A Second-order Strang Splitting Scheme for the Swift-Hohenberg Equation with Quadratic-cubic Nonlinearity

Ying Ye, Lingzhi Qian, Huiping Cai*

Abstract—**This paper primarily investigates the conservative Swift-Hohenberg (SH) equation with quadratic-cubic nonlinearity, where mass conservation is achieved by introducing a Lagrange multiplier. Based on Fourier spectral method and SSP-RK2 method, we ingeniously utilize operator splitting method to derive an accurate and efficient numerical scheme for solving the SH equation. The theoretical analysis of this numerical scheme, including mass conservation, convergence, and energy stability, is discussed in detail. Finally, a variety of numerical examples are presented to validate the precision and effectiveness of the scheme.**

Index Terms—**Swift-Hohenberg equation, Quadratic-cubic nonlinearity, Operator splitting, Fourier spectral method, SSP-RK2 method.**

I. Introduction

IN this paper, we pay attention to the following free energy functional [4], [15]:

$$\varepsilon(\phi) := \int_\Omega \left(\frac{1}{4}\phi^4 - \frac{g}{3}\phi^3 + \frac{1}{2}\phi\left(-\varepsilon + (1+\Delta)^2\right)\phi\right)dx. \quad (1)$$

where ϕ is the density field, g and ε are non-negative constants which have specific physical significance. We consider that ϕ and $\Delta\phi$ are periodic on $\partial\Omega$. By taking the variational derivative to Eq.(1), we obtain the following chemical potential

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= \Delta\mu,\\ \mu &= -\left(\phi^3 - g\phi^2 + \left(-\varepsilon + (1+\Delta)^2\phi\right)\right).\end{aligned} \quad (2)$$

The PFC model (2) is mass-conservative, i.e.,

$$\frac{d}{dt}\int_\Omega \phi dx = \int_\Omega \frac{\partial\phi}{\partial t}dx = \int_\Omega \Delta\mu dx = \int_\Omega \nabla\mu\cdot n ds = 0. \quad (3)$$

We note that the PFC model (2) includes sixth-order spatial derivative which is the real bottleneck in terms of computational time. To overcome the difficulty, Lee [9] recently developed a conservative SH equation by introducing a space-time dependent Lagrange multiplier. Based on the L^2-gradient flow, a conservative SH equation is proposed to be

$$\begin{cases}\frac{\partial\phi}{\partial t} = -\left(\phi^3 - g\phi^2 + \left(-\varepsilon + (1+\Delta)^2\phi\right)\right) + (I(\phi))^r\beta(t),\\ \phi|_{t=0} = \phi_0,\end{cases} \quad (4)$$

where $I(\phi) = \frac{1-\varepsilon}{2}\phi^2 + \left(\frac{1}{4}\phi^4 - \frac{1}{3}g\phi^3\right)$ and $\beta(t) = \frac{\int_\Omega I'(\phi)dx}{\int_\Omega (I(\phi))^r dx}$. Note that the above model can include both the overall information and the local effect. It's easy to see that the SH model (4) is mass-conservative and the energy functional $\varepsilon(\phi)$ is non-increasing in time.

The methods used to solve the SH equation are usually operator splitting [10], convex splitting [9] and so on. Lee [9] presented temporally first- and second-order energy stable methods for the SH equation with quadratic-cubic nonlinearity based on the Fourier spectral method. Moreover, by using operator splitting method, Lee [10] constructed mass-conservative first- and second-order methods for solving a new conservative SH equation.

In this study, we aim to solve the mass-conservative SH model in virtue of Strang splitting method [11], [17], [18], [21], [26]. Strang splitting is one of the operator splitting methods, which mainly simplifies the problem by dividing the original problem into two subproblems. Thus, Strang splitting method has been used to solve many complicated problems [6], [7], [13], [19], [22]. However, it's very challenging to apply the Strang splitting method to nonlinear partial differential equations on account of its multi-stage nature. To our best knowledge, there is little research for Strang splitting method for the SH model. Our main contribution, which is different from that in [10], includes two aspects. On the one hand, the nonlinear part is solved by second-order strong stability preserving Runge-Kutta(SSP-RK2) method. On the other hand, the theoretical analysis of convergence in L^2-norm is discussed.

The rest of the paper is organized as follows. In Section 2, we proposed an effective second-order numerical scheme for the SH model. In Section 3, we analyze the mass conservation, convergence and energy stability of our proposed scheme. A series of numerical examples are presented in Section 4. Finally, conclusions are drawn in Section 5.

II. An effective and easy-to-implement second-order scheme

In this section, we present an efficient second-order Strang splitting scheme for the SH equation (4).

To obtain an efficient scheme, we first rewrite the SH

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equation (4) into the following equivalent form

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= -\left(\phi^3 - g\phi^2 + (-\varepsilon + (1+\Delta)^2)\phi\right) + (I(\phi))^r\beta(t)\\ &= -\left(\Delta^2\phi + 2\Delta\phi\right) - (1-\varepsilon)\phi - \left(\phi^3 - g\phi^2\right) + (I(\phi))^r\beta(t)\\ &= H(\phi) + Q(\phi),\end{aligned} \tag{5}$$

where

$$H(\phi) = -\left(\Delta^2\phi + 2\Delta\phi\right)$$

and

$$Q(\phi) = -(1-\varepsilon)\phi - \left(\phi^3 - g\phi^2\right) + (I(\phi))^r\beta(t).$$

A. Strang splitting method

The main idea of our method is to transform the original problem into linear subproblem and nonlinear subproblem. Let's consider the following two subproblems:

$$\mathcal{S}_{\mathcal{L}}(\phi_1):\ \phi_t = H(\phi), \qquad \phi|_{t=0} = \phi_1, \tag{6}$$

$$\mathcal{S}_{N}(\phi_2):\ \phi_t = Q(\phi), \qquad \phi|_{t=0} = \phi_2. \tag{7}$$

The standard form of the Strang splitting method [21], [27] is given by

$$\phi^{n+1} = \mathcal{S}_{\mathcal{L}}(\frac{\tau}{2})\mathcal{S}_N(\tau)\mathcal{S}_{\mathcal{L}}(\frac{\tau}{2})\phi^n, \tag{8}$$

where $\tau > 0$ is the time step. Now we employ the standard Strang splitting (8) to approximate the SH equation (4).

B. Numerical approximation of $\phi_t = H(\phi)$

First, in order to obtain a numerical approximation of the SH equation (4), it's necessary to mesh the geometry domain Ω^{per}. The set of the grid points is defined as

$$\Omega_h^{per} = \left\{(x_i, y_j) = (a + ih, a + jh), 0 \le i, j \le N-1\right\},$$

where $h = \frac{b-a}{N}$ is the space step, N is the number of grid nodes. Let ϕ_{mn}^k denotes the numerical solution $\phi(x_m, y_n, t_k)$, where $t_k = k\tau$, $\tau = T/M$ is the time step and M is time iteration number. The discrete Fourier transform and its inverse transform are given by [12], [23]

$$\mathcal{F}_N:\quad \hat{\phi}_{pq}^k = \frac{h^2}{c_p c_q (b-a)^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\phi_{mn}^k e^{-i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-a)}{b-a}\right)}, \tag{9}$$

$$\mathcal{F}_N^{-1}:\quad \phi_{mn}^k = \sum_{p=-N/2}^{N/2}\sum_{q=-N/2}^{N/2}\hat{\phi}_{pq}^k e^{i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-a)}{b-a}\right)}, \tag{10}$$

where $p, q = 0, \pm1, \pm2, ...$ and c_p and c_q are respectively defined as

$$c_r = \begin{cases} 2, & |r| = \dfrac{N}{2},\\ 1, & |r| < \dfrac{N}{2}.\end{cases} \qquad r = p \text{ or } q.$$

In accordance with [12], [23], the discrete Laplacian term is recast to

$$\Delta\phi_{mn}^k = -\sum_{p=-N/2}^{N/2}\sum_{q=-N/2}^{N/2}(\xi_p^2 + \eta_q^2)\hat{\phi}_{pq}^k e^{i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-a)}{b-a}\right)}.$$

Similarly, we can handle the higher order derivative $\Delta^2\phi_{mn}^k$ as

$$\Delta^2\phi_{mn}^k = -\sum_{p=-N/2}^{N/2}\sum_{q=-N/2}^{N/2}(\xi_p^2 + \eta_q^2)^2\hat{\phi}_{pq}^k e^{i\left(\frac{2p\pi(x_m-a)}{b-a} + \frac{2q\pi(y_n-a)}{b-a}\right)}.$$

Substituting (9) for (6), we can get an ordinary differential equation for the (pq)-th Fourier coefficient

$$\frac{d\hat{\phi}_{pq}(t)}{dt} = K\hat{\phi}_{pq}(t), \tag{11}$$

where

$$K = -\left(\lambda_{pq}^2 + 2\lambda_{pq}\right)$$

and

$$\lambda_{pq} = -\left[\left(\frac{p\pi}{b-a}\right)^2 + \left(\frac{q\pi}{b-a}\right)^2\right]. \tag{12}$$

Employing the variable separation approach, the solution to the equation (11) is given by

$$\hat{\phi}_{pq}^{k+1} = exp\,(\tau K)\,\hat{\phi}_{pq}^k.$$

Thus, we obtain

$$\phi^{k+1} = \mathcal{F}_N^{-1}\left\{\exp(\tau K)\mathcal{F}_N[\phi^k](p,q)\right\}. \tag{13}$$

C. Numerical approximation of $\phi_t = Q(\phi)$

Now we focus on the nonlinear subproblem (7). By utilizing the second-order SSP-RK method [3], we obtain that

$$\begin{cases}\phi_{mn}^{(1)} = \phi_{mn}^k + \tau Q\left(\phi_{mn}^k\right),\\ \phi_{mn}^{k+1} = \dfrac{1}{2}\phi_{mn}^k + \dfrac{1}{2}\phi_{mn}^{(1)} + \dfrac{1}{2}\tau Q\left(\phi_{mn}^{(1)}\right).\end{cases} \tag{14}$$

In summary, an effective and easy-to-implement second-order scheme for problem (4) is given by

$$\begin{cases}\phi_{mn}^{(1)} = \mathcal{F}_N^{-1}\left\{exp(\dfrac{\tau}{2}K)\mathcal{F}_N[\phi_{mn}^k](p,q)\right\},\\ \phi_{mn}^{(2)} = \phi_{mn}^{(1)} + \tau Q\left(\phi_{mn}^{(1)}\right),\\ \phi_{mn}^{(3)} = \dfrac{1}{2}\phi_{mn}^{(1)} + \dfrac{1}{2}\phi_{mn}^{(2)} + \dfrac{1}{2}\tau Q\left(\phi_{mn}^{(2)}\right),\\ \phi_{mn}^{k+1} = \mathcal{F}_N^{-1}\left\{exp(\dfrac{\tau}{2}K)\mathcal{F}_N[\phi_{mn}^{(3)}](p,q)\right\}.\end{cases} \tag{15}$$

In the following, $\mathcal{S}_{\mathcal{L}}^h$ and $\mathcal{S}_N^h$ denote the numerical approximations of $\mathcal{S}_{\mathcal{L}}$ and $\mathcal{S}_N$, respectively.

III. Theoretical analysis of the mass conservation and convergence

In this section, we primarily conduct a detailed theoretical analysis of the proposed numerical scheme. To this end, it is necessary to establish some definitions and notations.

For any function

$$\phi \in H_{per}^m(\Omega) = \{u|_{\bar{\Omega}} : u \in H^m(\Omega) \text{ and } u \text{ is } \Omega - periodic\},$$

we define the following Fourier interpolation

$$(I_N\phi)(x,y) = \sum_{p=-N/2}^{N/2}\sum_{q=-N/2}^{N/2}\hat{\phi}_{pq}(t)\varphi_{pq}(x,y). \tag{16}$$

By leveraging the definition of Fourier interpolation (16), we can efficiently derive the first derivative by multiplying the appropriate Fourier coefficients by $i(\frac{p\pi}{b-a} + \frac{q\pi}{b-a})$. The process for higher order derivatives follows a similar pattern; for instance, the second derivative and fourth derivative can be obtained by multiplying the Fourier coefficients by $-\left[(\frac{p\pi}{b-a})^2 + (\frac{q\pi}{b-a})^2\right]$ and $\left[(\frac{p\pi}{b-a})^4 + (\frac{q\pi}{b-a})^4\right]$, respectively. Assuming that ϕ and its derivatives are continuous, the convergence of the interpolation is encapsulated by the following result:$(0 \le k \le m, m > \frac{d}{2})$

$$\left\|\partial^k \phi(x,y) - \partial^k I_N \phi(x,y)\right\|_{L^2} \le C \left\|\phi\right\|_{H^m} h^{m-k}. \quad (17)$$

For any periodic function ϕ, we denote its collocation approximation as $(0 \le i, j \le N-1)$

$$\phi(x_i, y_j) = (I_N\phi)_{i,j} = \sum_{p=-N/2}^{N/2} \sum_{q=-N/2}^{N/2} \hat{\phi}_{pq}(t)\varphi_{pq}(x_i, y_j). \quad (18)$$

Conversely, the discrete differentiation operator D_N is defined on the vector of grid values $\phi = \phi(x_i, y_j), 0 \le i, j \le N-1$. Essentially, the collocation coefficients $\hat{\phi}_{pq}$ are computed using FFT, as indicated by (9), and one could multiply them by the corresponding eigenvalues and perform the inverse FFT. This same methodology is applied to compute the higher order derivatives, such as D_N^2 and D_N^4.

A. Discrete mass conservation

Recall the equation (3), the exact solution ϕ_e to the SH equation (4) upholds the principle of mass conservation, expressed as:

$$\int_\Omega \phi_e(\cdot, t)\, dx = \int_\Omega \phi_e(\cdot, 0)\, dx = C, \quad \text{with } \forall t > 0.$$

This property is also preserved in the proposed operator splitting scheme, at a discrete level, as illustrated by the following theorem.

Theorem 3.1: The second-order scheme (15) preserves the discrete mass conservation property.

Proof: Using (12), we have

$$\lambda_{00} = -\left[\left(\frac{p\pi}{b-a}\right)^2 + \left(\frac{q\pi}{b-a}\right)^2\right] = 0, \qquad \text{when } p, q = 0.$$

Using the discrete Fourier transform (9), we bound the first expression of the scheme (15) as

$$\tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{(1)} = \hat{\phi}_{00}^{(1)} = \hat{\phi}_{00}^{k} = \tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{k}.$$

where $\tilde{C} = \frac{h^2}{c_p c_q (b-a)^2}$. Similarly, it's clear to see that

$$\tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{(2)} = \hat{\phi}_{00}^{(2)} = \hat{\phi}_{00}^{(1)} = \tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{(1)},$$

$$\tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{(3)} = \hat{\phi}_{00}^{(3)} = \hat{\phi}_{00}^{(2)} = \tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{(2)}.$$

Finally, we obtain that

$$\tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{k+1} = \hat{\phi}_{00}^{k+1} = \hat{\phi}_{00}^{(3)} = \hat{\phi}_{00}^{k} = \tilde{C} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \phi_{mn}^{k}.$$

Thus, our proposed scheme inherits the mass conservation.

B. Convergence analysis

In order to establish the optimal error estimate, the concept from [24] is adopted. First, we consider the following numerical scheme equivalent to the scheme (15):

Stage1: linear part, with $\frac{1}{2}\Delta t$ advance

$$\frac{\partial \phi_1}{\partial t} = A\phi_1, \quad \text{over } (t^k, t^{k+\frac{1}{2}}), \quad (19)$$

where

$$A = -\left(D_N^4 + 2D_N^2\right). \quad (20)$$

Integrating both sides of equation (19) with respect to time t, the numerical solution is explicitly expressed as

$$\phi^{k,(1)} = e^{\frac{1}{2}\Delta t A}\phi^k, \quad (21)$$

$$\phi^{k,(1)} = \phi_1(t^{k+\frac{1}{2}}), \phi^k = \phi_1(t^k). \quad (22)$$

Stage2: nonlinear part, with Δt advance

$$\frac{\partial \phi_2}{\partial t} = Q_N(\phi_2), \quad \text{over } (t^k, t^{k+1}), \quad (23)$$

in which

$$Q_N(\phi) = -(1-\varepsilon)\phi - \left(\phi^3 - g\phi^2\right) + (I(\phi))^r \frac{\frac{h^2}{b-a}\sum_{i,j=0}^{N-1}(I'(\phi))_{ij}}{\frac{h^2}{b-a}\sum_{i,j=0}^{N-1}(I(\phi))_{ij}^r}.$$

Using the second-order SSP-RK method to the aforementioned equation (23) yields

$$\begin{cases} \phi^* = \phi^{k,(1)} + \Delta t Q_N\left(\phi^{k,(1)}\right), \\ \phi^{k_*} = \frac{1}{2}\phi^{k,(1)} + \frac{1}{2}\phi^* + \frac{1}{2}\Delta t Q_N\left(\phi^*\right), \end{cases} \quad (24)$$

$$\phi^{k_*} = \phi_2(t^{k+1}), \phi^{k,(1)} = \phi_2(t^k). \quad (25)$$

Stage3: linear part, with $\frac{1}{2}\Delta t$ advance

$$\frac{\partial \phi_3}{\partial t} = A\phi_3, \quad \text{over } (t^k, t^{k+\frac{1}{2}}), \quad (26)$$

Mirroring the approach taken in the first stage, we are poised to deduce the numerical solution

$$\phi^{k+1} = e^{\frac{1}{2}\Delta t A}\phi^{k_*}, \quad (27)$$

$$\phi^{k+1} = \phi_3(t^{k+\frac{1}{2}}), \phi^{k_*} = \phi_3(t^k). \quad (28)$$

Let ϕ_e and $\phi_{\Delta t,h}$ be the solutions to the equations (5) and (19)-(28), respectively, then the convergence result concerning the numerical scheme is presented in the following theorem. The proof of theorem is along the lines of [24] with appropriate modifications.

Theorem 3.2: For any final time $T > 0$, assume the exact solution ϕ_e to the SH equation (5) is smooth enough. As $\Delta t, h \to 0$, the following convergence result is valid:

$$\left\|\phi_{\Delta t,h} - \phi_e\right\|_{L^2(\Omega)} \le C(\Delta t^2 + h^m), \quad (29)$$

provided that the time step Δt and the space grid size h are bounded by given constants only depending on the exact solution.

C. Energy stability

In this section, we present the energy stability of the numerical scheme. To streamline our analysis, let us define the spectral approximation of the L^2 (denoted as ℓ^2) inner product $(\cdot,\cdot)_N$ and the norm $\|\cdot\|_N$ as follows:

$$(f,g)_N = \frac{h^2}{b-a}\sum_{i,j=0}^{N-1} f_{ij}g_{ij}, \qquad \|f\|_N = \sqrt{(f,f)_N}. \tag{30}$$

Upon a simple verification, it can be concluded that

$$(f, A\cdot g)_N = (A\cdot f, g)_N.$$

The space-discrete scheme for (5) is designed to identify a function $\tilde{\phi}:[0,T]\to\Omega_h^{per}$ satisfying

$$\begin{cases} \dfrac{d\tilde{\phi}}{dt} = A\tilde{\phi} + Q_N(\tilde{\phi}), & t\in(0,T], \\ \tilde{\phi}(0) = \phi_0, & t = 0, \end{cases}$$

where $\phi_0 \in \Omega_h^{per}$ is determined by the initial data. We consider the following numerical scheme:

For any $k \geq 1$, our objective is to find $\phi_h^{k+1}(t) : [t^k, t^{k+1}] \to \Omega_h^{per}$ such that for any $t \in [t^k, t^{k+1}]$,

$$\frac{d\phi_h^{k+1}(t)}{dt} = A\phi_h^{k+1}(t) + Q_N(\phi_h^{k+1}(t)). \tag{31}$$

Let ϕ_h^k be denoted as $\phi_h(t^k)$ for $k \geq 0$. Subsequently, we introduce the modified energy

$$\tilde{\varepsilon}_N(\phi) = \left(\frac{\phi}{2}, -A\phi\right)_N - \left(\int Q_N(\phi)d\phi, 1\right)_N. \tag{32}$$

The principal outcome of this section is delineated below. The proof of theorem adheres to the methodology of [24], with suitable adjustments to fit our context.

Theorem 3.3: The numerical system (31) is energy-stable in the sense that for any $1 \leq k \leq M-1, \tilde{\varepsilon}_N(\phi_h^{k+1}) \leq \tilde{\varepsilon}_N(\phi_h^k)$.

IV. Numerical experiments

A. Temporal accuracy test

Firstly, we investigate the convergence rate of our proposed method with an initial condition [8], [16]

$$\phi(x,0) = 0.07 - 0.02\cos\left(\frac{2\pi(x-12)}{32}\right) + 0.02\cos^2\left(\frac{\pi(x+10)}{32}\right) - 0.01\sin^2\left(\frac{4\pi x}{32}\right).$$

We consider the SH model (4) posed on the domain $\Omega = [0, 32]$. In this simulation, we fix the grid size to $h = \frac{1}{3}$ and vary the time step size $\tau = 2^{-4}, 2^{-5}, ..., 2^{-8}$. The L^2-norm errors, temporal convergence rates and CPU times with $g = 0$ and 1 for $\varepsilon = 0.25$ are shown in Table I. We observe that the method is second-order accuracy in time.

Table I. L^2-errors, convergence rates and CPU times for SH model with $r = 0$ at $T = 1$.

	g=0			g=1		
τ	Err_{L^2}	Rate	CPU	Err_{L^2}	Rate	CPU
2^{-4}	9.79E-06	-	4.25E-03	5.89E-06	-	4.13E-03
2^{-5}	2.40E-06	2.03	6.27E-03	1.45E-06	2.02	5.85E-03
2^{-6}	5.96E-07	2.01	1.43E-02	3.60E-07	2.01	1.23E-02
2^{-7}	1.48E-07	2.01	2.09E-02	8.97E-08	2.01	1.94E-02
2^{-8}	3.68E-08	2.01	4.31E-02	2.23E-08	2.01	3.90E-02

B. Phase transition behaviors in 2D

In this section, we simulate the phase transition for the SH model (4) on domain $\Omega = [0, 32] \times [0, 32]$. The initial condition is given by $\phi(x, y, 0) = 0.07+(2\text{rand}(x, y) - 1)$, where rand(x,y) is a random number between 0 and 1. To perform the phase transition, we take $\tau = 2^{-3}$ and $h_x = h_y = \frac{1}{3}$. The other parameters are set to be $\varepsilon = 0.07$ and $r = 0$. (a)-(c) in Figure 1 display the snapshots at different moments. We can observe that the phase transition reaches a steady state until $T = 200$. It can be observed that the energy is non-increasing in time and the mass is conservative in Figure 2 and Figure 3.

V. Conclusion

In this work, we mainly use Fourier spectral method and SSP-RK2 method to provide a second-order operator splitting numerical scheme for the conservative SH equation. We conduct the theoretical analysis of the proposed scheme and validate its accuracy and efficiency by numerical experiments.

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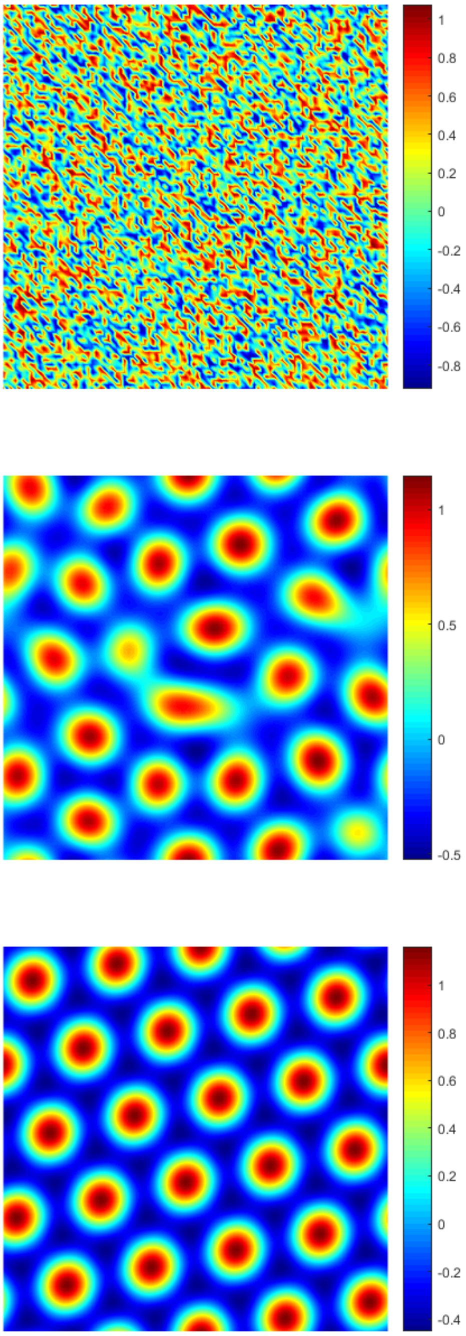

Fig. 1: (a)$T = 0$;(b)$T = 20$;(c)$T = 200$. The snapshots of phase transition for the SH equation with $g = 1$.

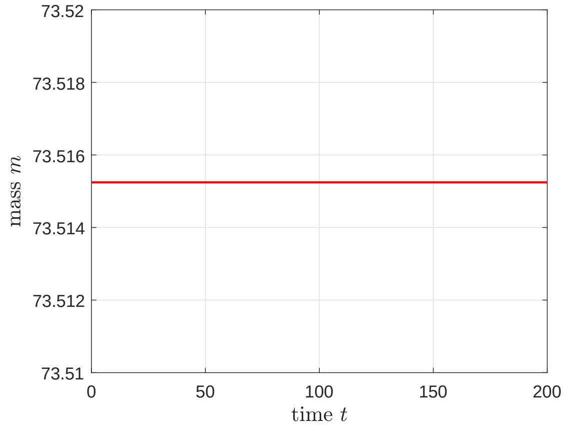

Fig. 2: Evolution of the mass with $\tau = 2^{-3}$ and $h_x = h_y = \frac{1}{3}$ using the second-order scheme.

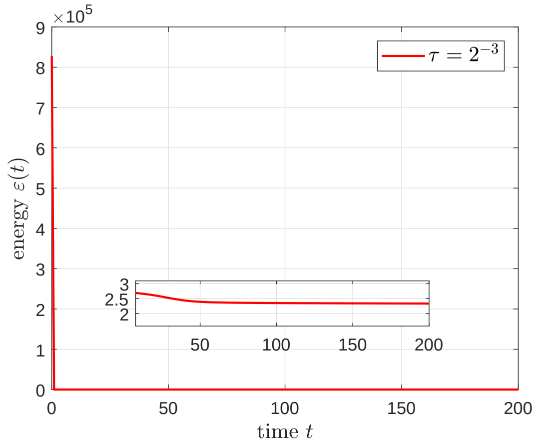

Fig. 3: Evolution of the energy with $\tau = 2^{-3}$ and $h_x = h_y = \frac{1}{3}$ using the second-order scheme.

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