Improved Stabilization Conditions for Singular Markovian Jump Systems by Dynamic Output-feedback Control

Chaohua Wang, Songhua Wang, Runzhang Zhang and Wenbin Chen*

Abstract—**In this study, the dynamic output-feedback (DOF) control problem for singular Markovian jump systems (SMJSs) with interval time-varying delay is thoroughly reexamined. With only a few decision variables and state decomposition components, a unique mode-dependent augmented Lyapunov-Krasovskii functional is put forth. Additionally, based on linear matrix inequalities, a delay-dependent mean-square exponentially (MSE) admissible criterion (corresponding to regularity, non-impulsiveness, and MSE stability) is established for the open–loop SMJSs. This serves as the foundation for designing a DOF controller for closed-loop SMJSs and obtaining the associated MSE admissibility requirements. The appropriate DOF controller parameters are ascertained by solving each parameter of the DOF controller decomposition component through the use of a state decomposition recombination approach. Remarkably, our findings can improve upon earlier findings, and the suggested approach has a lot of versatility. To demonstrate the superiority and viability of our method, some comparisons with results from the existing literature are presented in two numerical examples.**

Index Terms—**Singular Markovian jump system, Time-varying delay, Dynamic feedback control, Exponentially admissible.**

I. Introduction

SINGULAR systems (SSs) are a class of interesting dynamical systems composed of algebraic equations and differential equations, which have attracted the attention of many researchers, especially researchers from the control and mathematics communities [1]–[4]. One of the main reasons is that these systems have several real-world uses, such as robotics, mechanical engineering systems, chemical processes, and economics [5], [6]. Nevertheless, when random abrupt events arising from repairs, failures, disconnection, and connection of components, etc., occur in SSs, they cannot be described by a singular linear model [7]–[9]. In other words, most practical application systems are stochastic in SSs, so we can model them by utilizing continuous–time Markovian chains. One may refer to this type of system as singular Markovian jump systems (SMJSs).

Scholars have also turned their attention to SMJSs, and insightful study findings have been generated [10]–[15]. To name only a few, Sakthivel et al. [10] addressed the mixed H_∞ and passive control for SMJSs. Liu et al. [11] discussed the reliable exponential H_∞ filtering for SMJSs based on sensor failures. With nonlinear uncertainties and time-varying delay, Monhanapriya et al. [12] studied the disturbance rejection for SMJSs. Tao et al. [13] discussed the stochastic admissibility of SMJSs with time–delay by the sliding mode approach. In these literature examples, the stability of systems is the primary condition for almost all control systems to be designed. Thus, the stabilization issue for SMJSs has aroused scholars' attention [17]–[24]. For example, in [18], by the linear matrix inequality (LMI) approach, the dynamic output–feedback (DOF) control for SMJSs was discussed, and the necessary and sufficient condition was established. Moreover, for continuous–time SMJSs, Park et al. [19] considered the DOF H_∞ control. In [21], the DOF H_∞ control for SMJSs with partly unknown transition rates was concerned and a new stability criterion was acquired. Recently, the robust H_∞ control of SMJSs was investigated in [22] using a matrix decoupling technique, and the DOF guaranteed cost controller was designed. Nevertheless, it is easy to see that the effect of time-delay on the SMJSs is not covered in [18], [19], [21], [22], which is relatively conservative. We know that time–delay is an inevitable factor, which can reduce the system's performance and even destroy its stability. Thus, the consideration of time–delay is crucial. Recently, the stabilization for neutral SMJSs being composed of state feedback control was exploited in [20]; the same issue was discussed in [23]. However, their stabilization is based only on the design of state feedback control. This design may be too conservative because some technologies often fail to measure state variables. Fortunately, a suitable DOF control design can overcome this problem. In [24], by using the variable elimination technique, the H_∞ DOF control for SMJS with time–varying was examined, and certain stochastic admissible criteria were acquired. However, the criteria obtained are independent of time–delay size information, which is more conservative. In addition, in order to reduce the conservativeness of SMJS and ensure low computational complexity, the SDR technology in [25] has aroused our interest in the controller design method. The basic idea is to use a full-order filter to break down and reorganize the closed-loop SMJSs according to the properties of a singular matrix. This will yield the equivalent systems, which can

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then be used to assess the admissibility of the mean-square exponential (MSE). The ability to handle decision variables more flexibly and eliminate redundant decision variables to lower computing complexity is one of this strategy's main advantages.

In this study, the DOF control problem of SMJSs is rediscussed. The purpose is as follows : To begin with, by establishing a state decomposition and a mode-dependent Lyapunov-Krasovskii function (LKF), the MSE admissibility conditions of open-loop SMJSs are derived. Second, by using an SDR technique, for the closed-loop SMJSs with a DOF controller, some delay-dependent and mode-dependent conditions are produced based on the existing MSE admissibility criteria. Two numerical examples will be provided at the end to demonstrate the benefits and efficacy of the results presented. The DOF problem for SMJSs with interval time-varying delays has not received much attention in research that employs the SDR approach. We therefore close this gap in our paper. These are the principal contributions:

- In contrast to earlier studies [9]–[12], [15], [16], [18], [19], [21], [22], a new mode–dependent and state–decomposed LKF function has been developed. To be more precise, we build LKF from the standpoint of state decomposition instead of the perspective of the total state, based on the properties of singular matrices. This may lessen computing complexity to some degree. Combining tighter integral inequality approaches will make the benefits of decreasing conservatism more apparent.
- Using an SDR technique, the requirements for delay-dependent and mode-dependent MSE admissibility are provided for the closed-loop SMJSs using the DOF controller. Compared to [18], [19], [21], [24], [25], the results exhibit a lower degree of conservatism. Considering each decomposition element of the investigated controller parameters, in particular, can help acquire the necessary DOF controller parameters more precisely and flexibly.

Notations : T is the transpose of a matrix, and -1 is the inverse of a matrix. $\mathbb{A} > 0$ represents a positive definite matrix. $\begin{bmatrix} \mathbb{A} & \mathbb{Q} \\ \star & \mathbb{C} \end{bmatrix}$ stands for $\begin{bmatrix} \mathbb{A} & \mathbb{Q} \\ \mathbb{Q}^T & \mathbb{C} \end{bmatrix}$. $\bar{\chi}_1(t) = \begin{bmatrix} \chi_1(t) & \chi_{f1}(t) \end{bmatrix}$. $\bar{\chi}_2(t) = \begin{bmatrix} \chi_2(t) & \chi_{f2}(t) \end{bmatrix}$. $0_\imath, 0_{\imath\times\jmath}, I_\imath, I_{\imath\times\jmath}$ are $\imath\times\imath, \imath\times\jmath$ zero matrices and $\imath\times\imath, \imath\times\jmath$ identity matrices, respectively. $sym\{\mathbb{A}\} = \mathbb{A} + \mathbb{A}^T$. $col\{\mathbb{A}_1, \mathbb{A}_2, ..., \mathbb{A}_\kappa\} = [\mathbb{A}_1^T, \mathbb{A}_2^T, ..., \mathbb{A}_\kappa^T]^T$. $||\omega(t)||_{\bar{\kappa}} = \sup_{-\bar{\kappa}\leq t\leq 0} ||\omega(t)||$. $\mathbb{R}^\imath$ and $\mathbb{R}^{\imath\times\jmath}$ are $\imath-$dimensional Euclidean space and the set of $\imath\times\jmath$ real matrices, respectively.

II. Problem statements

Consider the SMJSs as follows:

$$\begin{cases} E\dot{x}(t) = \mathcal{W}(r_t)x(t) + \mathcal{W}_d(r_t)x(t-\kappa(t)) + \mathcal{H}(r_t)u(t), \\ y(t) = \mathcal{F}(r_t)x(t), \\ x(t) = \emptyset(t), \forall t \in [-\kappa_2, 0], \end{cases} \tag{1}$$

where $E \in \mathbb{R}^{n\times n}$ satisfies $rank(E) = r < n$. $y(t) \in \mathbb{R}^s$, $x(t) \in \mathbb{R}^n$, and $u(t) \in \mathbb{R}^q$ are control output, the state, and input, respectively. $\{r_t\}$ belongs to a finite set $\mathcal{S} = \{1, 2, ..., N\}$ and is a continuous Markovian process. $\emptyset(t)$ is an initial function. The transition probability matrix $\prod = \{\pi_{ij}\}$ is defined in [24], [25]. Let $r_t = i \in \mathcal{S}$, given matrices $\mathcal{W}(r_t), \mathcal{W}_d(r_t), \mathcal{H}(r_t), \mathcal{F}(r_t)$ are abbreviated as $\mathcal{W}_i, \mathcal{W}_{di}, \mathcal{H}_i, \mathcal{F}_i$. The time delay $\kappa(t)$ meets:

$$0 \leq \kappa_1 \leq \kappa(t) \leq \kappa_2, \dot{\kappa}(t) \leq \delta < 1, \tag{2}$$

where κ_1, κ_2 and δ are given constant scalars.

We will design a DOF controller as

$$\begin{cases} E\dot{x}_f(t) = \mathcal{W}_{fi}x_f(t) + \mathcal{H}_{fi}y(t), \\ u(t) = \mathcal{G}_{fi}x_f(t) + \mathcal{J}_{fi}y(t), \end{cases} \tag{3}$$

where the parameter matrices $\mathcal{W}_{fi}, \mathcal{H}_{fi}, \mathcal{G}_{fi}$ and $\mathcal{J}_{fi}$ for DOF controller need need to be verified, and $x_f(t) \in \mathbb{R}^n$ is the controller state.

According to (1) and (3), we get the following closed-loop SMJS,

$$\bar{E}\dot{\bar{x}}(t) = \bar{\mathcal{W}}_i\bar{x}(t) + \bar{\mathcal{W}}_{di}\bar{x}(t-\kappa(t)), \tag{4}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \ \bar{\mathcal{W}}_{di} = \begin{bmatrix} \mathcal{W}_{di} & 0 \\ 0 & 0 \end{bmatrix},$$
$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}, \ \bar{\mathcal{W}}_i = \begin{bmatrix} \mathcal{W}_i + \mathcal{H}_i\mathcal{J}_{fi}\mathcal{F}_i & \mathcal{H}_i\mathcal{G}_{fi} \\ \mathcal{H}_{fi}\mathcal{F}_i & \mathcal{W}_{fi} \end{bmatrix}.$$

Definition 1. *[24] The SMJS (1) with $u(t) \equiv 0$ is*

(i) regular and impulse-free if the pair $(E, \mathcal{W}_i)$ is regular and impulse free;

(ii) MSE stable if there are scalars $\mu > 0$ and $\nu > 0$ such that $\mathcal{E}\{||x(t)||^2\} \leq \mu e^{-\nu t} \sup_{-\kappa_2\leq t\leq 0} ||\emptyset(t)||$, $t > 0$;

(iii) MSE admissible if it is regular, impulse–free, and MSE stable.

Lemma 1. *[26] For given scalars $\imath < \jmath$, matrix $\aleph > 0 \in \mathbb{R}^{n\times n}$, and a differentiable function x, it has*

$$-(\jmath - \imath)\int_\imath^\jmath \dot{x}^T(s)\aleph\dot{x}(s)ds \leq -\sum_{i=1}^3 (2i-1)\triangle_i^T\aleph\triangle_i,$$

where

$$\begin{aligned} \triangle_1 =& x(\jmath) - x(\imath), \\ \triangle_2 =& x(\imath) + x(\jmath) - \frac{2}{\jmath-\imath}\int_\imath^\jmath x(s)ds, \\ \triangle_3 =& x(\jmath) - x(\imath) + \frac{6}{\jmath-\imath}\int_\imath^\jmath x(s)ds \\ &- \frac{12}{(\jmath-\imath)^2}\int_\imath^\jmath\int_\theta^\jmath x(s)dsd\theta. \end{aligned}$$

III. Main results

Theorem 1. *For some given scalars κ_1, κ_2 and δ, SMJS (1) under $u(t) \equiv 0$ satisfies MSE admissible if there exist any matrices $L_{1i}, L_{2i}, L_{3i}, L_{4i}$ with proper dimensions and $\mathcal{Q}_{\alpha i} > 0, \mathcal{P}_{\beta i} > 0, \mathcal{R}_{\beta i} > 0, \mathcal{R}_\beta > 0, \alpha = 1, 2, 3, \beta = 1, 2,$*

such that

$$\sum_{j=1}^{N}\pi_{ij}(\mathcal{Q}_{1j}+\mathcal{Q}_{2j}+\mathcal{Q}_{3j})<\mathcal{Q}_1+\mathcal{Q}_2, \tag{5}$$

$$\sum_{j=1}^{N}\pi_{ij}(\mathcal{Q}_{2j}+\mathcal{Q}_{3j})<\mathcal{Q}_2,\sum_{j=1}^{N}\pi_{ij}\mathcal{Q}_{2j}<\mathcal{Q}_2, \tag{6}$$

$$\sum_{j=1}^{N}\pi_{ij}\mathcal{R}_{1j}<\mathcal{R}_1,\sum_{j=1}^{N}\pi_{ij}\mathcal{R}_{2j}<\mathcal{R}_2, \tag{7}$$

$$\Gamma_i<0, \tag{8}$$

where

$$\begin{aligned}\Gamma_i = {} & sym\Big\{\Pi_1^T\mathcal{P}_{1i}\Pi_2\Big\}+sym\Big\{\Pi_3^T\mathcal{P}_{2i}\Pi_4\Big\}+\sum_{j=1}^{N}\pi_{ij}\Pi_1^T\mathcal{P}_{1j}\Pi_1+\sum_{j=1}^{N}\pi_{ij}\Pi_3^T\mathcal{P}_{2j}\Pi_3+\pi_1^T\mathcal{Q}_{1i}\pi_1-\\ & \pi_2^T\mathcal{Q}_{1i}\pi_2+\pi_1^T\mathcal{Q}_{2i}\pi_1-\pi_3^T\mathcal{Q}_{2i}\pi_3+\pi_1^T\mathcal{Q}_{3i}\pi_1-(1-\delta)\pi_4^T\mathcal{Q}_{3i}\pi_4+\kappa_1\pi_1^T\mathcal{Q}_1\pi_1+\kappa_2\pi_1^T\mathcal{Q}_2\pi_1+\frac{\kappa_1^3}{2}e_9^T\mathcal{R}_1e_9+\\ & \frac{\kappa_2^2-\kappa_1^2}{2}(\kappa_2-\kappa_1)e_9^T\mathcal{R}_2e_9+\kappa_1^2e_9^T\mathcal{R}_{1i}e_9-b_1^T\mathcal{R}_{1i}b_1-3b_2^T\mathcal{R}_{1i}b_2-5b_3^T\mathcal{R}_{1i}b_3+(\kappa_2-\kappa_1)^2e_9^T\mathcal{R}_{2i}e_9-\\ & b_4^T\mathcal{R}_{2i}b_4-3b_5^T\mathcal{R}_{2i}b_5-5b_6^T\mathcal{R}_{2i}b_6+sym\Big\{(e_1^TL_{1i}+e_9^TL_{2i})(-e_9+e_{si})\Big\}+sym\Big\{[e_4^TL_{3i}+e_{10}^TL_{4i}]e_{0i}\Big\},\end{aligned}$$

$$\begin{aligned}&\Pi_1=col\{e_1,\kappa_1e_5,\kappa_1^2e_6\},\\ &\Pi_2=col\{e_9,e_1-e_2,\kappa_1e_1-\kappa_1e_5\},\\ &\Pi_3=col\{e_1,(\kappa_2-\kappa_1)e_7,(\kappa_2-\kappa_1)^2e_8\},\\ &\Pi_4=col\{e_9,e_2-e_3,(\kappa_2-\kappa_1)e_2-(\kappa_2-\kappa_1)e_7\},\\ &b_1=e_1-e_2,b_2=e_1+e_2-2e_5,\\ &b_3=e_1-e_2+6e_5-12e_6,b_4=e_2-e_3,\\ &b_5=e_2+e_3-2e_7,b_6=e_2-e_3+6e_7-12e_8,\\ &\pi_1=col\{e_1,e_{10}\},\pi_2=col\{e_2,e_{11}\},\\ &\pi_3=col\{e_3,e_{12}\},\pi_4=col\{e_4,e_{13}\},\\ &e_{si}=\begin{bmatrix}\mathcal{W}_{11i} & \mathcal{W}_{12i}\end{bmatrix}\pi_1+\begin{bmatrix}\mathcal{W}_{d11i} & \mathcal{W}_{d12i}\end{bmatrix}\pi_4,\\ &e_{0i}=\begin{bmatrix}\mathcal{W}_{21i} & \mathcal{W}_{22i}\end{bmatrix}\pi_1+\begin{bmatrix}\mathcal{W}_{d21i} & \mathcal{W}_{d22i}\end{bmatrix}\pi_4,\\ &e_\varsigma=\begin{cases}\begin{bmatrix}0_{r\times(\varsigma-1)r} & I_r & 0_{r\times[(9-\varsigma)r+4(n-r)]}\end{bmatrix},\\ \varsigma=1,2,...,9,\\ \begin{bmatrix}0_{(n-r)\times[9r+(\varsigma-10)(n-r)]} & I_{n-r} \\ 0_{(n-r)\times[(13-\varsigma)(n-r)]}\end{bmatrix},\varsigma=10,11,...,13.\end{cases}\end{aligned}$$

Proof. *Given matrices $\mathcal{L}$ and $\mathcal{Z}$, which are nonsingular, system (1) is represented as*

$$\begin{cases}\dot{\chi}_1(t)=\begin{bmatrix}\mathcal{W}_{11i} & \mathcal{W}_{12i}\end{bmatrix}\chi(t)\\ \quad+\begin{bmatrix}\mathcal{W}_{d11i} & \mathcal{W}_{d12i}\end{bmatrix}\chi(t-\kappa(t))+\mathcal{H}_{1i}u(t),\\ 0=\begin{bmatrix}\mathcal{W}_{21i} & \mathcal{W}_{22i}\end{bmatrix}\chi(t)\\ \quad+\begin{bmatrix}\mathcal{W}_{d21i} & \mathcal{W}_{d22i}\end{bmatrix}\chi(t-\kappa(t))+\mathcal{H}_{2i}u(t),\\ y(t)=\bar{\mathcal{F}}_i\chi(t),\\ \chi(t)=\Phi(t),\forall t\in[-\kappa_2,0],\end{cases} \tag{9}$$

where

$$\begin{aligned}&\chi(t)=\mathcal{Z}^{-1}x(t)=\begin{bmatrix}\chi_1(t)\\ \chi_2(t)\end{bmatrix},\Phi(t)=\mathcal{Z}^{-1}\emptyset(t),\\ &\mathcal{L}E\mathcal{Z}=\begin{bmatrix}I_r & 0\\ 0 & 0\end{bmatrix},\mathcal{L}\mathcal{W}_i\mathcal{Z}=\begin{bmatrix}\mathcal{W}_{11i} & \mathcal{W}_{12i}\\ \mathcal{W}_{21i} & \mathcal{W}_{22i}\end{bmatrix},\\ &\mathcal{L}\mathcal{W}_{di}\mathcal{Z}=\begin{bmatrix}\mathcal{W}_{d11i} & \mathcal{W}_{d12i}\\ \mathcal{W}_{d21i} & \mathcal{W}_{d22i}\end{bmatrix},\mathcal{L}\mathcal{H}_i=\begin{bmatrix}\mathcal{H}_{1i}\\ \mathcal{H}_{2i}\end{bmatrix},\\ &\bar{\mathcal{F}}_i=\mathcal{F}_i\mathcal{Z}=[\mathcal{F}_{1i}\quad\mathcal{F}_{2i}].\end{aligned}$$

From $e_{10}\Gamma_ie_{10}^T$, we get

$$\mathcal{Q}_{1i4}+\mathcal{Q}_{2i4}+Q_{3i4}+L_{4i}\mathcal{W}_{22i}+\mathcal{W}_{22i}^TL_{4i}^T<0, \tag{10}$$

where $\begin{bmatrix}\mathcal{Q}_{ji1} & \mathcal{Q}_{ji2}\\ \star & \mathcal{Q}_{ji4}\end{bmatrix}=\mathcal{Q}_{ji},j=1,2,3$. *Noting (10) and $\mathcal{Q}_{ji}>0,j=1,2,3$,we have*

$$L_{4i}\mathcal{W}_{22i}+\mathcal{W}_{22i}^TL_{4i}^T<0.$$

Because $\mathcal{W}_{22i}$ is nonsingular, SMJS (9) with $u(t)\equiv 0$ is hence impulse free and regular. Next, think about a new stochastic LKF:

$$\bigvee(\chi_t,i,t)=\sum_{\kappa=1}^{3}\bigvee_{\kappa}(\chi_t,i,t),$$

where

$$\begin{aligned}\bigvee_1(\chi_t,i,t)=&\tau_1^T(t)\mathcal{P}_{1i}\tau_1(t)+\tau_2^T(t)\mathcal{P}_{2i}\tau_2(t),\\ \bigvee_2(\chi_t,i,t)=&\int_{t-\kappa_1}^{t}\chi^T(s)\mathcal{Q}_{1i}\chi(s)ds\\ &+\int_{t-\kappa_2}^{t}\chi^T(s)\mathcal{Q}_{2i}\chi(s)ds\\ &+\int_{t-\kappa(t)}^{t}\chi^T(s)\mathcal{Q}_{3i}\chi(s)ds,\\ &+\int_{t-\kappa_1}^{t}\int_{\beta}^{t}\chi^T(s)\mathcal{Q}_1\chi(s)dsd\beta\\ &+\int_{t-\kappa_2}^{t}\int_{\beta}^{t}\chi^T(s)\mathcal{Q}_2\chi(s)dsd\beta,\\ \bigvee_3(\chi_t,i,t)=&\kappa_1\int_{t-\kappa_1}^{t}\int_{\beta}^{t}\dot{\chi}_1^T(s)\mathcal{R}_{1i}\dot{\chi}_1(s)dsd\beta\\ &+(\kappa_2-\kappa_1)\int_{t-\kappa_2}^{t-\kappa_1}\int_{\beta}^{t}\dot{\chi}_1^T(s)\mathcal{R}_{2i}\dot{\chi}_1(s)dsd\beta\\ &+\kappa_1\int_{t-\kappa_1}^{t}\int_{\beta}^{t}\int_{v}^{t}\dot{\chi}_1^T(s)\mathcal{R}_1\dot{\chi}_1(s)dsdvd\beta\\ &+(\kappa_2-\kappa_1)\int_{t-\kappa_2}^{t-\kappa_1}\int_{\beta}^{t}\int_{v}^{t}\dot{\chi}_1^T(s)\mathcal{R}_2\dot{\chi}_1(s)dsdvd\beta\end{aligned}$$

with

$$\begin{aligned}\tau_1(t)=&col\Big\{\chi_1(t),\int_{t-\kappa_1}^{t}\chi_1(s)ds,\int_{t-\kappa_1}^{t}\int_{\theta}^{t}\chi_1(s)dsd\theta\Big\},\\ \tau_2(t)=&col\Big\{\chi_1(t),\int_{t-\kappa_2}^{t-\kappa_1}\chi_1(s)ds,\\ &\int_{t-\kappa_2}^{t-\kappa_1}\int_{\theta}^{t-\kappa_1}\chi_1(s)dsd\theta\Big\},\end{aligned}$$

and $\chi_t = \chi(t+\ell), -2\kappa_2 \le \ell \le 0$. *Assume the weak infinitesimal generator of the random process* $\{\chi_t, r_t\}$ *can be expressed as* Ł, *we get*

$$Ł\bigvee_1(\chi_t,i,t) = \Upsilon^T(t)\Big[sym\Big\{\Pi_1^T\mathcal{P}_{1i}\Pi_2\Big\} + sym\Big\{\Pi_3^T\mathcal{P}_{2i}\Pi_4\Big\} + \sum_{j=1}^{N}\pi_{ij}\Pi_1^T\mathcal{P}_{1j}\Pi_1 + \sum_{j=1}^{N}\pi_{ij}\Pi_3^T\mathcal{P}_{2j}\Pi_3\Big]\Upsilon(t). \quad (11)$$

By (2), (5), and (6), we have

$$Ł\bigvee_2(\chi_t,i,t) \le \Upsilon^T(t)\Big\{\pi_1^T\mathcal{Q}_{1i}\pi_1 - \pi_2^T\mathcal{Q}_{1i}\pi_2 + \pi_1^T\mathcal{Q}_{2i}\pi_1 - \pi_3^T\mathcal{Q}_{2i}\pi_3 + \pi_1^T\mathcal{Q}_{3i}\pi_1 - (1-\delta)\pi_4^T\mathcal{Q}_{3i}\pi_4 + \kappa_1\pi_1^T\mathcal{Q}_1\pi_1 + \kappa_2\pi_1^T\mathcal{Q}_2\pi_1\Big\}\Upsilon(t). \quad (12)$$

Via Lemma 1 and (7), it has

$$\begin{aligned}Ł\bigvee_3(\chi_t,i,t) \le \Upsilon^T(t)\Big[&\kappa_1^2e_9^T\mathcal{R}_{1i}e_9 + \kappa_{12}^2e_9^T\mathcal{R}_{2i}e_9 + \frac{\kappa_1^3}{2}e_9^T\mathcal{R}_1e_9 + \frac{\kappa_2^2-\kappa_1^2}{2}(\kappa_2-\kappa_1)e_9^T\mathcal{R}_2e_9\\ &- b_1^T\mathcal{R}_{1i}b_1 - 3b_2^T\mathcal{R}_{1i}b_2 - 5b_3^T\mathcal{R}_{1i}b_3\\ &- b_4^TR_{2i}b_4 - 3b_5^T\mathcal{R}_{2i}b_5 - 5b_6^T\mathcal{R}_{2i}b_6\Big]\Upsilon(t),\end{aligned} \quad (13)$$

where

$$\begin{aligned}\Upsilon(t) = col\Big\{&\chi_1(t), \chi_1(t-\kappa_1), \chi_1(t-\kappa_2), \chi_1(t-\kappa(t)),\\ &\frac{1}{\kappa_1}\int_{t-\kappa_1}^{t}\chi_1(s)ds, \frac{1}{\kappa_1^2}\int_{t-\kappa_1}^{t}\int_{\theta}^{t}\chi_1(s)dsd\theta,\\ &\frac{1}{\kappa_2-\kappa_1}\int_{t-\kappa_2}^{t-\kappa_1}\chi_1(s)ds,\\ &\frac{1}{(\kappa_2-\kappa_1)^2}\int_{t-\kappa_2}^{t-\kappa_1}\int_{\theta}^{t-\kappa_1}\chi_1(s)dsd\theta, \dot{\chi}_1(t),\\ &\chi_2(t), \chi_2(t-\kappa_1), \chi_2(t-\kappa_2), \chi_2(t-\kappa(t))\Big\}.\end{aligned}$$

In addition, there exist any matrices $L_{\nu i}, \nu = 1,2,3,4$ *satisfying*

$$2\Upsilon^T(t)(e_1^TL_{1i} + e_9^TL_{2i})(-e_9 + e_{si})\Upsilon(t) = 0, \quad (14)$$

$$2\Upsilon^T(t)[e_4^TL_{3i} + e_{10}^TL_{4i}]e_{0i}\Upsilon(t) = 0. \quad (15)$$

Together with (11)-(15), we get

$$Ł\bigvee(\chi_t,i,t) \le \Upsilon^T(t)\Gamma_i\Upsilon(t),$$

Thus, by (8), a scalar $\ell > 0$ *exists such that*

$$Ł\bigvee(\chi_t,i,t) \le -\ell||\chi(t)||^2.$$

Moreover, SMJS (9) with $u(t) \equiv 0$ *can be expressed as*

$$\begin{cases}\dot{\chi}_1(t) = \mathcal{W}_{11i}\chi_1(t) + \mathcal{W}_{12i}\chi_2(t) + \mathcal{W}_{d11i}\chi_1(t-\kappa(t))\\ +\mathcal{W}_{d12i}\chi_2(t-\kappa(t)),\\ -\chi_2(t) = [\mathcal{W}_{22i}]^{-1}\mathcal{W}_{21i}\chi_1(t)\\ +[\mathcal{W}_{22i}]^{-1}\mathcal{W}_{d21i}\chi_1(t-\kappa(t))\\ +[\mathcal{W}_{22i}]^{-1}\mathcal{W}_{d22i}\chi_2(t-\kappa(t)).\end{cases} \quad (16)$$

Pre-multiplying the second equation of (16) by $\chi_2^T(t)L_{4i}W_{22i}$ *and set* $e_i(t) = [\mathcal{W}_{22i}]^{-1}\mathcal{W}_{21i}\chi_1(t) + [\mathcal{W}_{22i}]^{-1}\mathcal{W}_{d21i}\chi_1(t-\kappa(t))$, *we obtain*

$$0 = 2\Big[\chi_2^T(t)L_{4i}\mathcal{W}_{22i}\chi_2(t) + \chi_2^T(t)L_{4i}\mathcal{W}_{d22i}\chi_2(t-\kappa(t)) + \chi_2^T(t)L_{4i}\mathcal{W}_{22i}e_i(t)\Big]. \quad (17)$$

Let

$$J_i(t) = \chi_2^T(t)\mathcal{Q}_{3i4}\chi_2(t) - \chi_2^T(t-\kappa(t))\mathcal{Q}_{3i4}\chi_2(t-\kappa(t)). \quad (18)$$

By (17) and (18), we get

$$\begin{aligned}J_i(t) \le\, &\hbar\chi_2^T(t)\chi_2(t) + \begin{bmatrix}\chi_2(t)\\ \chi_2(t-\kappa(t))\end{bmatrix}^T\\ &\begin{bmatrix}L_{4i}\mathcal{W}_{22i} + (L_{4i}\mathcal{W}_{22i})^T + \mathcal{Q}_{3i4} & L_{4i}\mathcal{W}_{d22i}\\ \star & -\mathcal{Q}_{3i4}\end{bmatrix}\\ &\begin{bmatrix}\chi_2(t)\\ \chi_2(t-\kappa(t))\end{bmatrix}\\ &+ \hbar^{-1}e_i^T(t)(L_{4i}\mathcal{W}_{22i})^TL_{4i}\mathcal{W}_{22i}e_i(t),\end{aligned} \quad (19)$$

where scalar $\hbar > 0$. *From* $\begin{bmatrix} e_{10}^T & e_{13}^T\end{bmatrix}^T\Gamma\begin{bmatrix} e_{10}^T & e_{13}^T\end{bmatrix}$, *it's can be deduced that*

$$\begin{bmatrix}L_{4i}\mathcal{W}_{22i} + (L_{4i}\mathcal{W}_{22i})^T + \mathcal{Q}_{3i4} & L_{4i}\mathcal{W}_{d22i}\\ \star & -\mathcal{Q}_{3i4}\end{bmatrix} < 0. \quad (20)$$

Similar to [25] and applying (16), (19), and (20), SMJS (1) under $u(t) \equiv 0$ *is MSE admissible.*

Remark 1. *For SMJSs, most of the LKFs contain both the singular matrix's non-integral and integral non-integral terms [9]–[12], [15], [16], [18], [19], [21], [22], such as a non-integral term* $\chi^T(t)E^T\mathcal{P}_iE\chi(t)$ *or* $\chi^T(t)E^T\mathcal{P}_i\chi(t)$, *and an integral term* $\int_{t-\kappa_1}^{t}\int_{\theta}^{t}\dot{\chi}^T(s)E^T\mathcal{P}_iE\dot{\chi}(s)dsd\theta$ *or* $\int_{t-\kappa_2}^{t-\kappa_1}\int_{\theta}^{t}\dot{\chi}^T(s)E^T\mathcal{P}_iE\dot{\chi}(s)dsd\theta$ *(E is a singular matrix,* $\mathcal{P}_i > 0$ *). But the LKF we constructed in this piece isn't the same as it was in the others. The primary differentiation is expressed in the elimination of several redundant decision variables using singular matrix decomposition. A portion of the computational complexity can be reduced by using the mode-dependent LKFs with state decomposition components provided, such as* $\bigvee_1(\chi_t,i,t)$ *and* $\bigvee_3(\chi_t,i,t)$.

Next, using Theorem 1, the DOF control issue for SMJSs (4) will be explored.

Theorem 2. *For some given scalars* $\kappa_1, \kappa_2, a_{1i}, a_{2i}$ *and* δ, *system (4) is MSE admissible if there are any matrices* $\bar{L}_{1i}$, Λ_{a11i}, Λ_{a12i}, Λ_{a21i}, Λ_{a22i}, Λ_{b1i}, Λ_{b2i}, Λ_{c1i}, Λ_{c2i}, Λ_{di} *with proper dimensions and* $\bar{\mathcal{Q}}_{\alpha i} > 0$, $\bar{\mathcal{P}}_{1i} > 0, \bar{\mathcal{P}}_{2i} >$

$0, \bar{\mathcal{R}}_{\beta i} > 0, \bar{\mathcal{R}}_{\beta} > 0, \alpha = 1, 2, 3, \beta = 1, 2,$ *such that*

$$\sum_{j=1}^{N} \pi_{ij}(\bar{\mathcal{Q}}_{1j} + \bar{\mathcal{Q}}_{2j} + \bar{\mathcal{Q}}_{3j}) < \bar{\mathcal{Q}}_1 + \bar{\mathcal{Q}}_2, \tag{21}$$

$$\sum_{j=1}^{N} \pi_{ij}(\bar{\mathcal{Q}}_{2j} + \bar{\mathcal{Q}}_{3j}) < \bar{\mathcal{Q}}_2, \sum_{j=1}^{N} \pi_{ij}\bar{\mathcal{Q}}_{2j} < \bar{\mathcal{Q}}_2, \tag{22}$$

$$\sum_{j=1}^{N} \pi_{ij}\bar{\mathcal{R}}_{1j} < \bar{\mathcal{R}}_1, \sum_{j=1}^{N} \pi_{ij}\bar{\mathcal{R}}_{2j} < \bar{\mathcal{R}}_2, \tag{23}$$

$$\bar{\Gamma}_i < 0, \tag{24}$$

where

$$\bar{\Gamma}_i = \hat{\Gamma}_i + sym\Big\{(b_1 + b_{17})^T a_{1i}\bar{e}_{s1i} + (b_{19} + b_7)^T a_{2i}\bar{e}_{s2i} + (b_2 + b_{18})^T \bar{L}_{1i}\bar{e}_{01i} + (b_8 + b_{20})^T \bar{e}_{02i}\Big\},$$

$$\hat{\Gamma}_i = sym\Big\{\bar{\Pi}_1^T \bar{\mathcal{P}}_{1i}\bar{\Pi}_2\Big\} + sym\Big\{\bar{\Pi}_3^T \bar{\mathcal{P}}_{2i}\bar{\Pi}_4\Big\} + \sum_{j=1}^{N}\pi_{ij}\bar{\Pi}_1^T\bar{\mathcal{P}}_{1j}\bar{\Pi}_1 + \sum_{j=1}^{N}\pi_{ij}\bar{\Pi}_3^T\bar{\mathcal{P}}_{2j}\bar{\Pi}_3 + \bar{\pi}_1^T\bar{\mathcal{Q}}_{1i}\bar{\pi}_1 - \bar{\pi}_2^T\bar{\mathcal{Q}}_{1i}\bar{\pi}_2 + \bar{\pi}_1^T\bar{\mathcal{Q}}_{2i}\bar{\pi}_1 - \bar{\pi}_3^T\bar{\mathcal{Q}}_{2i}\bar{\pi}_3 + \bar{\pi}_1^T\bar{\mathcal{Q}}_{3i}\bar{\pi}_1 - (1-\delta)\bar{\pi}_4^T\bar{\mathcal{Q}}_{3i}\bar{\pi}_4 + \kappa_1\bar{\pi}_1^T\bar{\mathcal{Q}}_1\bar{\pi}_1 + \kappa_2\bar{\pi}_1^T\bar{\mathcal{Q}}_2\bar{\pi}_1 + \frac{\kappa_1^3}{2}\bar{e}_9^T\bar{\mathcal{R}}_1\bar{e}_9 + \frac{\kappa_2^2-\kappa_1^2}{2}(\kappa_2-\kappa_1)\bar{e}_9^T\bar{\mathcal{R}}_2\bar{e}_9 + \kappa_1^2\bar{e}_9^T\bar{\mathcal{R}}_{1i}\bar{e}_9 - \bar{b}_1^T\bar{\mathcal{R}}_{1i}\bar{b}_1 - 3\bar{b}_2^T\bar{\mathcal{R}}_{1i}\bar{b}_2 - 5\bar{b}_3^T\bar{\mathcal{R}}_{1i}\bar{b}_3 + (\kappa_2-\kappa_1)^2\bar{e}_9^T\bar{\mathcal{R}}_{2i}\bar{e}_9 - \bar{b}_4^T\bar{\mathcal{R}}_{2i}\bar{b}_4 - 3\bar{b}_5^T\bar{\mathcal{R}}_{2i}\bar{b}_5 - 5\bar{b}_6^T\bar{\mathcal{R}}_{2i}\bar{b}_6,$$

$$\begin{aligned}
&\bar{e}_1 = col\{b_1, b_2\}, \bar{e}_2 = col\{b_3, b_4\},\\
&\bar{e}_3 = col\{b_5, b_6\}, \bar{e}_4 = col\{b_7, b_8\},\\
&\bar{e}_5 = col\{b_9, b_{10}\}, \bar{e}_6 = col\{b_{11}, b_{12}\},\\
&\bar{e}_7 = col\{b_{13}, b_{14}\}, \bar{e}_8 = col\{b_{15}, b_{16}\},\\
&\bar{e}_9 = col\{b_{17}, b_{18}\}, \bar{e}_{10} = col\{b_{19}, b_{20}\},,\\
&\bar{e}_{11} = col\{b_{21}, b_{22}\}, \bar{e}_{12} = col\{b_{23}, b_{24}\},\\
&\bar{e}_{13} = col\{b_{25}, b_{26}\}, \bar{\Pi}_1 = col\{\bar{e}_1, \kappa_1\bar{e}_5, \kappa_1^2\bar{e}_6\},\\
&\bar{\Pi}_2 = col\{\bar{e}_9, \bar{e}_1 - \bar{e}_2, \kappa_1\bar{e}_1 - \kappa_1\bar{e}_5\},\\
&\bar{\Pi}_3 = col\{\bar{e}_1, (\kappa_2-\kappa_1)\bar{e}_7, (\kappa_2-\kappa_1)^2\bar{e}_8\},\\
&\bar{\Pi}_4 = col\{\bar{e}_9, \bar{e}_2 - \bar{e}_3, (\kappa_2-\kappa_1)\bar{e}_1 - (\kappa_2-\kappa_1)\bar{e}_7\},\\
&\bar{\pi}_1 = col\{\bar{e}_1, \bar{e}_{10}\}, \bar{\pi}_2 = col\{\bar{e}_2, \bar{e}_{11}\},\\
&\bar{\pi}_3 = col\{\bar{e}_3, \bar{e}_{12}\}, \bar{\pi}_4 = col\{\bar{e}_4, \bar{e}_{13}\},\\
&\bar{b}_1 = \bar{e}_1 - \bar{e}_2, \bar{b}_2 = \bar{e}_1 + \bar{e}_2 - 2\bar{e}_5,\\
&\bar{b}_3 = \bar{e}_1 - \bar{e}_2 + 6\bar{e}_5 - 12\bar{e}_6, \bar{b}_4 = \bar{e}_2 - \bar{e}_3,\\
&\bar{b}_5 = \bar{e}_2 + \bar{e}_3 - 2\bar{e}_7, \bar{b}_6 = \bar{e}_2 - \bar{e}_3 + 6\bar{e}_7 - 12\bar{e}_8,
\end{aligned}$$

$$\bar{e}_{s1i} = \begin{bmatrix} \mathcal{W}_{11i} + \mathcal{H}_{1i}\Lambda_{di}\mathcal{F}_{1i} & \mathcal{W}_{12i} + \mathcal{H}_{1i}\Lambda_{di}\mathcal{F}_{2i} & \mathcal{H}_{1i}\Lambda_{c1i} & \mathcal{H}_{1i}\Lambda_{c2i} \end{bmatrix}\bar{\pi}_1 + \begin{bmatrix} \mathcal{W}_{d11i} & \mathcal{W}_{d12i} & 0 & 0 \end{bmatrix}\bar{\pi}_4 - b_{17},$$

$$\bar{e}_{s2i} = \begin{bmatrix} \mathcal{W}_{21i} + \mathcal{H}_{2i}\Lambda_{di}\mathcal{F}_{1i} & \mathcal{W}_{22i} + \mathcal{H}_{2i}\Lambda_{di}\mathcal{F}_{2i} & \mathcal{H}_{2i}\Lambda_{c1i} & \mathcal{H}_{2i}\Lambda_{c2i} \end{bmatrix}\bar{\pi}_1 + \begin{bmatrix} \mathcal{W}_{d21i} & \mathcal{W}_{d22i} & 0 & 0 \end{bmatrix}\bar{\pi}_4,$$

$$\bar{e}_{01i} = \begin{bmatrix} \Lambda_{b1i}\mathcal{F}_{1i} & \Lambda_{b1i}\mathcal{F}_{2i} & \Lambda_{a11i} & \Lambda_{a12i} \end{bmatrix}\bar{\pi}_1 - b_{18},$$

$$\bar{e}_{02i} = \begin{bmatrix} \Lambda_{b2i}\mathcal{F}_{1i} & \Lambda_{b2i}\mathcal{F}_{2i} & \Lambda_{a21i} & \Lambda_{a22i} \end{bmatrix}\bar{\pi}_1,$$

$$b_\varsigma = \begin{cases} \begin{bmatrix} 0_{r\times(\varsigma-1)r} & I_r & 0_{r\times[(18-\varsigma)r+8(n-r)]} \end{bmatrix}, \\ \varsigma = 1, 2, ..., 18, \\ \begin{bmatrix} 0_{(n-r)\times[18r+(\varsigma-19)(n-r)]} & I_{n-r} & 0_{(n-r)\times[(26-\varsigma)(n-r)]} \end{bmatrix}, \varsigma = 19, 20, ..., 26. \end{cases}$$

The DOF controller (3)'s settings are

$$\mathcal{W}_{fi} = \mathcal{L}^{-1}\begin{bmatrix} L_{1i}^{-1}\Lambda_{a11i} & L_{1i}^{-1}\Lambda_{a12i} \\ \Lambda_{a21i} & \Lambda_{a22i} \end{bmatrix}\mathcal{Z}^{-1},$$

$$\mathcal{H}_{fi} = \mathcal{L}^{-1}\begin{bmatrix} L_{1i}^{-1}\Lambda_{b1i} \\ \Lambda_{b2i} \end{bmatrix},$$

$$\mathcal{G}_{fi} = \begin{bmatrix} \Lambda_{c1i} & \Lambda_{c2i} \end{bmatrix}\mathcal{Z}^{-1}, \mathcal{J}_{fi} = \Lambda_{di}, E_f = E.$$

Proof. *Via (9), rewritten DOF controller (3) as*

$$\begin{cases} \dot{\chi}_{f1}(t) = \begin{bmatrix} \mathcal{W}_{11fi} & \mathcal{W}_{12fi} \end{bmatrix}\chi_f(t) + +\mathcal{H}_{1fi}y(t), \\ 0 = \begin{bmatrix} \mathcal{W}_{21fi} & \mathcal{W}_{22fi} \end{bmatrix}\chi_f(t) + \mathcal{H}_{2fi}y(t), \\ u(t) = \bar{\mathcal{G}}_{fi}\chi_f(t) + \mathcal{J}_{fi}y(t), \end{cases} \tag{25}$$

where

$$\chi_f(t) = \mathcal{Z}^{-1}x_f(t) = \begin{bmatrix} \chi_{f1}(t) \\ \chi_{f2}(t) \end{bmatrix},$$

$$\mathcal{L}\mathcal{W}_{fi}\mathcal{Z} = \begin{bmatrix} \mathcal{W}_{11fi} & \mathcal{W}_{12fi} \\ \mathcal{W}_{21fi} & \mathcal{W}_{22fi} \end{bmatrix},$$

$$\mathcal{L}\mathcal{H}_{fi} = \begin{bmatrix} \mathcal{H}_{1fi} \\ \mathcal{H}_{2fi} \end{bmatrix}, \bar{\mathcal{G}}_{fi} = \mathcal{G}_{fi}\mathcal{Z} = [\mathcal{G}_{1fi} \quad \mathcal{G}_{2fi}].$$

Let

$$\mathcal{M}(t) = \begin{bmatrix} \mathcal{M}_1(t) \\ \mathcal{M}_2(t) \end{bmatrix}, \mathcal{M}_1(t) = \begin{bmatrix} \chi_1(t) \\ \chi_{f1}(t) \end{bmatrix},$$

$$\mathcal{M}_2(t) = \begin{bmatrix} \chi_2(t) \\ \chi_{f2}(t) \end{bmatrix},$$

$$\mathcal{M}_{11i} = \begin{bmatrix} \mathcal{W}_{11i} + \mathcal{H}_{1i}\mathcal{J}_{fi}\mathcal{F}_{1i} & \mathcal{H}_{1i}\mathcal{G}_{1fi} \\ \mathcal{H}_{1fi}\mathcal{F}_{1i} & \mathcal{W}_{11fi} \end{bmatrix},$$

$$\tilde{\mathcal{M}}_{12i} = \begin{bmatrix} \mathcal{W}_{12i} + \mathcal{H}_{1i}\mathcal{J}_{fi}\mathcal{F}_{2i} & \mathcal{H}_{1i}\mathcal{G}_{2fi} \\ \mathcal{H}_{1fi}\mathcal{F}_{2i} & \mathcal{W}_{12fi} \end{bmatrix},$$

$$\mathcal{M}_{21i} = \begin{bmatrix} \mathcal{W}_{21i} + \mathcal{H}_{2i}\mathcal{J}_{fi}\mathcal{F}_{1i} & \mathcal{H}_{2i}\mathcal{G}_{1fi} \\ \mathcal{H}_{2fi}\mathcal{F}_{1i} & \mathcal{W}_{21fi} \end{bmatrix},$$

$$\tilde{\mathcal{M}}_{22i} = \begin{bmatrix} \mathcal{W}_{22i} + \mathcal{H}_{2i}\mathcal{J}_{fi}\mathcal{F}_{2i} & \mathcal{H}_{2i}\mathcal{G}_{2fi} \\ \mathcal{H}_{2fi}\mathcal{F}_{2i} & \mathcal{W}_{22fi} \end{bmatrix},$$

$$\mathcal{M}_{d11i} = \begin{bmatrix} \mathcal{W}_{d11i} & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{M}_{d12i} = \begin{bmatrix} \mathcal{W}_{d12i} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_{d21i} = \begin{bmatrix} \mathcal{W}_{d21i} & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{M}_{d22i} = \begin{bmatrix} \mathcal{W}_{d22i} & 0 \\ 0 & 0 \end{bmatrix}.$$

Using (9) and (25), system (4) according to the SDR approach is equal to

$$\begin{cases} \dot{\mathcal{M}}_1(t) = \begin{bmatrix} \mathcal{M}_{11i} & \mathcal{M}_{12i} \end{bmatrix}\mathcal{M}(t) \\ \quad + \begin{bmatrix} \mathcal{M}_{d11i} & \mathcal{M}_{d12i} \end{bmatrix}\mathcal{M}(t-\kappa(t)), \\ 0 = \begin{bmatrix} \mathcal{M}_{21i} & \mathcal{M}_{22i} \end{bmatrix}\mathcal{M}(t) \\ \quad + \begin{bmatrix} \mathcal{M}_{d21i} & \mathcal{M}_{d22i} \end{bmatrix}\mathcal{M}(t-\kappa(t)). \end{cases} \tag{26}$$

Next, using the following LKF and analogous to Theorem 1, it has

$$\tilde{\bigvee}(\mathcal{M}_t, i, t) = \sum_{\kappa=1}^{3}\tilde{\bigvee}_{\kappa}(\mathcal{M}_t, i, t), \tag{27}$$

where $\tilde{\bigvee}(\mathcal{M}_t, i, t)$ in (27) substitute for $\bigvee(\chi_t, i, t)$ in Theorem 1. Let $\mathcal{H}(t), \bar{\mathcal{P}}_{1i}, \bar{\mathcal{P}}_{2i}, \bar{\mathcal{Q}}_{1i}, \bar{\mathcal{Q}}_{2i}, \bar{\mathcal{Q}}_{3i}, \bar{\mathcal{Q}}_1, \bar{\mathcal{Q}}_2, \bar{\mathcal{R}}_{1i}, \bar{\mathcal{R}}_{2i}, \bar{\mathcal{R}}_1, \bar{\mathcal{R}}_2$ replace $\chi(t), \mathcal{P}_{1i}, \mathcal{P}_{2i}, \mathcal{Q}_{1i}, \mathcal{Q}_{2i}, \mathcal{Q}_{3i}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{R}_{1i}, \mathcal{R}_{2i}, \mathcal{R}_1, \mathcal{R}_2$, respectively, we get

$$Ł\tilde{\bigvee}(\mathcal{M}_t, i, t) \leq \bar{\Upsilon}^T(t)\hat{\Gamma}\bar{\Upsilon}(t), \tag{28}$$

where

$$\begin{aligned}\bar{\Upsilon}(\varrho) =col\Big\{ & \bar{\chi}_1(\varrho), \bar{\chi}_1(\varrho-\kappa_1), \bar{\chi}_1(\varrho-\kappa_2), \bar{\chi}_1(\varrho-\kappa(\varrho)),\\ & \frac{1}{\kappa_1}\int_{\varrho-\kappa_1}^{\varrho}\bar{\chi}_1(s)ds, \frac{1}{\kappa_1^2}\int_{\varrho-\kappa_1}^{\varrho}\int_{\theta}^{\varrho}\bar{\chi}_1(s)dsd\theta,\\ & \frac{1}{\kappa_2-\kappa_1}\int_{\varrho-\kappa_2}^{\varrho-\kappa_1}\bar{\chi}_1(s)ds,\\ & \frac{1}{(\kappa_2-\kappa_1)^2}\int_{\varrho-\kappa_2}^{\varrho-\kappa_1}\int_{\theta}^{\varrho-\kappa_1}\bar{\chi}_1(s)dsd\theta, \dot{\bar{\chi}}_1(\varrho),\\ & \bar{\chi}_2(\varrho), \bar{\chi}_2(\varrho-\kappa_1), \bar{\chi}_2(\varrho-\kappa_2), \bar{\chi}_2(\varrho-\kappa(\varrho))\Big\}.\end{aligned}$$

Applying (26), for any given $\bar{L}_{1i}$, assume $\bar{L}_{1i}\mathcal{W}_{11fi} = \Lambda_{a11i}, \bar{L}_{1i}\mathcal{W}_{12fi} = \Lambda_{a12i}, \mathcal{A}_{21fi} = \Lambda_{a21i}, \mathcal{W}_{22fi} = \Lambda_{a22i}, \bar{L}_{1i}\mathcal{H}_{1fi} = \Lambda_{b1i}, \mathcal{H}_{2fi} = \Lambda_{b2i}, \mathcal{G}_{1fi} = \Lambda_{c1i}, \mathcal{G}_{2fi} = \Lambda_{c2i}, \mathcal{J}_{fi} = \Lambda_{di}$, there exist two scalars $a_{1i} > 0, a_{2i} > 0$ such that

$$\bar{\Upsilon}^T(t)[sym\{(b_1+b_{17})^T a_{1i}\bar{e}_{s1i}\}]\bar{\Upsilon}(t) = 0, \tag{29}$$
$$\bar{\Upsilon}^T(t)[sym\{(b_{19}+b_7)^T a_{2i}\bar{e}_{s2i}\}]\bar{\Upsilon}(t) = 0, \tag{30}$$
$$\bar{\Upsilon}^T(t)[sym\{(b_2+b_{18})^T \bar{L}_{1i}\bar{e}_{01i}\}]\bar{\Upsilon}(t) = 0, \tag{31}$$
$$\bar{\Upsilon}^T(t)[sym\{(b_8+b_{20})^T \bar{e}_{02i}\}]\bar{\Upsilon}(t) = 0. \tag{32}$$

Together with (29)–(32), we have

$$Ł\tilde{\bigvee}(\mathcal{M}_t, i, t) \leq \bar{\Upsilon}^T(t)\bar{\Gamma}\bar{\Upsilon}(t), \tag{33}$$

From (33) and similar to Theorem 1, we see system (4) is MSE admissible.

Remark 2. *First, for SMJSs, the DOF control issue was researched in [18], [19], [21] by the LMI approach, but the influence of time–delay on the performance of the SMJSs was not considered. This undoubtedly limits the application of the results they obtain in practice. Sun et al. [24] considered the H_∞ DOF control for time-varying SMJSs. But the admissible criteria are independent of the time–delay information, which makes the application scope of the results very narrow. Fortunately, to sum up, this paper complements these gaps for SMJSs. Second, through the SDR method, formulas (29)–(32) are derived, and system (4) can be changed into system (26). Interestingly, by varying the settings, we can efficiently and flexibly acquire the gain matrix components of the controller. Furthermore, it is easy to show that the properties of the equation solution are unchanged by this method, which just modifies the order of each sub-equation. It follows that this suggested approach is both workable and efficient.*

Remark 3. *For uncertain SMJSs with time–varying delays, the robust DOF stabilization was studied in [17]. Some admissible criteria with nonlinear matrix inequalities were derived, which brings some challenges to the verification of the results presented. However, in this paper, we can establish criteria in view of strict LMI conditions. Furthermore, we build the relevant augmented LKF using Theorem 1. Additionally, we build the zero-equation using the properties of each subsystem and use free matrix method to search for each DOF controller decomposition component separately in order to acquire the relevant DOF controller parameters. In contrast to earlier work [18], [19], [21], [24], we address individual DOF controller components as opposed to the entire system. Processing can lessen conservatism and improve flexibility in this way.*

IV. Numerical Examples

Example 1. *The following parameters are present in system (1) with $u(t) \equiv 0$:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{W}_1 = \begin{bmatrix} -2.5 & 2.1 \\ 2.2 & 5.6 \end{bmatrix},$$
$$\mathcal{W}_{d1} = \begin{bmatrix} 2.2 & 1.3 \\ 2.5 & -1.2 \end{bmatrix}, \mathcal{W}_2 = \begin{bmatrix} a & 2.2 \\ 4.2 & 7.3 \end{bmatrix},$$
$$\mathcal{W}_{d2} = \begin{bmatrix} 2.1 & 1.2 \\ 1.6 & -1.4 \end{bmatrix},$$

where a is a given scalar.

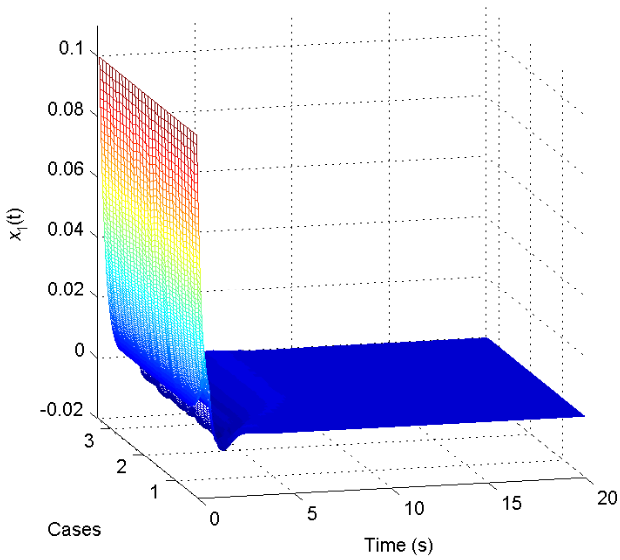

Fig. 1: State trajectories of $x_1(t)$.

TABLE I: Comparison of the results' feasibility

Methods	[25]	Theorem 1
$a = -0.5$	Feasible	Feasible
$a = -2.5$	Infeasible	Feasible
NDVs($n = 2, r = 1$)	71	62

Let $\kappa_1 = 0.3, \kappa_2 = 3.35$, $\mathcal{L} = \mathcal{Z} = I$ and $\prod = \begin{bmatrix} -0.2 & 0.2 \\ 0.4 & -0.4 \end{bmatrix}$. The feasibility comparison of the conclusions in [25] and Theorem 1 for different scalar $a \in \{-0.5, -2.5\}$ may be shown in Table I. Applying Theorem 1, we discover that our outcomes surpass those in [25]. It is also important to note that our results have fewer number

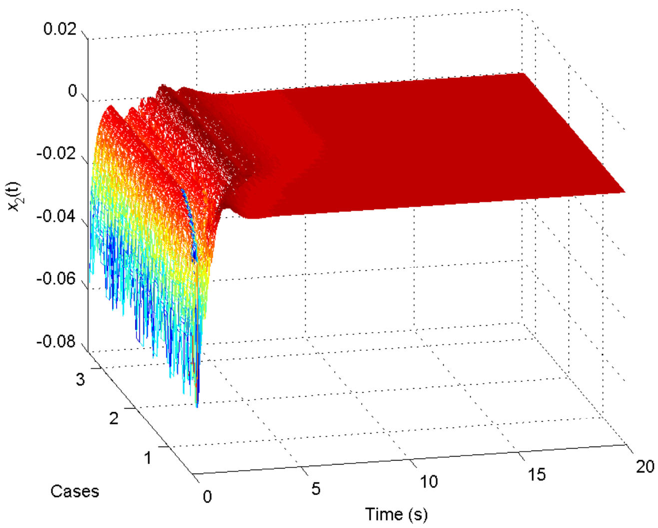

Fig. 2: State trajectories of $x_2(t)$.

of decision variables (NDVs) than those in [25]. These comparisons demonstrate that our approach is better in some situations and that our solutions have less conservatism and computational complexity.

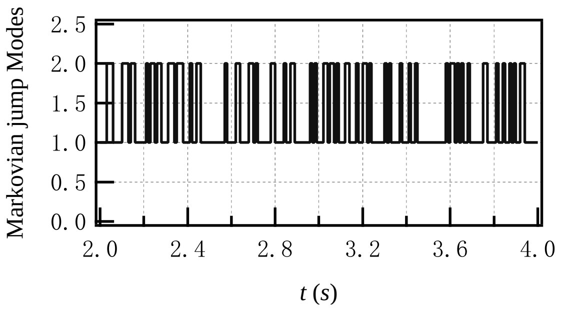

Fig. 3: Markovian jump models when $\kappa_1 = 0.3, \kappa_2 = 3.35$ in Example 1.

Furthermore, we apply 2000 sets of randomly selected examples with time-varying delays $\kappa(t)$ that satisfy an interval range $\kappa(t) \in [0.3, 3.35]$. In certain instances, the systems are asymptotically stable according to the state trajectories shown in Figs. 1 and 2. Simultaneously, we provide in Fig. 3 the Markovian jump modes scenario in part time $t \in [2, 4]$.

Example 2. There is a DC motor model available [27], which can be written as

$$\begin{cases} V_a(t) = R_a i_a(t) + K_b \omega_m(t) + L_a \frac{di_a(t)}{dt}, \\ \frac{d\omega_m(t)}{dt} = \frac{K_t}{J_m} i_a(t) - \frac{B_w}{J_m}\omega_m(t) + \frac{T_L(t)}{J_m}, \end{cases} \tag{34}$$

where K_t, K_b, L_a, R_a denote the torque constant, the back-EMF constant, inductance and the armature resistance, respectively. $T_L(t), i_a(t), \omega_m(t), V_a(t)$ are the unknown load torque, the armature current, the rotor angular velocity and the input voltage, respectively. When [27] is treated similarly and the effect of time delay is taken into account, system (34) becomes

$$E\dot{x}(t) = \mathcal{W}x(t) + \mathcal{W}_d x(t - \kappa(t)) + \mathcal{H}u(t), \tag{35}$$

where $\mathcal{W}, \mathcal{W}_d$ and $\mathcal{H}$ are given matrices. E is a singular matrix. Using the same processing technique as in [19] and taking into account that the load shift is abrupt and erratic, the system (35) can be recast as

TABLE II: Comparison of the results' feasibility

Methods	[18], [19], [21]	[24]	Theorem 2
Delay category	No time–delay	Time-varying	Time-varying
TSI	Independent	Independent	Dependent
Criteria	Infeasible	Feasible	Feasible
$\kappa_1 = 0.1$	–	–	5.0357
$\kappa_1 = 0.3$	–	–	5.0410

$$\begin{cases} E\dot{x}(t) = \mathcal{W}(r_t)x(t) + \mathcal{W}_d(r_t)x(t - \kappa(t)) + \mathcal{H}(r_t)u(t), \\ y(t) = \mathcal{F}(r_t)x(t), \end{cases} \tag{36}$$

where $\mathcal{W}(r_t), \mathcal{W}_d(r_t), \mathcal{H}(r_t), \mathcal{F}(r_t)$ refer to system (1). Let the subsequent matrix parameters to be [19]:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathcal{W}_1 = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix},$$

$$\mathcal{W}_{d1} = \begin{bmatrix} -0.3649 & 0.6192 \\ 0.4381 & 0.042 \end{bmatrix}, \mathcal{W}_2 = \begin{bmatrix} -1.7 & 1.5 \\ 1 & 1 \end{bmatrix},$$

$$\mathcal{W}_{d2} = \begin{bmatrix} -0.9503 & -1.1842 \\ 0.0672 & 0.3443 \end{bmatrix}, \mathcal{H}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$\mathcal{H}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \mathcal{F}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, \mathcal{F}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T,$$

$$\prod = \begin{bmatrix} -0.0193 & 0.0193 \\ 0.0307 & -0.0307 \end{bmatrix}.$$

Choosing $a_{11} = 4.57, a_{12} = 0.55, a_{21} = -0.05, a_{12} = -0.45, \delta = 0.1$ and $\mathcal{L} = \mathcal{Z} = I$, Table II displays the feasibility comparison between [18], [19], [21], [24] and Theorem 2 ("TSI" means time–delay size information). It is evident that Theorem 2's calculations yield superior results to [18], [19], [21]'s results in the absence of time variation. Although the results in [24] are time-varying and feasible, they are independent of the TSI. Thus, our results are more advantageous. Furthermore, when choosing $\mathcal{W}_{d1} = \mathcal{W}_{d2} = 0$, the system (36) can degenerate into the case studied in [18] and, obviously, our conclusion is still feasible by Theorem 2, which shows that our results generalize partly the conclusion of [18]. We select $\kappa_1 = 0.3, \kappa_2 = 2$ to demonstrate the effectiveness and viability of the DOF controller we designed. Using Matlab's LMI toolbox and Theorem 2, we can obtain the DOF controller as follow:

$$\mathcal{W}_{f1} = \begin{bmatrix} 3.3852 \cdot 10^{-7} & 1.8388 \cdot 10^{-7} \\ -0.0485 & -0.0609 \end{bmatrix},$$

$$\mathcal{W}_{f2} = \begin{bmatrix} 1.0545 \cdot 10^{-5} & 1.3381 \cdot 10^{-5} \\ -0.5387 & -0.8471 \end{bmatrix},$$

$$\mathcal{H}_{f1} = \begin{bmatrix} -5.6399 \cdot 10^{-4} \\ 0.2068 \end{bmatrix}, \mathcal{H}_{f2} = \begin{bmatrix} -8.7767 \cdot 10^{-5} \\ 0.2719 \end{bmatrix},$$

$$\mathcal{G}_{f1} = \begin{bmatrix} -18.2276 & -1.0007 \end{bmatrix}, \mathcal{J}_{f1} = -1.4352,$$

$$\mathcal{G}_{f2} = \begin{bmatrix} -1.2108 & -1.5413 \end{bmatrix}, \mathcal{J}_{f2} = 1.4851.$$

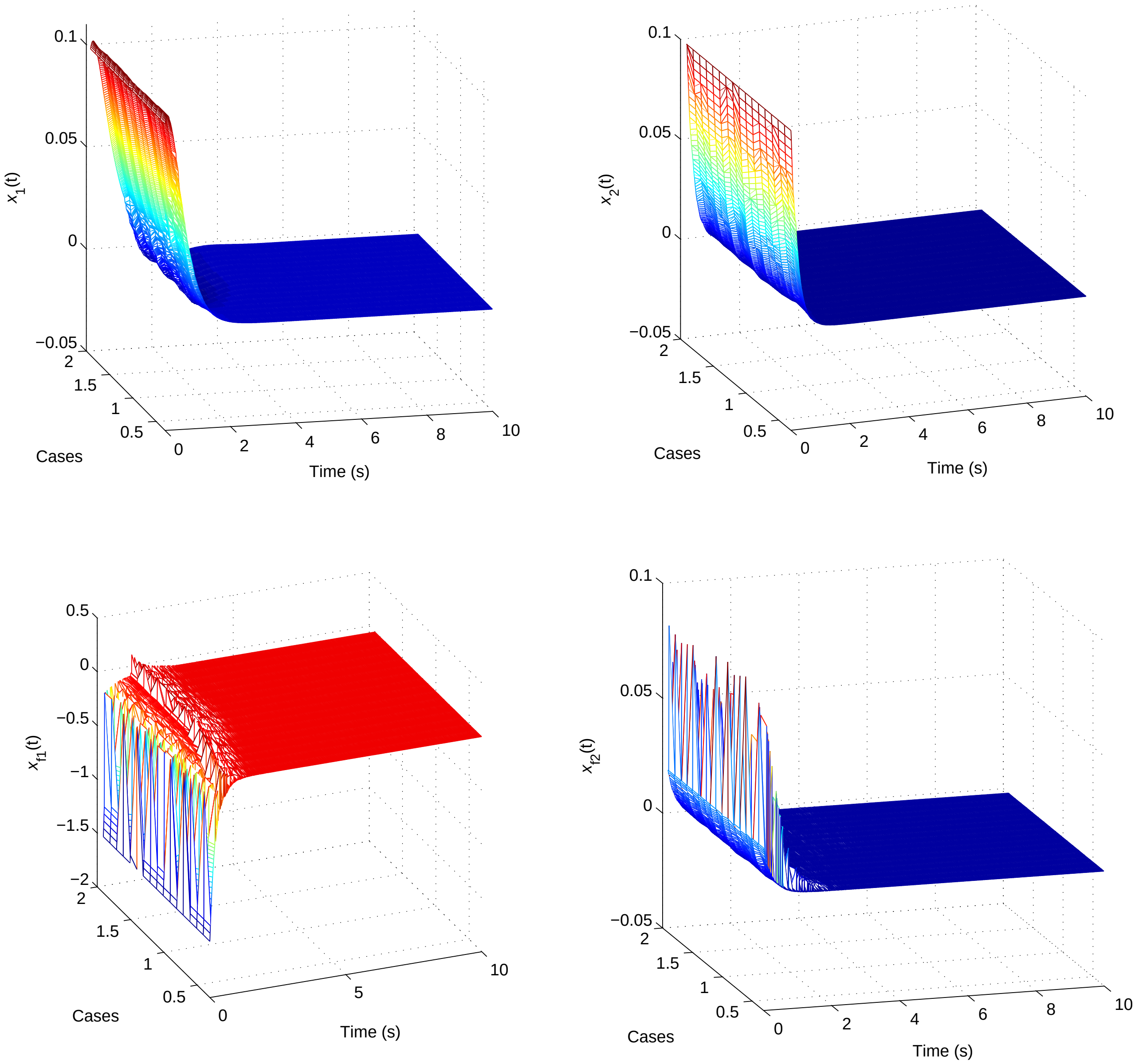

Fig. 4: State trajectories of the closed–loop (4)

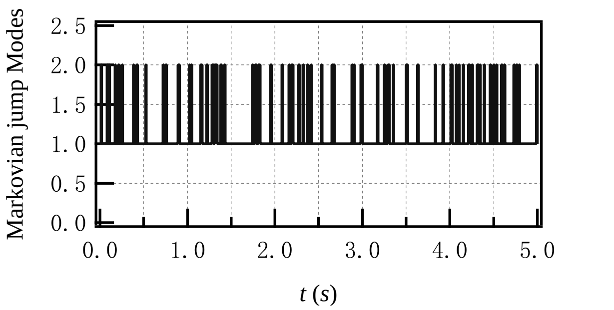

Fig. 5: Markovian jump models when $\kappa_1 = 0.3, \kappa_2 = 2$ in Example 2.

Additionally, we put into practice 1000 sets of randomly selected situations with time-varying delays $\kappa(t)$ that satisfy a range of intervals $\kappa(t) \in [0.3, 2]$ for simulation tests. The state trajectories in Fig. 4 demonstrate the asymptotic stability of the closed-loop systems with respect to MSE in those circumstances. In the meantime, we provide the Markovian jump modes scenario in part time $t \in [0, 5]$ in Fig. 5.

V. Conclusions

Sufficient thought has been given to the DOF control problem for SMJSs. First, the MSE admissible criterion for the open–loop SMJSs has been created, and certain adequate requirements, such as a small number of decision variables comparatively, have been obtained by creating a new state decomposed LKF. Second, the DOF controller is designed using the SDR approach, and the established MSE admissible results are used to determine the MSE admissible criterion. It is also possible to precisely determine the desired DOF controller settings by solving each parameter decomposition component of the DOF controller. Lastly, a numerical analysis shows how well the suggested approach enhances the findings of earlier studies. The SDR method is also theoretically important for the study of singular horse-hopping systems with variable probability and will be implemented in the future.

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