Inventory Systems with Present Value and Credit Period

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***Abstract*—We examined the article of Chung and Huang to show that their challenge of Chung is questionable. Chung and Huang extended a paper of Chung with four inventory models with credit period and present value. Chung and Huang claimed that they developed new theoretical results to locate an approximated lower bound for the optimal solution. Moreover, Chung and Huang mentioned that they proved that the objective functions are convex up such that the minimum solution exists and is unique. In this study, we showed that the convex properties of this kind of inventory models already proved by Rachamadugu. On the other hand, their proposed approximated lower bound is not workable for the starting point of the Newton-Raphson algorithm. Consequently, their new theoretical findings of convex properties and a new approximated lower bound are not useful for academic society. Our examinations will help researchers realize these two paper of Chung, and Chung and Huang.**

***Index Terms*—Approximated solution, Present value, Inventory systems, Trade credit**

I. Introduction

THIS paper is a response to the article of Chung and Huang [1] which tried to improve the paper of Chung [2]. Chung and Huang [1] claimed that four objective functions are convex, to derive a new lower bound for the optimal cycle time and then pointed out the relative errors for the inventory cycle time based on the approximated solution proposed by Chung [2] may up to 40%. The goal of this paper is threefold. First, we point out the convex property already derived by Rachamadugu [3]. Second, for the same numerical examples, we check the relative errors for the objective functions to reveal that are not significant with average of 0.3% and maximum 2.8% to point out they estimation for the relative errors for the inventory cycle time which contained questionable results. Third, according to their application of the Newton-Raphson method to find the optimal cycle time, we show that their new lower bound is useless. Our finding will improve the understanding of future cash outflows in inventory models and financial credit environment. Chung and Huang [1] has been referred to by several articles. For example, Chang et al. [4] indicated that some researchers have noticed Chung and Huang [1]. However, Chang et al. [4] have cited Chung and Huang [1] in their Introduction and then these Chang et al. [4] did not provide further comments for Chung and Huang [1]. Other researchers also paid attention to these kind inventory systems. Hou and Lin [5] examined inventory models with a finite planning horizon under time value, the discounted cash flow, selling price, and inflation to locate the optimal economic order quantity.Sana et al. [6] constructed a brand new inventory model with imperfect quality products to consider out-of-control production procedure with deteriorated items.Underan advance purchase discount, You and Su [7] studied a traditional inventory system to maximize the average profit between manufacturing system and retailers.Nobilet al. [8] considered an economic production quantity model with a single machine and several items under budget restriction, combined manufacture policy, and discrete delivery arrangement.Hence, we can claim this paper is an important article to present a detailed examination for Chung and Huang [1].

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II. Assumptions and Notation

We use the same assumptions and notation of Chung and Huang [1] and Chung [2].

Notation

p is the price per unit.
M denotes the credit period
T denotes the inventory cycle time, a decision variable.
S is the setup cost.
r is the discount rate.
K is the ordering cost.
h is the inventory holding cost, exclusive of capital charges.
D is the demand rate.

Assumptions

There are four inventory models in this paper.

The first model, denoted as $PV_1(T)$, is the classical inventory system with future cash depletions, which is denoted as model $(C1)$.

The second model, denoted as $PV_2(T)$, under the assumption that the inventory cycle time is less than the credit period, which is denoted as model $(C2)$.

The third model, denoted as $PV_3(T)$, is under the condition that the inventory cycle time is longer than the credit period, which is denoted as model $(C3)$.

The fourth model, denoted as $PV_4(T)$, assumed a fixed credit, which is denoted as model $(C4)$.

III. Review of Related Inventory Models

Chung [2] examined the discounted cash flows approach for the analysis of inventory policy in the presence of trade credit to construct the following four inventory systems:
(C1) Instantaneous cash flows: the model of the basic EOQ model;
(C2) Credit only on units in stock when $T \le M$;
(C3) Credit only on units in stock when $T \ge M$;
(C4) Fixed credit.

For model (C1), Chung [2] derived that the present value function of all future cash outflows,

$$PV_1(T)=\frac{(r+h)CDT+rK}{r\left(1-e^{-rT}\right)}-\frac{hCD}{r^2}, \quad (3.1)$$

and then assumed that

$$T_{1C}=\sqrt{\frac{2K}{(h+r)CD}}, \quad (3.2)$$

as the optimal cycle time, by the following approximation for the exponential function,

$$e^x \approx 1+x+\frac{x^2}{2}. \quad (3.3)$$

For model (C2), Chung [2] derived that the present value function of all future cash outflows, for $T \le M$,

$$PV_2(T)=\frac{rK+hCDT}{r\left(1-e^{-rT}\right)}+\frac{(r-h)CD}{r^2}, \quad (3.4)$$

and then assumed that

$$T_{2C}=\sqrt{\frac{2K}{hCD}}, \quad (3.5)$$

as the optimal cycle time

For model (C3), Chung [2] derived that the present value function of all future cash outflows, for $T \ge M$,

$$\mathrm{PV}_3(\mathrm{T})=\frac{-\mathrm{DCh}}{\mathrm{r}^2}, + \frac{(1-e^{-rM}-e^{-rM}rM)DC+Kr+TDC(e^{-rM}r+h)}{(1-e^{-rT})r}, \quad (3.6)$$

and then assumed that

$$T_{3C}=\sqrt{\frac{2\left[K+\left(1-e^{-rM}-rMe^{-rM}\right)(CD/r)\right]}{\left(h+re^{-rM}\right)CD}}, \quad (3.7)$$

as the optimal cycle time.

For model (C4), Chung [2] derived that the present value function of all future cash outflows,

$$PV_4(T)=\frac{hCDT+rK+rTCDe^{-rM}}{r\left(1-e^{-rT}\right)}-\frac{hCD}{r^2}, \quad (3.8)$$

and then assumed that

$$T_{4C}=\sqrt{\frac{2K}{\left(h+re^{-rM}\right)CD}}, \quad (3.9)$$

as the optimal cycle time.

Based on those results of Chung [2], Chung and Huang [1] first extended the domain of $\mathrm{PV}_2(\mathrm{T})$ and $\mathrm{PV}_2(\mathrm{T})$ to $\mathrm{T}>0$, and then they develop the following results.

Theorem 1 of Chung and Huang [1].
$\mathrm{PV}_i(\mathrm{T})$ is convex for $\mathrm{T}>0$, with i=1,2,3, and 4.
Before proving their Theorem 1, Chung and Huang [1] needed the next lemma.

Lemma 1 of Chung and Huang [1].
(a) $1/(1-e^{-rT})$ is convex for $T>0$.
(b) $T/(1-e^{-rT})$ is convex for $T>0$.

Chung and Huang [1] rewrote the objective functions of equations (3.1), (3.4), (3.6), and (3.8) as follows,

$$\mathrm{PV}_1(\mathrm{T})=\frac{-\mathrm{DCh}}{\mathrm{r}^2}+\frac{\mathrm{K}}{1-e^{-rT}} + \left(\frac{T}{1-e^{-rT}}\right)\left(\frac{DC(h+r)}{r}\right), \quad (3.10)$$

$$\mathrm{PV}_2(\mathrm{T})=\frac{\mathrm{DC}(r-h)}{\mathrm{r}^2}+\frac{\mathrm{K}}{1-e^{-rT}} + \left(\frac{T}{1-e^{-rT}}\right)\left(\frac{DCh}{r}\right), \quad (3.11)$$

$$\mathrm{PV}_3(\mathrm{T})=\frac{-\mathrm{DCh}}{\mathrm{r}^2}+\left(\frac{T}{1-e^{-rT}}\right)\left(\frac{DC(e^{-rM}r+h)}{r}\right), + \frac{(1-e^{-rM}-e^{-rM}rM)DC+Kr}{(1-e^{-rT})r}, \quad (3.12)$$

and

$$\mathrm{PV}_4(\mathrm{T})=\frac{-\mathrm{DCh}}{\mathrm{r}^2}+\frac{\mathrm{K}}{1-e^{-rT}} + \left(\frac{T}{1-e^{-rT}}\right)\left(\frac{DC(e^{-rM}r+h)}{r}\right), \quad (3.13)$$

so that Chung and Huang [1] can apply their Lemma 1 to prove the convex property of $\mathrm{PV}_i(\mathrm{T})$, for i=1,2,3, and 4.
After deriving the convex property, Chung and Huang [1] considered the comparison of the exact cycle time and the cycle time proposed by Chung [2]. Following the same arguments as in Rachamadugu [3], with

$$T_{iC}=\frac{1}{r}\left[2\left(e^{rT_i^*}-1-rT_i^*\right)\right]^{1/2}, \quad (3.14)$$

Chung and Huang [1] found that
(i) T_{iC} is an upper bound, that is,

$$T_i^* \le T_{iC}, \quad (3.15)$$

for i=1,2,3, and 4.
(ii) a new lower bound, where $\mathrm{LT_i} \le \mathrm{T_i^*}$, with

$$\mathrm{LT_i}=\frac{rT_{iC}^2}{\left[2\left(e^{rT_{iC}}-1-rT_{iC}\right)\right]^{0.5}}, \quad (3.16)$$

for i=1,2,3, and 4, respectively.
Chung and Huang [1] mentioned that they used the Newton-Raphson method to locate the optimal cycle time, T_1^*, and then used

$$\frac{T_{iC}-T_i^*}{T_i^*} \quad (3.17)$$

to compute the relative errors for the cycle time.

Chung and Huang [1] have run a detailed numerical examples to indicate that when the ordering cost, K, increases, then the relative errors also increases.

IV. Our Revisions

Now, we begin to discuss our first revision to show that the convex property proposed by Chung and Huang [1] had been verify by Rachamadugu [3].

We recall the inventory model to discount all future costs in Rachamadugu [3], then the objective function, ANN(T), is denoted as

$$\mathrm{ANN(T)} = \frac{-\mathrm{Dh}}{\mathrm{r}} + \frac{\mathrm{rS}}{1-\mathrm{e}^{-\mathrm{rT}}} + \frac{\mathrm{T}}{1-\mathrm{e}^{-\mathrm{rT}}}(\mathrm{h}+\mathrm{rp})\mathrm{D}, \quad (4.1)$$

where S is the setup cost; r is the discount rate; T is the reorder interval (decision variable); D is the demand rate; p is the price per unit; h is the inventory holding cost, exclusive of capital charges.

If we carefully examine the objective functions of equations (3.1), (3.4), (3.6), (3.8) and (4.1), then we may abstractly rewrite those five objective functions as follows

$$f(T) = a\frac{T}{1-e^{-rT}} + b\frac{1}{1-e^{-rT}} + c, \quad (4.2)$$

with a, b and c are constant, with respect to different inventory models, respectively.

Moreover, in Rachamadugu [3], he already proved that ANN(T) is convex. Consequently, if Chung and Huang [1] really understand the results of Rachamadugu [3], then they should realize that their Lemma 1 and Theorem 1 can be directly imply by the findings of Rachamadugu [3].

Consequently, we can claim that Lemma 1 and Theorem 1 of Chung and Huang [1] are redundant.

Next, we consider the improvement demonstrated by Chung and Huang [1] to derive the relative errors between the approximated solution, T_{iC} (un upper bound) proposed by Chung [2] and the optimal solution, T_i^* as computed in equation (3.16). It is strange to prepare so detailed study for the relative errors for cycle time so that in Chung and Huang [1], there are 48 examples to compare the relative errors between T_{iC} and T_i^*.

Intuitively, the costs, not the cycle times, are concerned by decision makers. On the other hand, it has been experimented and documented that the cost is more robust than the cycle time. For example, in a traditional textbook, the relative error of cycle time is 50%, but the relative error of cost is 4%. Therefore, we begin to consider the relative error of two costs: (a) based on the upped bound proposed by Chung [2], and (b) based on the optimal cycle time.

First, we examine Table 3 of Chung and Huang [1] in the following Table 1 with $i = 1$, $R = 0.2$, $h = 0.3$, $M = 0.75$, $D = 15$, and $C = 1$. Moreover, in Table 3 of Chung and Huang [1], the abbreviation, denoted as (1), is defined as

$$(1) = (T_{1C} - T_1^*)/T_1^*, \quad (4.3)$$

and the other abbreviation, denoted as (2), is defined as

$$(2) = (T_1^* - T_L)/T_1^*, \quad (4.4)$$

If we carefully examine the results of T_{iC} in the fifth column, for example, for model (C1), then we discover the following Table 2.

Based on Table 2, we derive that

$$\frac{T_{1C}(K=1)}{T_{1C}(K=0.1)} = \frac{0.516398}{0.163299} = \sqrt{10}, \quad (4.5)$$

$$\frac{T_{1C}(K=10)}{T_{1C}(K=1)} = \frac{1.632993}{0.516398} = \sqrt{10}, \quad (4.6)$$

and

$$\frac{T_{1C}(K=100)}{T_{1C}(K=10)} = \frac{5.163978}{1.632993} = \sqrt{10}. \quad (4.7)$$

The results of Equations (4.5-4.7) can be directly derive, because the definition of T_{1C} in Equation (4.2).

Moreover, the more trivial relation appeared in Table 2. We point out that

$$\frac{T_{1C}(K=10)}{T_{1C}(K=1)} = \frac{1.632993}{0.163299} = 10, \quad (4.8)$$

and

$$\frac{T_{1C}(K=100)}{T_{1C}(K=1)} = \frac{5.163978}{0.516398} = 10. \quad (4.9)$$

Similarly, the findings of Equations (4.8-4.9) are trivial, owing to the definition of T_{1C} in Equation (4.2).

On the other hand, we evaluate that

$$\frac{T_L(K=1)}{T_L(K=0.1)} = \frac{0.507509}{0.162410} = 3.125, \quad (4.10)$$

$$\frac{T_L(K=10)}{T_L(K=1)} = \frac{1.544158}{0.507509} = 3.043, \quad (4.11)$$

and

$$\frac{T_L(K=100)}{T_L(K=10)} = \frac{4.280767}{1.544158} = 2.772. \quad (4.12)$$

The ratios of Equations (4.10-4.12) are all less than

$$\sqrt{10} = 3.162. \quad (4.13)$$

The results of Equations (4.10-4.12) also can be directly predicted, because based on Equation (3.16), we show that

$$LT_i = T_{ic}/\sqrt{1 + (rT_{ic}/3) + (r^2T_{ic}^2/12)+\ldots} \quad (4.14)$$

If T_{ic} increases at the rate of $\sqrt{10}$, then the denominator of Equation (4.14) will be bigger than 1 more far away to result is the ratio of Equations (4.10-4.12) more and more less than $\sqrt{10}$ of Equation (4.13).

Based on our above discovery, those sensitivity findings can be predicted by analytical observations such that we can claim that the sensitivity analysis of Chung and Huang [1] with respect to K to vary its value among 0.1, 1, 10, and 100 are redundant.

It reveals that, when the magnitude of K become 10 times, then the magnitude of T_{1C} become $\sqrt{10}$ times. It is consistent with equation (3.2). It also implicitly influences the magnitude of T_i^* to increase around $\sqrt{10}$ times. Consequently, the relative error between T_{1C} and T_i^* will also increase around $\sqrt{10}$ times to imply that the increasing of the sixth column in Table 1 around $\sqrt{10}$ times that can be predicted. Therefore, we can claim that Table 3 of Chung and Huang [1] did not provide any useful information for sensitivity analysis of K. Hence, we conclude that Table 3 of Chung and Huang [1] is redundant. For completeness, we run a detailed sensitivity analysis with respect to $PV_1(LT_1)$, $PV_1(T_1^*)$, $PV_1(T_{1C})$, $PV_2(LT_2)$, $PV_2(T_2^*)$, $PV_2(T_{2C})$, $PV_3(LT_3)$, $PV_3(T_3^*)$, $PV_3(T_{3C})$, $PV_4(LT_4)$, $PV_4(T_4^*)$, and $PV_4(T_{4C})$, and list them in the Table 3.

V. A Future Study

In the future, we suggest researchers reconsider the numerical examples 1 and 2 of Chung and Huang [1], and then the relative error of costs are computed as

$$\frac{PV_i(T_{iC}) - PV_i(T_i^*)}{PV_i(T_i^*)}, \quad (5.1)$$

for i=1,2,3 and 4, respectively, and then listed them in the future tables, such that the first table for Example 1, and the second table for Example 2. We believe that those computation results will be interesting for further examinations. We can predict that will reveal the costs are more robust than the cycle time. Therefore, Chung and Huang [1] considered the relative errors between T_{iC} and T_i^* may be exaggerated the effects of the influence of the approximated solution, T_{iC}, proposed by Chung [2]. Those future discussion based on objective function, present value function of costs, will be the proper measure to reflect that T_{iC} still is a good approximation for the optimal solution, T_i^*.

Moreover, we point out that it is valuable to examine the effect of the new lower bound, T_L, of equation (3.14), proposed by Chung and Huang [1]. They mentioned that they will apply the Newton-Raphson method to find the optimal solution. It indicates that they need a starting point, say x_0, to execute the following iterative process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{5.2}$$

with

$$f(x) = e^x - 1 - x - \alpha, \tag{5.3}$$

under the relation of variable,

$$rT = x, \tag{5.4}$$

where α denotes a constant that corresponds to the four models of trade credit,

$$\frac{r^2 K}{(r+h)CD}, \tag{5.5}$$

$$\frac{r^2 K}{hCD}, \tag{5.6}$$

$$\frac{r\left[rK + CD\left(1 - e^{-rM} - rMe^{-rM}\right)\right]}{\left(re^{-rM} + h\right)CD}, \tag{5.7}$$

and

$$\frac{r^2 K}{\left(re^{-rM} + h\right)CD}, \tag{5.8}$$

for models (C1), (C2), (C3), and (C4), respectively.

Chung and Huang [1] did not tell us how they will select the starting point, x_0. In their paper, there are only two formulated approximations, the upper bound, T_{iC}, proposed by Chung [2], and a lower bound, T_L, proposed by Chung and Huang [1]. Hence, T_{iC} and T_L are the only two choice for the starting point. If researchers select rT_L as the starting point to run the Newton-Raphson method in the future, and then we predict that x_1 will be a point far away from the desire solution, since $f(x) = e^x - 1 - x - \alpha$ of Equation (5.3) is a convex and increasing function. Moreover, we assumed that the sequence $(x_n)_{n\geq 1}$ will converge to rT_i^*. On the other hand, if researchers choose rT_{iC} as x_0 and then we predict that it will yield a decreasing sequence converge to rT_i^*.

To clearly explain our prediction, we believe that two examples of Newton-Raphson method with starting point: (a) rT_L, and (b) rT_{iC} in the future which will be useful to reveal that to take rT_L as the starting point to execute the Newton-Raphson method is a bad choose. We will predict that the future studies will point out that if Chung and Huang [1] really tried to run the Newton-Raphson method, then the upper bound, T_{iC}, proposed by Chung [2], is a good formulated one. Their new derivation, the lower bound, T_L, is useless for the Newton-Raphson method. Based on our above predictions, it will reveal that the new lower bound, T_L, proposed by Chung and Huang [1] is redundant.

VI. A RELATED PROBLEM

We study a related inventory model that was considered by Wu and Ouyang [9] and Tuan et al. [10] with defective items and partial item inspection.

To be compatible with the results of Wu and Ouyang [9], we apply the same assumptions and notation as them in the following.

X is the lead time demand with finite mean μL and standard deviation $\sigma\sqrt{L}$ for lead time L.

L is the length of the lead time, a decision variable.

Q is the lot size (order quantity), a decision variable.

f is the proportion of order quantity which was inspected.

p is the defective rate in an order lot (independent of lot size), $0 \leq p < 1$ and it is a random variable.

g(p) is the probability density function (p.d.f.) of p.

β is the backordered rate of the demand during the stockout period, under the restriction $0 \leq \beta \leq 1$.

w is the unit penalty cost for uninspected defective items.

v is the unit inspection cost per item.

π_0 is the marginal profit per unit item.

π is the shortage cost per shortaged item.

h is the non-defective (including uninspected defective items) holding cost per unit item per unit year.

A is the setup cost per replenishment.

D is the expected demand per unit year.

r is the reorder point, with

$$r = \mu L + k\sigma\sqrt{L}, \tag{6.1}$$

where k is the safety factor that is a decision variable.

The lead time L has m mutually independent components. The j-th component has a minimum duration a_j, normal duration b_j, and a crashing cost per unit time c_j. Further, for convenience, researchers rearrange c_j such that

$$c_1 \leq c_2 \leq \cdots \leq c_m. \tag{6.2}$$

The components of lead time are crashed one at a time from the first component 1, (owing to it has the smallest unit crashing cost), and then to the second component 2, etc.

To simplify the expression, we assume that

$$L_0 = \sum_{j=1}^{n} b_j, \tag{6.3}$$

and

$$L_i = \sum_{j=i+1}^{n} b_j + \sum_{j=1}^{i} a_j, \tag{6.4}$$

so the lead time crashing cost, denoted as R(L) per cycle for a given $L \in [L_j, L_{j-1}]$, is given by the next formula,

$$R(L) = c_j(L_{j-1} - L) + \sum_{k=1}^{j-1} c_k(b_k - a_k). \tag{6.5}$$

VII. OUR FURTHER DISCUSSIOM

In this section, we provide a detailed explanation for the formula of Equation (6.5).

We recall that

$$L_0 = b_1 + b_2 + \cdots + b_n, \tag{7.1}$$

is the lead time without extra investment to reduce the lead time. Consequently, we derive that

$$L_1 = a_1 + b_2 + \cdots + b_n, \tag{7.2}$$

is the lead when the investment with c_1 is complete to reduce the duration from b_1 to a_1. Similarly, we show that

$$L_2 = a_1 + a_2 + b_3 + \cdots + b_n, \tag{7.3}$$

is the lead when the investment with c_2 is complete to reduce the duration from b_2 to a_2. Consequently, the definition of L_i as $L_i = \sum_{j=i+1}^{n} b_j + \sum_{j=1}^{i} a_j$ of Equation (6.4) is beyond doubt.

Wu and Ouyang [9] and Tuan et al. [10] applied two different approaches by (i) numerical approach of iterative method, and (ii) by analytical approach, derived the almost identical optimal solution.

It points out that both papers have the right computer program to locate the optimal solution. The benefit of analytical approach proposed by Tuan et al. [10] is that it ensures the uniqueness of the optimal solution. The numerical method of Wu and Ouyang [9] was based on two iterated sequences. However, the convergences of these two sequences, proposed by the first partial derivations, did not prove in Wu and Ouyang [9]. The analytical approach proposed by Tuan et al. [10] is to merge two equations, proposed by two first partial derivations, into one equation such that Tuan et al. [10] can analysis it to prove that had unique solution under our conditions.

Consequently, the analytical approach constructed by Tuan et al. [10] may be viewed as a proof for the convergence of the two iterated sequences of Wu and Ouyang [9]. The solution procedure of Tuan et al. [10] is to demonstrate that sequence sometimes did not converge to the desired root of the original equation. It is just an illustration to point out that iterated sequence is not always reliable. The solution procedure of Tuan et al. [10] only contains one variable. The iterated two sequences, proposed by Wu and Ouyang [9], contain two variables Q and k. The solution procedure of Tuan et al. [10] is simple than that of Wu and Ouyang [9]. If a method can not handle a simple problem then we may conclude that this method may not be suitable for more complicate problems.

On the other hand, we will provdie a beief literature reviewing for related published articles to indicate the research trend. Chang et al. [11] provide a further analysis for algebraic procedures for economic ordering quantity models and economic production quantity systems without using calculus. Chu and Chung [12] provide a detailed analytic examination for the sensitivity analysis with respect to the inventory systems under partial backordering.
Grubbstrom and Erdem [13] was the first paper to apply algebraic methods to study inventory models without using calculus. With backorder rates, decay, and arbitrary demand rate, Hung [14] developed inventory systems.
Under condition of partial backorders, Park [15] constructed inventory models. Referring to algebraic approaches, Ronald et al. [16] examined economic production quantity model and economic ordering quantity model under shortages. According to sensitivity analysis, Yang [17] developed inventory systems with partial backorders. Cardenas-Barron [18] developed an economic production quantity inventory system through a pure algebraic process.

VIII. Numerical Example and Further Discussion

In this section, we will try to present numerical examples for discussed inventory systems. Our study is a further examination of Chung and Huang [1] which tried to revise the article of Chung [2]. Chung and Huang [1] claimed that four objective functions are convex, to derive a new lower bound for the optimal cycle time and then pointed out the relative errors for the inventory cycle time based on the approximated solution proposed by Chung [2] may up to 40%. The goal of our study try to finish the following examinations. In the beginning, we showed the convex property already derived by Rachamadugu [3]. Next, based on the same numerical examples, we checked the relative errors for the objective functions to reveal that are not significant with average of 0.3% and maximum 2.8% to point out their estimation for the relative errors for the inventory cycle time which contained questionable findings. Finally, according to their application of the Newton-Raphson method to locate the optimal cycle period interval, we show that their new lower bound is redundant.

Our finding will improve the understanding for future researchers with respect to cash outflows in inventory systems and financial credit environment. The relative errors for the inventory replenishment period range from 3.04% to 8.60% which is a very small difference between upper bound and optimal solution.

We will raise the following two challenges for Chung and Huang [1]: Why did they only consider lower bound, optimal cycle time, and upper bound?

They should examine the interrelationship among $\mathrm{PV}(\mathrm{T_L})$, $\mathrm{PV}(\mathrm{T}^*)$, and $\mathrm{PV}(\mathrm{T_U})$.
Consequently, we evaluate those values in the following Tables 3-5. In Table 3, we take the variation of R with $\mathrm{R} = 0.1$, $\mathrm{R} = 0.2$, $\mathrm{R} = 0.25$, and $\mathrm{R} = 0.3$.
In Table 4, we change the value of K from $\mathrm{K} = 5$ to $\mathrm{K} = 50$, and then run the variation of R with $\mathrm{R} = 0.1$, $\mathrm{R} = 0.2$, $\mathrm{R} = 0.25$, and $\mathrm{R} = 0.3$.
In Table 5, we keep the value of R as $\mathrm{R} = 0.2$, and then we run the variation of K with $\mathrm{K} = 0.1$, $\mathrm{K} = 1.0$, $\mathrm{K} = 10$, and $\mathrm{K} = 100$.
We assume that

$$\Sigma_1 = \{\Phi_i : \mathrm{K} = 5, \mathrm{R} \in \mathrm{A}, i = 1,2,3,4\}, \tag{8.1}$$

$$\Sigma_2 = \{\Phi_i : \mathrm{K} = 50, \mathrm{R} \in \mathrm{A}, i = 1,2,3,4\}, \tag{8.2}$$

$$\Sigma_3 = \{\Phi_i : \mathrm{K} \in \mathrm{B}, \mathrm{R} = 0.2, i = 1,2,3,4\}, \tag{8.3}$$

and

$$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3. \tag{8.4}$$

where

$$\Phi_i = \left(\mathrm{PV}_i(\mathrm{T}_{iC}) - \mathrm{PV}_i(\mathrm{T}_i^*)\right)/\mathrm{PV}_i(\mathrm{T}_i^*), \tag{8.5}$$

$$\mathrm{A} = \{\mathrm{R} = 0.1, \mathrm{R} = 0.2, \mathrm{R} = 0.25, \mathrm{R} = 0.3\}, \tag{8.6}$$

$$\mathrm{B} = \{\mathrm{K} = 0.1, \mathrm{K} = 1, \mathrm{K} = 10, \mathrm{K} = 100\}. \tag{8.7}$$

We derive that the mean of Σ is 0.003 and the maximum value of Σ is 0.028.
We define that

$$\Theta_1 = \left\{\psi_i : \mathrm{K} = 5, \mathrm{R} \in \mathrm{A}, i = 1,2,3,4\right\}, \tag{8.8}$$

$$\Theta_2 = \left\{\psi_i : \mathrm{K} = 50, \mathrm{R} \in \mathrm{A}, i = 1,2,3,4\right\}, \tag{8.9}$$

$$\Theta_3 = \left\{\psi_i : \mathrm{K} \in \mathrm{B}, \mathrm{R} = 0.2, i = 1,2,3,4\right\}, \tag{8.10}$$

and

$$\Theta = \Theta_1 \cup \Theta_2 \cup \Theta_3. \tag{8.11}$$

where

$$\psi_i = |\mathrm{T}_{iC} - \mathrm{T}_i^*|/\mathrm{T}_i^*. \tag{8.12}$$

We obtain that the minimum value of Θ is 0.0304 and the maximum value of Θ is 0.0860.

Chung and Huang [1] have formulated a lower bound and an upper bound, but did not use bisection method. Instead, Chung and Huang [1] applied Newton-Raphson method.

To the best of our memory, applying the Newton Raphson method, researchers only needed a starting point. Deriving (i) a lower bound and (ii) an upper bound is another redundant result in Chung and Huang [1].

IX. Some Related Problems

In this section we try to answer the following three questions to describe the figure of a mapping without actually drawing the graph. Question (a): For $0 \leq x \leq 1$, the graph of $\sqrt{x}+\sqrt{y}=1$. Question (b): The graph of $x^p+y^p=1$. Question (c): Why does the graph of $x^{0.3}+y^{0.3}=1$ lies inside $x^{0.5}+y^{0.5}=1$.

For the first problem, we assume that $\sqrt{y}=1-\sqrt{x}$, and then we derive that

$$y=\left(\sqrt{y}\right)^2=\left(1-\sqrt{x}\right)^2=1-2\sqrt{x}+x, \qquad (9.1)$$

Hence, we assume an auxiliary function,

$$f(x)=1-2\sqrt{x}+x. \qquad (9.2)$$

We obtain that $df/dx=1-(1/\sqrt{x})<0$ for $0<x<1$ so f(x) is decreasing function and then we find that

$$\frac{d^2 f}{dx^2}=\frac{1}{2x^{3/2}}>0 \qquad (9.3)$$

for $0<x<1$, so f(x) is concave up function.

For the second problem and for a given parameter, p, which is satisfying $0<p<1$, we begin to describe the graph of $x^p+y^p=1$ such that we did not sketch the picture. We derive that

$$y^p=1-x^p, \qquad (9.4)$$

and then we obtain that

$$y=\left(1-x^p\right)^{\frac{1}{p}}, \qquad (9.5)$$

we assume another auxiliary function,

$$f(x)=\left(1-x^p\right)^{\frac{1}{p}}. \qquad (9.6)$$

We show that

$$\frac{df}{dx}=\frac{1}{p}\left(1-x^p\right)^{\frac{1}{p}-1}\frac{d}{dx}\left(1-x^p\right)=\frac{1}{p}\left(1-x^p\right)^{\frac{1}{p}-1}(-1)\,px^{p-1}$$

$$=(-1)\left(1-x^p\right)^{\frac{1}{p}-1}x^{p-1}. \qquad (9.7)$$

Since $0<x<1$, we derive that $0<x^p<1$, that is to claim that $0<x^{p-1}<1$, so that we know $df/dx<0$. we obtain that

$$\frac{d^2 f}{dx^2}=(-1)\left[\left(1-x^p\right)^{\frac{1}{p}-2}(-1)\,px^{p-1}\right]x^{p-1}+(-1)\left(1-x^p\right)^{\frac{1}{p}-1}(p-1)x^{p-2}.$$

Owing to $(-1)(-1)>0$ and $(-1)(p-1)>0$, such that we derive that $d^2f/dx^2>0$ to indicate that the picture of our second problem is decreasing and concave up.

For the third problem, we consider that why does the graph of $x^{0.3}+y^{0.3}=1$ lies inside $x^{0.5}+y^{0.5}=1$?
For a given x with $0<x<1$, for example $x=0.4$, we derive that

$$0.4^{0.3}+y^{0.3}\big|_{x=0.4}=1, \qquad (9.8)$$

and

$$0.4^{0.5}+y^{0.5}\big|_{x=0.4}=1, \qquad (9.9)$$

Owing to $0.4^{0.3}>0.4^{0.5}$, and then we show that

$$y^{0.3}\big|_{x=0.4}<y^{0.5}\big|_{x=0.4}, \qquad (9.10)$$

, that is, $(0.4, y^{0.3}|_{x=0.4})$ is located below the location of $(0.4, y^{0.5}|_{x=0.4})$. Consequently, we show that the graph of $x^{0.3}+y^{0.3}=1$ lies below than the graph of $x^{0.5}+y^{0.5}=1$, without sketching the picture.

X. A Related Problem

We study a related problem of Gallego and Moon [19] for the solution procedure of inventory systems with distribution free demand during the shortage period. The original problem was constructed by Gallego [20] and then this kind of inventory systems has been examined and extended by Luo et al. [21], Hu et al. [22], Kumar and Goswami [23], Lin [24], Hung et al. [25], Lin et al. [26], and Chu [27]. However, the proof for the two-point distribution inventory system needs further discussion which did not examine by those published articles. In this section, we will present a detailed examination.

Our study problem is related to the uniqueness of the distribution achieving the upper bound of Gallego and Moon [19]. If our problem is solvable and then we can show that for two point distribution, the uniqueness is verified. We follow Gallego and Moon [19] to assume the following,

$$\alpha=\frac{\sqrt{\sigma^2+(Q-\mu)^2}+(Q-\mu)}{2\sqrt{\sigma^2+(Q-\mu)^2}}, \qquad (10.1)$$

and then we derive that

$$1-\alpha=\frac{\sqrt{\sigma^2+(Q-\mu)^2}-(Q-\mu)}{2\sqrt{\sigma^2+(Q-\mu)^2}}, \qquad (10.2)$$

and

$$\alpha(1-\alpha)=\frac{\sigma^2}{\left(2\sqrt{\sigma^2+(Q-\mu)^2}\right)^2}. \qquad (10.3)$$

According to Equation (10.3), we derive that

$$\sqrt{\alpha(1-\alpha)}=\frac{\sigma}{\left(2\sqrt{\sigma^2+(Q-\mu)^2}\right)}. \qquad (10.4)$$

Hence, we compute that

$$(d_2-Q)(1-\alpha)=\left[\left(\mu+\sqrt{\frac{\alpha}{1-\alpha}}\sigma\right)-Q\right](1-\alpha)$$

$$=(\mu-Q)\frac{\sqrt{\sigma^2+(Q-\mu)^2}-(Q-\mu)}{2\sqrt{\sigma^2+(Q-\mu)^2}}$$

$$+\frac{\sigma^2}{2\sqrt{\sigma^2+(Q-\mu)^2}}$$

$$=\frac{-(Q-\mu)}{2}+\frac{\sigma^2+(Q-\mu)^2}{2\sqrt{\sigma^2+(Q-\mu)^2}}$$

$$=\frac{\sqrt{\sigma^2+(Q-\mu)^2}-(Q-\mu)}{2}. \qquad (10.5)$$

Referring to Equation (10.5), the desire valve of

$$E\left[(D-Q)^+\right]=\frac{\sqrt{\sigma^2+(Q-\mu)^2}-(Q-\mu)}{2}, \quad (10.6).$$

is attained.

For two point distribution at two points, denoted as d_1 and d_2 with $d_2 > d_1$, under the restrictions of $\Pr(d_1) = \alpha$, $\Pr(d_2) = 1 - \alpha$, $d_1\alpha + d_2(1 - \alpha) = \mu$, and

$$d_1^2\alpha + d_2^2(1-\alpha) = \mu^2 + \sigma^2, \qquad (10.7)$$

and then Gallego and Moon [19] mentioned that

$$d_1 = \mu - \sqrt{\frac{1-\alpha}{\alpha}}\sigma, \qquad (10.8)$$

and

$$d_2 = \mu + \sqrt{\frac{\alpha}{1-\alpha}}\sigma. \qquad (10.9)$$

We will divide our discussion into three different cases: (i) $d_2 > d_1 \geq Q$, (ii) $Q > d_2 > d_1$, and (iii) $d_2 \geq Q > d_1$.

For case (i), we derive that

$$E\left[(D-Q)^+\right] = E(D-Q)$$
$$= E[D] - Q = \mu - Q, \qquad (10.10)$$

If the restriction of

$$\mu - Q = \frac{\sqrt{\sigma^2 + (Q-\mu)^2} - (Q-\mu)}{2}, \qquad (10.11)$$

is satisfied, and then we show that

$$2(\mu - Q) = \sqrt{\sigma^2 + (Q-\mu)^2} - (Q-\mu). \qquad (10.12)$$

Based on Equation (10.12), we derive that

$$\mu - Q = \sqrt{\sigma^2 + (Q-\mu)^2}. \qquad (10.13)$$

The results of Equation (10.13) is violated with $\sigma > 0$, such that we conclude that case (i) will not happen.

For case (ii), we obtain that

$$E\left[(D-Q)^+\right] = 0. \qquad (10.14)$$

However, the findings of Equation (10.14) is violated our assumption.

For case (iii), we compute that

$$E\left[(D-Q)^+\right] = \left[\left(\mu + \sqrt{\frac{\alpha}{1-\alpha}}\sigma\right) - Q\right](1-\alpha)$$
$$= \frac{\sqrt{\sigma^2 + (Q-\mu)^2} - (Q-\mu)}{2}, \qquad (10.15)$$

which is the desired result.

XI. A Further Study

The open challenge in the further direction of Chang et al. [11] is mentioned in the following formula, to minimize the goal mapping,

$$f(s) = -s + \sqrt{v + s^2(1+u)}, \qquad (11.1)$$

for $0 < s < \infty$ with two restrictions, u>0, and v>0. Under the consideration of an analytical process, Chang et al. [11] had showed that the minimum point,

$$s^* = \sqrt{\beta/(\alpha + \alpha^2)}, \qquad (11.2)$$

and the minimum value,

$$f(s^*) = \alpha(s^*) = \sqrt{\alpha\beta/(1+\alpha)}. \qquad (11.3)$$

We recall that Gallego and Moon [19], they tried to minimize the following problem,

$$g(Q) = Qd + \frac{d+m}{2}\left(\left((Q-\mu)^2 + \sigma^2\right)^{1/2} - (Q-\mu)\right), \qquad (11.4)$$

for $Q \geq \mu$ and then with respect to analytical procedure, they solve $dg(Q)/dQ = 0$ to imply the minimum solution,

$$Q^S = \left[\left((m/d)^{1/2} - (d/m)^{1/2}\right)\sigma/2\right] + \mu. \qquad (11.5)$$

Motivated by the second derivative with respect to Q,

$$d^2 g(Q)/dQ^2$$
$$= \sigma^2(m+d)/2\left((Q-\mu)^2 + \sigma^2\right)^{3/2}, \qquad (11.6)$$

which has a positive sign, then Gallego and Moon [19] knew that g(Q) is strictly convex so the optimal solution is solved as the maximum of two candidates,

$$Q^* = \max\{Q^S, 0\}, \qquad (11.7)$$

and the minimum value, when $m \leq d$, the minimum value is

$$g(0) = \frac{m+d}{2}\left((\sigma^2 + \mu^2)^{1/2} + \mu\right), \qquad (11.8)$$

and when m>d, the minimum value is

$$g(Q^S) = d\mu + (md)^{1/2}\sigma. \qquad (11.9)$$

We check whether or not $g(0) = g(Q^S)$ if and only if m=d.
We assume that $g(0) = g(Q^S)$ with the condition m=d, then

$$\sqrt{\sigma^2 + \mu^2} = (\sigma/2). \qquad (11.10)$$

Equation (11.10) is not valid. It will reveal that $Q \geq 0$ that is, $Q^* = \max\{0, Q^S\}$ of Equation (11.7) is questionable.
It is the lower bound when $Q^* = 0$. Since, $g(0) = g(Q^S)$ doesn't have to be satisfied for this problem. On the other hand, the satisfaction of $Q^* = \max\{0, Q^S\}$ is to make sure the optimal Q has to be positive.

Q^S is the maximum solution for equation (11.4) of Gallego and Moon [19]. If $\mu/\sigma \geq \sqrt{d/m}$ is satisfied then the profit for the lower bound for the worst case is nonnegative, so Q^S is optimal for $Q \geq [(\mu^2 + \sigma^2)/2\mu]$.

We predict that Gallego and Moon [19] did not solve the following problems:

(a) Under $\mu/\sigma \geq \sqrt{d/m}$, the optimal solution for the restricted domain, $0 \leq Q \leq [(\mu^2 + \sigma^2)/2\mu]$.
(b) Under $\mu/\sigma < \sqrt{d/m}$, the optimal solution for the entire domain, $0 \leq Q < \infty$.

Our prediction is mentioned whether or not $Q^S \geq 0$ is not related to $m \geq d$. Our observation is assumed that $Q^S \geq \mu$ is related to $m \geq d$.

From Equation (11.5) in this article, m does not have to greater than or equal to d.
$((m/d)^{0.5} - (d/m)^{0.5})(\sigma/2)$ can be positive or negative to represent the deviation of μ.

If our result will use $m \geq d$ and m<d to denote two cases then the possible result will be $Q^* = \max\{0, Q^S\}$.
[Proof] We assume that the domain of g(Q) is $Q \geq \mu$ (Intuitively, $Q \geq 0$, or those Q satisfies $\pi^G(Q) \geq 0$).

Our lemma 1. After we derive that $Q^* = \max\{\mu, Q^S\}$ such that there are following three cases:
(i) If m>d then $Q^* = Q^S$, we can show that

$$g(\mu) = d\mu + \frac{m+d}{2}\sigma,$$

$$> d\mu + \sigma\sqrt{md} = g(Q^S). \quad (11.11)$$

(ii) If m=d, then

$$Q^* = \mu = Q^S. \quad (11.12)$$

and

$$g(\mu) = d\mu + \frac{m+d}{2}\sigma =,$$

$$m(\mu+\sigma) = d\mu + \sigma\sqrt{md} = g(Q^S). \quad (11.13)$$

(iii) If m<d, then

$$Q^* = \mu. \quad (11.14)$$

We state our possible proposed problem to locate a lower bound for Q, say $Q^\#$, such that $Q^* = \max\{Q^\#, Q^S\}$.
Our goal is to find new criterion, say $\Delta \geq M$, and then $Q^* = Q^\#$ and $\Delta < M$, and then $Q^* = Q^S$.
Moreover, when $\Delta = M$, we have $g(Q^\#) = g(Q^S)$.
The purpose of our discussion is obtained the results of Equations (11.11-11.14) by an algebraic process such that those readers who are not familiar with analytic calculus can accept the inventory systems examined by Gallego and Moon [19].

XII. Our Algebraic Process

To simplify the expression, we assume a new variable, say

$$x = Q - \mu, \quad (12.1)$$

and a new objective function, say

$$h(x) = 2(g(x) - d\mu), \quad (12.2)$$

to imply the minimum problem

$$h(x) = (m+d)\sqrt{\sigma^2 + x^2} - (m-d)x, \quad (12.3)$$

for $x \geq 0$.

From

$$h(x) + (m-d)x = (m+d)\left(\sqrt{\sigma^2 + x^2}\right), \quad (12.4)$$

we square on both sides of Equation (12.4) and then arrange them in the descending order of x to imply that

$$4mdx^2 - 2x(m-d)h(x)^2$$
$$= (h(x))^2 - (m+d)^2\sigma^2. \quad (12.5)$$

We complete the square on the right hand side of Equation (12.5) for variable x and treat h(x) as a constant for the present time being, then

$$4md\left(x - \frac{m-d}{4md}h(x)\right)^2 + (m+d)^2\sigma^2$$
$$= \frac{(m+d)^2}{4md}(h(x))^2. \quad (12.6)$$

Based on Equation (12.6), we can minimize h(x) and then it yields that

$$x = \frac{m-d}{4md}h(x), \quad (12.7)$$

and

$$h(x) = 2\sigma\sqrt{md}. \quad (12.8)$$

XIII. Some Unsolved Issues

According to Equation (12.3), we derive that $\pi^G(Q)$ has a lower bound as

$$m\mu - \sigma\sqrt{md} = m\mu\left(1 - \frac{\sigma}{\mu}\sqrt{\frac{d}{m}}\right)$$
$$= \pi^G(Q^S) \leq \pi^G(Q). \quad (13.1)$$

If the lower bound is negative. It is too underestimated then Gallego and Moon [19] recalled $\pi^G(Q=0)$so that the minimum problem of max $\pi^G(Q)$ cannot be estimated by the lower bound,

$$m\mu - \sigma\sqrt{md} = m\mu\left(1 - \left(\sigma\sqrt{d}/\mu\sqrt{m}\right)\right)$$
$$= \pi^G(Q^S) < 0. \quad (13.2)$$

Gallego and Moon [19] mentioned that (i) if $(m/d) \geq (\sigma^2/\mu^2)$, then $\widehat{Q}^S = Q^S$, and (ii) if $(m/d) < (\sigma^2/\mu^2)$, then $\widehat{Q}^S = 0$.

We claim that the solution procedure of Gallego and Moon [19]. If the lower bound is underestimated to imply a negative lower bound, then Gallego and Moon [19] gave up the lower bound, directly take $Q^* = 0$ with $\pi^G(Q^* = 0) = 0$.

We list several unsolved problems in the following.
(i) How to verify $g(0) \geq g(Q^S)$?
where

$$g(0) = \frac{m+d}{2}\left((\sigma^2 + \mu^2)^{1/2} + \mu\right), \quad (13.3)$$

and

$$g(Q^S) = d\mu + \sigma\sqrt{md}. \quad (13.4)$$

(ii) If $(m/d) < (\sigma^2/\mu^2)$, then $Q^S < 0$.
(iii) If $(m/d) < (\sigma^2/\mu^2)$, then there should be a point, say $\widetilde{Q}$, that satisfies

$$(m+d)\mu - g(\widetilde{Q}) = 0, \quad (13.5)$$

where

$$g(Q) = dQ +$$
$$+ \frac{m+d}{2}\left((\sigma^2 + (Q-\mu)^2)^{1/2} - (Q-\mu)\right). \quad (13.6)$$

XIV. Direction for future Research

Moreover, in this section, we consider several recently published papers to help practitioners to find possible hot research trend in the following. According to Clique polynomial, for singular Lane-Emden type equations, Jummannavar et al. [28] studied numerical solutions. To learn adversarial generative system, Pandey et al. [29] examined tomato sheet illness pattern recognition. Referring to Java training associated knowledge models, with respect to cipher coding system, Wai et al. [30] applied trial information generation to solve policy validation course. Owing to spot protein composite, Zhang, and Guo [31] developed sexual characteristics dissimilarity with multiple firefly approach. To realize intense sensation machinery, Tong et al. [32] constructed broadcast forecast system for virtual community attitude. Based on linear least-squares linear process and k-means group, Zhang et al. [33] considered voracious crash approach with arbitrary coordinate arrangement. Through a intangible learning procedure, Slime et al. [34] obtained prediction system for heart illness hazard estimation. Under Fibonacci sequence, Blasiyus, and Christy [35] acquired combined characters. With respect to Omicron disease extension, Bahri et al. [36], gained neighborhood replication and solidity examination. For vague ramp best possibility, Li et al. [37] derived project information contact policy to share main know-how system. In Lagrange reduction, An et al. [38] employed transit route timetabling with respect to speedy train optimization. Related to time value of money and two storehouses, Pathak et al. [39] found out inventory systems with partial backordered, changeable holding cost, deteriorated goods, and exponential demand. Based on our above discussion, researchers can locate interesting topics for their further examinations.

XV. Conclusion

In this study, we present a detailed examination of Chung and Huang [1] to reveal that their Lemma 1 and Theorem 1 are corollary of the theoretical results of Rachamadugu [3]. On the other hand, their new result of an approximated lower bound for the optimal solution which cannot be used as the initial point to run the iterative method named after Newton and Raphon. Therefore, their lower bound is useful for the bisection method in the future. However, based on literature reviewing, the bisection method is inferior to the Newton-Raphon algorithm. Therefofre, we can conclude that the approximated lower bound developed by Chung and Huang [1] is not useful for further numerical computation and theoretical development.

Table 1. Reproduction of Table 3 of Chung and Huang [1].

K	T_{1C}	T_1^*	T_L	(1)%	(2)%
100	4.280767	4.408612	5.163978	17.13	2.90
10	1.544158	1.548737	1.632993	5.44	0.30
1	0.507509	0.507660	0.516398	1.72	0.03
0.1	0.162410	0.162415	0.163299	0.51	0

Table 2. The inherent relation in Example 3 of Chung and Huang [1].

K	0.1	1	10	100
T_{1C}	0.163299	0.516398	1.632993	5.163978
T_L	0.162410	0.507509	1.544158	4.280767

Table 3. Varation of R, when $K = 5$.

$K = 5$	$R = 0.1$	$R = 0.2$	$R = 0.25$	$R = 0.3$
$PV_1(LT_1)$	208.155109	111.954055	92.407798	79.255646
$PV_1(T_1^*)$	208.155084	111.953944	92.407622	79.255395
$PV_1(T_{1C})$	208.181157	111.997163	92.457917	79.312119
$PV_2(LT_2)$	192.132534	97.832872	78.991067	66.439623
$PV_2(T_2^*)$	192.132462	97.832169	78.989839	66.437448
$PV_2(T_{2C})$	192.169706	97.908785	79.086644	66.554628
$PV_3(LT_3)$	201.264277	105.100405	85.590800	72.481057
$PV_3(T_3^*)$	201.264248	105.100258	85.590553	72.480677
$PV_3(T_{3C})$	201.292115	105.149823	85.650254	72.550317
$PV_4(LT_4)$	200.168183	103.738034	84.118618	70.910689
$PV_4(T_4^*)$	200.168157	103.737911	84.118418	70.910392
$PV_4(T_{4C})$	200.194562	103.872624	84.171059	70.970474

Table 4. Varation of R, when $K = 50$.

$K = 50$	$R = 0.1$	$R = 0.2$	$R = 0.25$	$R = 0.3$
$PV_1(LT_1)$	358.032337	216.751841	187.673817	167.971623
$PV_1(T_1^*)$	358.023670	216.711509	187.609943	167.879717
$PV_1(T_{1C})$	358.884849	218.149828	189.282541	169.761747
$PV_2(LT_2)$	307.869037	173.467069	147.144069	129.954239
$PV_2(T_2^*)$	307.843376	173.240019	146.687845	129.150523
$PV_2(T_{2C})$	307.081532	175.741986	149.764032	132.740764
$PV_3(LT_3)$	349.048219	206.885086	177.503661	157.558163
$PV_3(T_3^*)$	349.039128	206.839560	177.428694	157.445938
$PV_3(T_{3C})$	349.916352	208.343493	179.201008	159.466380
$PV_4(LT_4)$	348.609255	206.266729	176.803093	156.779317
$PV_4(T_4^*)$	348.600247	206.222016	176.729766	156.669981
$PV_4(T_{4C})$	349.472646	207.710066	178.479217	158.659883

Table 5. Varation of K, when R = 0.2.

$R = 0.2$	$K = 0.1$	$K = 1.0$	$K = 10$	$K = 100$
$PV_1(LT_1)$	81.190572	95.037244	143.077922	340.430407
$PV_1(T_1^*)$	81.190572	95.037243	143.077626	340.322939
$PV_1(T_{1C})$	81.190663	95.040159	143.172638	343.484275
$PV_2(LT_2)$	79.810316	90.673945	129.322936	298.127660
$PV_2(T_2^*)$	79.810316	90.673944	129.322284	297.887504
$PV_2(T_{2C})$	79.810434	90.677725	129.446242	301.988089
$PV_3(LT_3)$	82.624461	90.735055	133.666468	325.932548
$PV_3(T_3^*)$	82.624461	90.735051	133.666078	325.8126.3
$PV_3(T_{3C})$	82.626869	90.742121	133.775548	329.106485
$PV_4(LT_4)$	70.570633	84.043297	130.905452	324.647296
$PV_4(T_4^*)$	70.570633	84.043296	130.905129	324.529678
$PV_4(T_{4C})$	70.570727	84.046298	131.003012	327.875810

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