0\ge\beta,\ j=1,2,\cdots,m',\\ x_i\ge 0, i=1,2,\cdots,n',\end{cases}$$

where the possibility of the fuzzy event $\tilde{A}\ge k$ is

$Pos\left\{\tilde{A} \geq k\right\} = \sup_{\mu \geq k} \mu_{\tilde{A}}(x)$. The necessity of the fuzzy event is $C_k\left\{\tilde{A} \geq k\right\} = \frac{1}{2}\left(Pos\left\{\tilde{A} \geq k\right\} + Nes\left\{\tilde{A} \geq k\right\}\right)$, and $Nes\left\{\tilde{A} \geq k\right\} = 1 - \sup_{\mu \geq k} \mu_{\tilde{A}}(x)$. The expectation of the fuzzy variable η is $E[\eta] = \int_0^{+\infty} C_k\{\eta \geq k\} dk - \int_{-\infty}^0 C_k\{\eta \leq k\} dk$.

In summary, fuzzy optimization and stochastic programming are two pivotal optimization methods for tackling uncertainty in optimization problems. Their primary distinction lies in how uncertain parameters are modeled and described. Fuzzy optimization treats uncertain parameters as fuzzy numbers and constraints as fuzzy sets, some of which allow violations and define the satisfaction degree of constraints as the membership function of constraints [66]. Conversely, stochastic programming represents uncertain parameters as discrete or continuous probability distributions. Although interval fuzziness effectively describes uncertain variables, measuring uncertainty through the degree of membership functions largely relies on the expertise of analysts. Therefore, fuzzy optimization has an inevitable subjectivity [70].

IV. Interval optimization

Obtaining appropriate membership functions or accurate probability distributions is difficult in uncertain and complex environments. In recent years, interval optimization (IO) has been developed to address optimization problems that involve uncertain parameters. This approach requires only the knowledge of the boundaries of uncertain parameters without the need for detailed probability distributions or membership functions. Rommelfanger [81] proposed a linear programming problem with interval parameters in the objective function in 1989, marking a significant step forward in this field. Ishibuchi [82] further advanced this concept by transforming mathematical programming problems with interval parameters in the objective function into multi-objective problems, utilizing ordinal relationships. Sengupta [83] proposed an interval linear programming problem with inequality constraints containing interval parameters and simplified it to an equivalent form for the solution. Jiang et al. [84] introduced a nonlinear interval number programming problem with uncertain coefficients of nonlinear objective functions and constraints. The general form of the nonlinear interval number programming (NINP) problem with uncertain interval parameters in the objective function and constraints is presented below:

$$\begin{aligned} &\min_{x \in X} \varphi(x;\xi) \\ &s.t. \begin{cases} g_k(x;\xi) \leq \left[v_k^l, v_k^R\right], \ k = 1, \cdots, K, \\ \xi = \left[\xi^l, \xi^R\right], \ \xi_i = \left[\xi_i^l, \xi_i^R\right], \ i = 1, \cdots, q. \end{cases} \end{aligned} \tag{10}$$

Consider the vector ξ as a q-dimensional uncertain vector with interval numbers components. The objective function or constraint condition is represented as intervals rather than a real number for each specific x. The general objective function and constraint conditions are typically nonlinear. Traditional deterministic optimization and linear interval programming methods cannot solve this problem. Considering robustness, Jiang [84] transformed the uncertain objective function into two deterministic objective functions. Calvete [85] discussed the optimization problem of stochastic linear bilevel programming, demonstrating that when the coefficients of the two objective functions are interval parameters, optimal solutions are found at the extreme points of the polytope defined by the common constraints. Interval optimization has a broad range of applications. Simic V [86] proposed a two-stage stochastic programming model with interval parameter conditional value-at-risk for end-of-life vehicle management. Wan [87] decomposed the interval parameter model into two standard nonlinear uncertain programming problems and developed a two-step sampling method.

Interval optimization methods provide a new approach to solving general optimization problems with uncertainty. IO provides an alternative to stochastic programming techniques and offers a viable method to solve some problems without sufficient uncertainty information.

V. Polymorphic uncertain optimization

Previous literature generally only concentrated on a situation, considering random variables without considering fuzzy parameters, or vice versa. The optimization problems simultaneously considering random, fuzzy, or interval parameters have not been mentioned. The concept of polymorphic uncertain optimization (PUO) was first introduced by Wan et al. [88] in 2011, representing a significant advancement in this area. They designed a piecewise inference method to infer the probability distribution. This method can offer specific analytical expressions for these distributions and makes a breakthrough in prediction methods.

For uncertain analysis, mixed techniques such as fuzzy randomness are also introduced [89]. Contemporary literature employs an approach to model uncertainty by simultaneously considering random and fuzzy methods using fuzzy random variables (FRVs) theory. This approach constructs an appropriate distribution ambiguity set probability measure, with crucial contributions from researchers such as Puri et al. [90], Klement [91], and Colubi [92]. Zhang et al. [93] applied the polymorphic uncertain optimization model to comprehensively address the fuzziness in objective function parameters and the randomness in constraint condition parameters within an end-of-life vehicle recycling network model. By transforming the original problem into a deterministic multi-objective mixed integer linear optimization problem to offset the uncertainty of the model, they developed a straightforward, interactive fuzzy method to find the compromise solution of the uncertain model.

Further advancing this field, Wan et al. [94] used the developed polymorphic uncertain equilibrium model to solve the problem of decentralized supply chain management. In their approach, consumer demand was regarded as a continuous random variable, while the holding costs of retailers and the transaction costs between manufacturers and retailers are described by fuzzy sets. They first derived a deterministic equivalent formula (DEF) through a compromise optimization method. Then, standard smoothing techniques are used to polish it partially. Ultimately, an approximate equilibrium point for the uncertain problem is found by applying robust algorithms based on gradient information. Notably, although the DEF in their study is a nonlinear complementarity problem, a particular type of variational inequality, their method effectively addresses the assumption of monotonicity of the variational inequality.

In the realm of optimization problems characterized by fuzzy and interval parameters, Wan notably developed a linear programming model incorporating interval and random coefficient constraints derived from the sintering to address the optimal mixture slurry problem associated with the production process of alumina [95]. Zhang et al. [96] developed a polymorphic uncertain nonlinear programming model (PUNP) consisting of a nonlinear objective function and constraints with uncertain parameters. For a given satisfaction level, this definition is a nonlinear optimization that only involves interval parameters. They also proposed a sampling-based interactive method to estimate parameters under polymorphic uncertainty and obtain a robust solution to the original model.

The PUO algorithm represents a synthesis of deterministic optimization techniques and uncertainty processing techniques, incorporating stochastic programming, fuzzy optimization, and robust optimization to manage various forms of parameter uncertainty.

VI. Conclusion and Prospect

This article provides a comprehensive overview of the state-of-the-art techniques in uncertain optimization. It succinctly introduces methods and fundamental principles for addressing uncertain optimization problems. Each method exhibits unique adaptations and advantages, as Table IV details. In practice, selecting the most appropriate algorithms depends on the specific characteristics of the problem to achieve the optimal solutions.

TABLE IV
SUMMARY OF UNCERTAIN OPTIMIZATION MODELS AND METHODOLOGIES

Model	Uncertainty variables	Set	Methodologies
SP	Random variable	Uncertainty/ ambiguity set	EM, CCPM, SAA, RO, DRO, and DDM
FO	Fuzzy variable	Fuzzy set	α-Cut Set Method, Fuzzy Expectation
IO	Interval parameter	Interval	Interval search
PUO	Random variable and Fuzzy variable	Support set and fuzzy set	Integrated algorithm

Stochastic programming, an integral component of uncertain optimization, has yielded substantial research advancements in tackling complex practical problems. Extensive literature has contributed significantly to this field, documenting notable progress in stochastic programming. Table V comprehensively discusses and summarizes various widely adopted models and methodologies for addressing stochastic programming.

TABLE V
SUMMARY OF STOCHASTIC PROGRAMMING AND METHODOLOGIES

Model	Set	Distribution known	Methodologies
EM	Uncertainty set	√	Take the expectation
CCPM	Uncertainty set	√	Satisfy a confidence level
SAA	Support set	×	Take the sample average
RO	Uncertainty set	×	The worst-case realization
DRO	Ambiguity set	×	Take the expectation of the worst-case distribution
DDA	Data sets	×	Train and test the data

Fig. 7 presents an analysis of six methods evaluated on robustness and complexity. Robustness scores indicate that robust optimization and distributionally robust optimization exhibit superior performance in the face of worst-case scenarios. In contrast, the expectation method scores the lowest, suggesting limited robustness. Regarding complexity, RO and EM stand out with the highest scores when dealing with high-dimensional random variables, highlighting their computational demands. Overall, these illustrate a notable trade-off between robustness and complexity. Selecting appropriate methods based on specific application needs and computational resources is crucial.

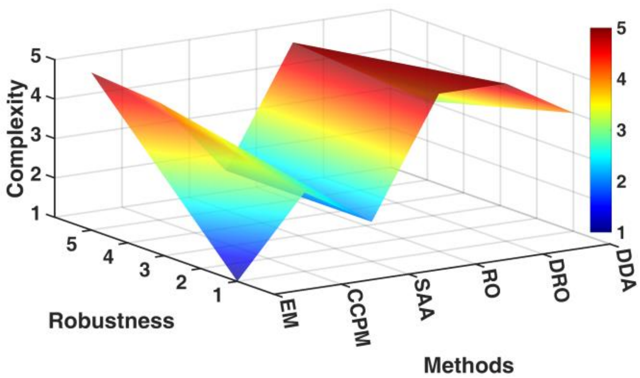


Fig. 7. The robustness and complexity of different models.

As documented in the literature, significant progress has been achieved in stochastic programming. The expectation method generally converts the model into a mixed-integer nonlinear model, and efficient methods for such models are currently a prominent research hotspot and challenge. Distributionally robust optimization models address uncertainty by defining an ambiguity set for the probability distribution. The distributionally robust optimization problems are relatively tractable while the moment information is obtainable. However, distributionally robust optimization problems with unknown numerical characteristics are challenging to solve. For instance, most optimization problems with unknown distributions may be confronted with the dilemma of nonconvex or non-closed solutions. Nevertheless, such problems are more common in practical modelling. Data-driven methods have good numerical performance and can solve the limitations of chance-constrained programming methods and distributionally robust optimization methods. By leveraging existing data, these methods can constrain the range of random variables and reduce the excessive conservatism often associated with traditional robust optimization approaches.

Despite these advancements, several notable deficiencies persist in current methodologies. For example, the expectation and chance-constraint methods require extensive data to fit the probability distribution accurately. In distributionally robust optimization methods, determining the optimal radius and bounds of the ambiguity set is challenging. Although heuristic algorithms have effectively addressed various uncertain programming models, their applicability is generally limited to smaller-scale problems. Other data-driven methods exhibit strong numerical performance but face challenges in ensuring guaranteed out-of-sample performance and consistent convergence.

Fuzzy optimization, interval optimization, and polymorphic uncertain optimization have been achieved in numerous research results on uncertain optimization models and algorithms. However, each approach presents inherent challenges. Fuzzy optimization methods, while valuable, introduce a degree of subjectivity. This subjectivity arises from relying on membership functions to represent uncertainty, which depends on expert judgment. Interval optimization, which deals with parameter ranges rather than precise values, can expand the search space and limit accuracy. Polymorphic uncertain optimization involves multiple uncertainty variables, making the problem more complex. Collectively, these methodologies are computationally tricky and time-consuming.

To effectively address larger-scale problems, further improvements and new attempts are necessary in method design to address the limitations of traditional methods. Based on the analysis of previous literature, future research in uncertain optimization models can focus on the following areas:

Applying the theory of distributionally robust optimization and stochastic programming to construct abstract optimization models is necessary to tackle complex environments scientifically. Secondly, considering various methods have limitations, their accuracy is still unsatisfactory, suggesting room for significant improvement. Researchers combine recently developed data-driven methodologies to design efficient, novel techniques based on data-driven techniques. Construct new optimization methods suitable for random parameters and independent of specific distribution assumptions. These advancements assist in making optimal decisions in complex and uncertain environments. Evaluating the latest techniques in terms of convergence, convergence speed, operating cost, solution quality, and overall performance is crucial. Thirdly, most current methods for solving these models are single. There is a notable gap in research addressing the polymorphic uncertain optimization problems that integrate stochastic programming and fuzzy optimization models. New flexible methods that consider both are lacking. Further exploration using diverse techniques is needed to effectively address these complex optimization challenges. Fourthly, the distributionally robust optimization problems, particularly those involving two-stage or newsvendor problems, can be further studied. Transforming these problems into deterministic models amenable to numerical solutions could enhance performance and practical implementation. Finally, the potential application prospects of uncertain optimization models are broad. They are crucial tools for solving real-world problems, with great practicality and significant implications. For instance, the supply chain network was optimized for medical relief supplies management under severe crises, considering the uncertainty of affected or infected populations. Effective optimization can improve the management of medical disaster relief materials after common emergencies like home fires, emergency treatment, vehicle accidents, and more catastrophic situations such as earthquakes, tsunamis, and explosions.

Based on the analysis, uncertain optimization problems demonstrate extensive applicability and significant potential for further development. This paper comprehensively analyzes various models and methodologies in this domain, highlighting their specific applications and inherent limitations. To address particular challenges, it is necessary to creatively construct efficient uncertain optimization models and develop practical algorithms in complex environments. Exploring the application of uncertain optimization models and their associated algorithms in practice presents a formidable challenge and a highly academic research value. The ongoing advancement in this field is crucial for driving progress in related disciplines and tackling complex, real-world problems with greater efficacy. The continuous advancement of uncertain optimization models and algorithms is vital in enhancing their practical utility and promoting related fields.

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1) Date of modification - October 4, 2024
2) Brief description of the changes
- Updated the optimal solution to the optimal value in Table III.
- Corrected a typo by updating '8' in the optimization problem to '80'.