Clique Transversal-Critical, Fixed, Free and Totally Free Elements

Smitha Ganesh Bhat* and Ravishankar Bhat

***Abstract*—A vertex v clique dominates a clique l if v is incident on l. A set $D \subseteq V$ is a clique transversal set if every clique in G is clique dominated by some vertex in D. The clique transversal number $\tau_c = \tau_c(G)$ is the cardinality of a minimum clique transversal set of G. This paper explores properties of vertices and edges based on their membership in all, at least one but not all, or none of the clique transversal sets. A graph G is defined as τ_c-dot-critical if contracting any edge reduces the clique transversal number. We establish bounds for τ_c-dot-critical graphs and a lower bound for the full open domination number of a graph in terms of the maximum signature.**

***Index Terms*—τ_c-critical, τ_c-fixed, τ_c-free elements, clique radius, clique diameter, full open domination number.**

I. Introduction

FOR any undefined terminologies we refer [4], [13]. By a graph we mean a connected finite simple graph with p vertices and q edges. A vertex $v \in V$ is a $cut-vertex$ of a graph G, if $G - v$ is disconnected and such an edge is a $bridge$ or a $cut-edge$. A graph G is $separable$ if it has a cut-vertex otherwise it is $nonseparable$. A maximal nonseparable subgraph is a $block$ of G. A maximal complete subgraph is a $clique$. A vertex v clique dominates a clique l if v is incident on l. A set $D \subseteq V$ is said to be a $clique$ $transversal$ set if every clique in G is clique dominated by some vertex in D. The $clique$ $transversal$ $number$ $\tau_c = \tau_c(G)$ is the cardinality of a minimum clique transversal set of G. A detailed study of this literature is done by Tuza, Erdos and Gallai [15] in 1990 and [3] in 1992.

This passage highlights the work of E. Sampathkumar and Neeralagi [6], who introduced fundamental concepts related to domination number and neighborhood number in 1992. In their research, they explored the significance of certain vertices and edges concerning these graph parameters. Specifically, they investigated the criticality of vertices and edges in relation to domination number and neighborhood number.

Building upon their work, we propose an extension of the notion of criticality to the clique transversal number. This suggests that similar to the critical aspects identified for domination number and neighborhood number, there are elements within graphs that significantly influence the clique transversal number. By extending this concept, the study aims to explore and understand the critical elements that impact the clique transversal number of graphs.

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Furthermore Surekha et al. [9], [10], [11], [12], Sayinath Udupa N V [7], and Tana et al. [14] have conducted comprehensive investigation into the characteristics of cliques in graph structures. This implies a detailed study focusing on understanding various properties and behaviors of cliques within graph theory. Isabel Cristina Lopes et al. [5] have also explored the topic of cliques in graph structures, indicating another independent study on this subject.

II. τ_c-Critical, Fixed, Free and Totally Free Elements

Let G be a graph and x be any element of the graph G. Then the element x is said to be

(i) τ_c-critical, if $\tau_c(G - x) \neq \tau_c(G)$
(ii) τ_c^+-critical, if $\tau_c(G - x) > \tau_c(G)$
(iii) τ_c^--critical, if $\tau_c(G - x) < \tau_c(G)$
(iv) τ_c-reduntant if $\tau_c(G - x) = \tau_c(G)$
(v) τ_c-fixed, if x belongs to every τ_c-set
(vi) τ_c-free, if x belongs to some τ_c-set but not all τ_c-set.
(vii) τ_c-totally free, if x belongs to no τ_c-set

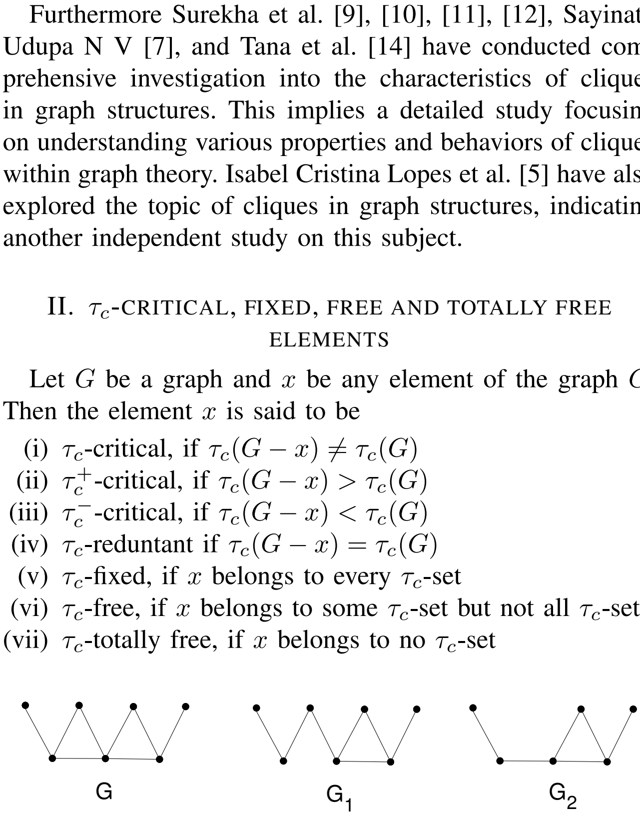

Fig. 1. A Graph G with removal of an edge and a vertex

Example II.1. *The graph G_1 in Fig. 1 represents removal of an edge from G.Whereas G_2 represents removal of a vertex from G. Thus $\tau_c(G) = 2,\ \tau_c(G_1) = 3,\ \tau_c(G_2) = 2$. Thus the edge of G_1 is τ_c^+-critical. And the vertex of G_2 is τ_c-reduntant.*

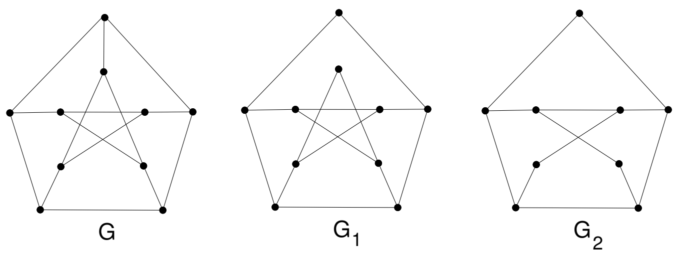

Fig. 2. Peterson Graph G

Example II.2. *From the Fig.2, $\tau_c(G) = 6, \tau_c(G_1) = 6$ and $\tau_c(G_2) = 5$. Thus the edge of G_1 is τ_c-reduntant. And the vertex of G_2 is τ_c^--critical.*

Theorem II.1.

1) For any cycle C_n, $\tau_c(C_n - v) = \left\lfloor \frac{n-1}{2} \right\rfloor$

2) *For any wheel* W_n,

$$\tau_c(W_n - v) = \begin{cases} \left\lfloor \dfrac{n-1}{2} \right\rfloor, & v \in V(K_1) \\ 1, & v \in V(C_{n-1}) \end{cases}$$

3) *Let* V_1 *and* V_2 *be the partite sets of a complete bipartite graph* $K_{m,n}$ *with* $|V_1| > |V_2|$. *Let* v *be any vertex of* $K_{m,n}$. *Then*

$$\tau_c(K_{m,n} - v) = \begin{cases} \tau_c(K_{m,n}), & v \in V_1 \\ \tau_c(K_{m,n}) - 1, & v \in V_2 \end{cases}$$

Corollary II.1.1. *If x is any edge of a graph G, then*

i *For any cycle* C_n, $\tau_c(C_n - x) = \left\lfloor \dfrac{n}{2} \right\rfloor$

ii *For any wheel* W_n,

$$\tau_c(W_n - x) = \begin{cases} 1, & x \in C_{n-1} \\ 2, & x \in K_1 \end{cases}$$

Theorem II.2. *Every* τ_c^-*-critical vertex of a graph G belongs to a* τ_c*-set and* τ_c^+*-critical vertex of G is* τ_c*-fixed.*

Proof: Let v be a τ_c^--critical vertex and S be a τ_c-set of $G - v$. Since $\tau_c(G) > \tau_c(G - v)$ it follows that $S \cup \{v\}$ is a τ_c-set of G.

Let v be τ_c^+-critical and D is a τ_c-set of G. If $v \notin D$ then D is a τ_c-set of $G - v$ also. Hence $\tau_c(G - v) \leq \tau_c(G)$ a contradiction. ■

Theorem II.3. τ_c^-*-critical vertex is* τ_c*-fixed if it is isolated and* τ_c*-free otherwise.*

Proof: If S is any τ_c-set of $G - v$, then $S \cup \{u\}$ is a τ_c-set of G for any $u \in N[v]$. Any isolated vertex is τ_c-fixed. ■

Theorem II.4. *If a polycliqual vertex v is* τ_c*-fixed, then v is* τ_c^+*-critical.*

Proof: If v is a polycliqual vertex which is τ_c-fixed. Then v is in every τ_c-set. Note that $\tau_c(G - v) \geq \tau_c(G)$. For otherwise a τ_c-set of $G - v$ could be extended to a vertex clique dominating set of G which avoids v and has cardinality atmost $\tau_c(G)$. ■

Theorem II.5. *An edge x is* τ_c*-critical if and only if there is no* τ_c*-set of* $G - x$ *with* $\tau_c(G)$ *vertices.*

Proof: Suppose an edge x is τ_c-critical. So $\tau_c(G-x) \neq \tau_c(G)$. Then $\tau_c(G - x) = \tau_c(G) \pm 1$ and there exists a τ_c-set of $G - x$ with τ_c vertices. Then $\tau_c(G - x) = \tau_c(G)$, a contradiction.

Conversely if there is no τ_c-set of $G - x$ with τ_c vertices, then $\tau_c(G - x) \pm \tau_c(G)$ and x is τ_c-critical. ■

Theorem II.6. *The support vertices of a path* P_n *with even number of vertices are always* τ_c*-fixed.*

Proof: Let $P = v_1v_2v_3 \ldots v_{n-1}v_n$ be a path on n vertices. Then the vertices v_2 and v_{n-1} are the support vertices of P_n. Clearly $P_n - v_i \cong K_1 \cup P_{n-2}$ for $i = 2, n-1$ which contains an isolated vertex. Since $\tau_c(P_n - v_i) = \tau_c(K_1 \cup P_{n-2}) = \tau_c(K_1) + \tau_c(P_{n-2}) < \tau_c(P_n)$. ■

Theorem II.7. *Every vertex v of a regular graph G is* τ_c^-*-critical.*

Proof: It may be noted that removal of any vertex from a regular graph G reduces the minimum degree of graph by 1. Hence there exists atleast two vertices of degree $\delta(G) - 1$ say u and v in $G - x$ where x is a vertex in G. Then the set $V - u$ or $V - v$ forms a clique transversal set of the graph $G - x$. Thus $\tau_c(G - x) \leq |V| - 1 < \tau_c(G)$. ■

The open neighborhood of a vertex v is $N(v) = \{u | u \text{ } is \text{ } adjacent \text{ } to \text{ } v\}$. The closed neighborhood is $N(v) \cup \{v\}$.

Theorem II.8. *Let G be a graph of order n such that* $\tau_c(G) < n$. *If an edge* $x = uv$ *of G is* τ_c^+*-critical, then for every* τ_c*-set D any one of the following conditions holds*

(i) $u \in D$ *and* $v \in V - D \Rightarrow N(v) \cap D = \{u\}$

(ii) $u, v \in D$

Proof: Let $x = uv$ be a τ_c^+ -critical edge of the graph G. Assume that none of the above two condition holds. Then there exists a τ_c-set D of G such that $u \in D$ and $v \in V - D$ but $N(v) \cap D \neq u$. Since D is a VC-dominating set one must have $|N(v) \cap D| \geq 2$. Thus v is vertex of $V - D$ which has atleast two neighbors in D and hence removal of the edge x doesnot affect the clique transversal property of D. Hence $\tau_c(G - x) = \tau_c(G)$, a contradiction. Thus any one of the conditions in the statements must be true. ■

Theorem II.9. *An edge* $x = uv$ *of a graph G is* τ_c*-fixed edge if and only if both the end vertices u and v of x are* τ_c*-fixed vertices of G.*

Proof: Let $x = uv$ be a τ_c-fixed edge of G. Then the edge lies in every $\tau_c$set of the graph G. Hence the end vertices of the edge also lies in every τ_c set of the graph. Thus u and v are τ_c-fixed vertices of G.

Conversely, if the end vertices u and v are τ_c-fixed vertices then the edge x also lies in every τ_c-set of the graph. So x is a τ_c-fixed edge of G. ■

Theorem II.10. *An edge* $x = uv$ *is a* τ_c*-free edge of G if and only if both the end vertices of x share atleast one* τ_c*-set in common but not all.*

Proof: Since $x = uv$ is a τ_c-free edge of the graph, the edge lies in some τ_c-set butnot in all. Thus both the end vertices must lie in atleast one τ_c-set of the graph. If u and v share all the τ_c-sets, then the edge $x = uv$ is a τ_c-fixed edge of the graph, a contradiction.

Conversely, if both the end vertices share atleast one τ_c-set in common but not all, then the edge x lies in atleast one τ_c-set, but not all. Hence edge is τ_c-free edge. ■

Theorem II.11. *An edge* $x = uv$ *is* τ_c*-totally free edge, if both the end points doesnot share a* τ_c*-set in common.*

Proof: Since x is a τ_c-totally free edge of the graph, the edge doesnot lie in any τ_c-set of the graph. If the end vertices share atleast one τ_c-set in common then the edge is a τ_c-free edge of the graph, a contradiction. ■

Theorem II.12. *A vertex* v_i *of a path* P_n *with even number of vertices is* τ_c*-reduntant if* $i = 2, n - 1$

Proof: The removal of support vertices from a path results in a graph which contains isolated vertices. The support vertices of a path P_n with even number of vertices is τ_c-fixed. (Refer Theorem 1.6). Thus the removal of the support vertices doesnot alter the clique transversal number. ■

Theorem II.13. *A τ_c-free vertex v of a graph G, is always τ_c-reduntant.*

Proof: If v is τ_c-free vertex of the graph G, Then G always contains atleast one τ_c-set D_1 such that $v \notin D_1$. So τ_c remains unaltered. Hence v is τ_c-reduntant. ■

Theorem II.14. *If v is a pendant vertex of a graph G, then $\tau_c(G-v) \leq \tau_c(G)$.*

Proof: If v is a pendant vertex of a graph G then $\delta(G) = 1$. If S is any τ_c-set of $G-v$, then $S \cup \{u\}$ is a τ_c-set of G for any $u \in N[v]$. ■

Theorem II.15. *If G is a $\tau_c EC$ graph and v is a vertex of G that is not a support vertex, then $\tau_c(G-v) \leq \tau_c(G)+1$*

Proof: If G is a complete graph K_n, then $n \geq 3$ and $\tau_c(G-v) = \tau_c(G) = 1$. Therefore, assume $G \neq K_n$. Suppose the neighborhood $N(v)$ forms a complete subgraph. Let S be a τ_c-set. To dominate a clique, S includes a neighbor u of v, and $N[v] \subseteq N[u]$. If $v \in S$, replace v with a vertex from $N[u]-\{v\}$, so we assume $v \notin S$. Thus, S is also a τ_c-set for $G-v$, giving $\tau_c(G-v) \leq \tau_c(G)$.

Now, assume $N(v)$ contains two non-adjacent vertices u and w. Consider the edge $x = uw \in X(\bar{G})$. Since G is a $\tau_c EC$ graph, we have $\tau_c(G-x) = \tau_c(G) \pm 1$. Let D be a $\tau_c(G-x)$-set. Assume $u \in D$. If $v \in D$, then D is a τ_c-set of G, which contradicts G being a $\tau_c EC$ graph. Hence, $v \notin D$.

Since v is not a support vertex and both u and w have degree at least 2, if $w \notin D$, then let $w' \in N(w)-\{v\}$ and note that $D \cup \{w'\}$ is a τ_c-set for $G-v$, so $\tau_c(G-v) \leq \tau_c(G)$. If $w \in D$ and u and w share a common neighbor $w' \neq v$, then $D \cup \{w'\}$ is a τ_c-set for $G-v$. Assume $N(u) \cap N(w) = \{v\}$, and let $u' \in N(u)-\{v\}$, $w' \in N(w)-\{v\}$. Then $D \cup \{u', w'\}$ is a τ_c-set for $G-v$, giving $\tau_c(G-v) \leq \tau_c(G)+1$. ■

III. Dot-Critical

Identifying or Contracting Vertices: Given two adjacent vertices v and u in a graph G, when these two vertices are identified (merged into one), the result is a new graph $G.vu$. In this new graph, the vertices v and u are replaced by a single vertex (vu), and this new vertex is adjacent to all vertices that were adjacent to either v or u in the original graph.

A graph G is called τ_c dot-critical if, for any pair of adjacent vertices v and u, contracting the edge between them (i.e., identifying v and u) decreases the clique transversal number by exactly 1. Mathematically, for any adjacent vertices v and u,

$$\tau_c(G.vu) = \tau_c(G) - 1.$$

This means that contracting an edge between adjacent vertices always reduces the clique transversal number, but only by 1. Therefore, τ_c dot-critical graphs are those for which every edge contraction has a precise effect on the clique transversal number, lowering it by exactly 1.

A graph is called totally τ_c dot-critical if, for any pair of vertices v and u (whether they are adjacent or not), identifying these two vertices reduces the clique transversal number by exactly 1. In other words, for any vertices v and u,

$$\tau_c(G.vu) = \tau_c(G) - 1.$$

This is a stronger condition than the τ_c dot-critical property because it applies to any pair of vertices, not just adjacent ones.

A τ_c dot-critical graph only requires that the clique transversal number decreases by 1 when contracting adjacent vertices. A totally τ_c dot-critical graph requires the same reduction for any pair of vertices, whether they are adjacent or not.

For any two vertices v and u in a graph G, the graph $G.vu$ is the result of identifying the two vertices. This process can be viewed as:
- Deleting both v and u from the graph G.
- Introducing a new vertex (vu), which is adjacent to all the neighbors of v and u in the original graph.

If v and u are adjacent in the original graph, then this process is equivalent to contracting the edge between them, forming the new vertex (vu) with the appropriate adjacencies.

Every totally τ_c dot-critical graph is also a τ_c dot-critical graph because the totally τ_c dot-critical property is a stronger condition. If the clique transversal number decreases by 1 for any pair of vertices (the totally τ_c dot-critical property), it will certainly decrease by 1 when contracting adjacent vertices (the τ_c dot-critical property). However, not every τ_c dot-critical graph is totally τ_c dot-critical because the latter requires the reduction to happen for any pair of vertices, not just adjacent ones.

Theorem III.1. *Let $u, v \in V(G)$ for a graph G. Then $\tau_c(G.vu) < \tau_c(G)$ if and only if either there exists an minimum clique transversal set S of G such that $u, v \in S$ or at least one of u or v is critical in G.*

Proof: Let $u, v \in V(G)$ such that $\tau_c(G.vu) < \tau_c(G)$. Consider a minimum clique transversal set S for $G.vu$.

If $(vu) \in S$, then the set

$$S^* = [S - (vu)] \cup \{v, u\}$$

serves as a minimum clique transversal set for G that includes both u and v.

If $(vu) \notin S$, there must be some $t \in S$ that is adjacent to (vu). If t is adjacent to both u and v, S would then clique-dominate G, contradicting $\tau_c(G.vu) < \tau_c(G)$. Thus, t can only be adjacent to one of u or v, say u. This implies S clique-dominates $G-v$, leading to $v \in G'$.

Conversely, if u and v are both in a common minimum clique transversal set S for G, then

$$S^* = (S - \{u, v\}) \cup (uv)$$

is a clique transversal set for $G.vu$ with size $\tau_c(G) - 1$. Additionally, if $u \in G'$, any minimum clique transversal set for $G-u$ will also serve as a clique transversal for $G.vu$ of the same size.

Thus, we conclude that $\tau_c(G.vu) < \tau_c(G)$. ■

Theorem III.2. *If u and v are adjacent vertices in a graph G on at least three vertices, then $\tau_c(G) - 1 \leq \tau_c(G.uv) \leq \tau_c(G)$*

Proof: We begin by establishing that

$$\tau_c(G.uv) \leq \tau_c(G).$$

Let S be a minimum clique transversal set for G. If S does not include u or v, then S remains a valid transversal set for $G.uv$, yielding

$$\tau_c(G.uv) \leq |S| = \tau_c(G).$$

Assuming $v \in S$, if $u \notin S$, we can form a new set

$$S_{uv} = (S - \{v\}) \cup \{(uv)\}.$$

If both u and v are in S, we select a neighbor w of either u or v (not including u or v) and create

$$S_{uv} = (S - \{u, v\}) \cup \{(uv), w\}.$$

In either scenario, $|S_{uv}| \leq |S|$, thus confirming that

$$\tau_c(G.uv) \leq \tau_c(G).$$

Next, we demonstrate that

$$\tau_c(G) - 1 \leq \tau_c(G.uv).$$

Let S_{uv} be a minimum clique transversal set for $G.uv$. If $(uv) \in S_{uv}$, we can reconstruct a set

$$S = (S_{uv} - \{(uv)\}) \cup \{u, v\},$$

maintaining the size. If $(uv) \notin S_{uv}$, we find a vertex w in S_{uv} adjacent to (uv) and include an element x from the neighborhood of w that is either u or v. In both cases, this results in

$$|S| = |S_{uv}| + 1,$$

confirming

$$\tau_c(G) \leq |S_{uv}| + 1 = \tau_c(G.uv).$$

■

Mixed block domination, as introduced by Surekha and P. G. Bhat [2], [8], is a concept in graph theory aimed at understanding the domination properties within a graph. This concept builds upon the notion of block degrees, which are defined in Surekha et. al previous work.

In their paper, Surekha and P. G. Bhat [8] defined different types of block degrees, laying the groundwork for further exploration of domination within graph structures.

The concept of clique walk, clique path and clique cycles is introduced in [9]. Let $K(G)$ denote the set of all cliques of G. Let $C_m(G)$ denote the set of all polycliqual vertices of G. Two cliques l_1 and l_2 are adjacent if there is a polycliqual vertex incident on l_1 and l_2. Again two polycliqual vertices c_{m_1} and c_{m_2} are adjacent if there is a common clique incident on c_{m_1} and c_{m_2}. A *clique graph* $K_G(G)$ is a graph with vertex set $K(G)$ and two cliques l_1 and l_2 are adjacent in $K_G(G)$ if there is a polycliqual vertex incident on l_1 and l_2. A *polycliqual vertex graph* $C_{mG}(G)$ is a graph with vertex set $C_m(G)$ and two polycliqual vertices c_{m_1} and c_{m_2} are adjacent in $C_{mG}(G)$ if they are adjacent in G. A polycliqual vertex graph $C_{mG}(G)$ is a subgraph of G without unicliqual vertices. Further a *clique polycliqual vertex graph* $(CPV - graph)$ $KC_m(G)$ is a bigraph with vertex set $K(G) \cup C_m(G)$ and a polycliqual vertex $c_m \in C_m(G)$ and a clique $l \in K(G)$ are adjacent if and only if c_m is incident on the clique l.

IV. Clique Walks and Cliqe Paths and Clique Trees

A $clique - walk$ $(C - walk)$ is a sequence of cliques and polycliqual vertices say, $l_1, c_{m_2}, l_3, c_{m_4}, l_5, \ldots, l_{m-2}, c_{m_{m-1}}, l_m$ beginning and ending with cliques in which each polycliqual vertex c_{m_i} is incident with the cliques l_{i-1}, l_{i+1}. The *length* of a clique walk is the number of polycliqual vertices in a clique-walk. If all the polycliqual vertices are distinct in a clique walk, then such a walk is called a *clique path*. A clique path with l cliques is denoted as C_{pl}. A graph G is said to be *clique connected* if there is a clique path between any two cliques of G.

A graph G is said to be a *clique tree* if there exists a unique clique path between any two cliques of a graph. A graph G is a clique tree if and only if G is a block graph. A *clique cycle* denoted by C_{cl} is a clique path in which the starting and terminal cliques are same.

A graph G with l-cliques is $Clique - Complete\ graph$ Q_l if any two cliques of G are adjacent. A graph G is a $Clique - Star$ denoted by $C_{l_1, l_2, \ldots, l_k}$, if there exists a clique l with c cutvertices and i^{th} cutvertex is incident with $l_i + 1$ cliques, $l_i \in N$, $1 \leq i \leq k$. The clique l is then called the central clique. Clique graph of a clique star is clique complete. Clique graph of a clique complete graph is a complete graph K_l.

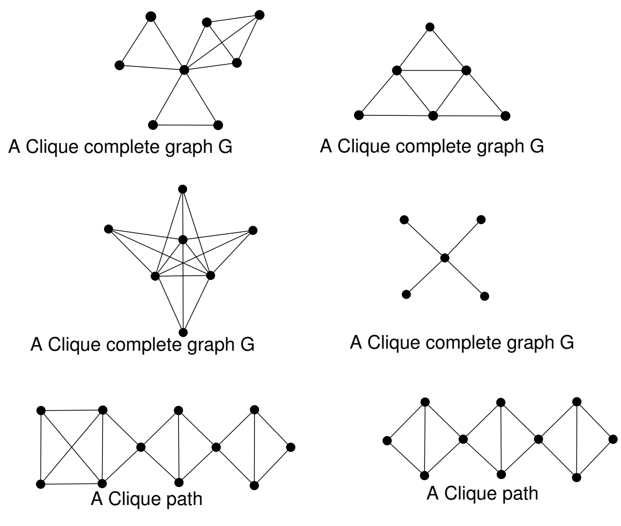

Fig. 3. Different types of clique graphs

V. Clique-distance between two cliques

For any two cliques $l_1, l_2 \in K(G)$, the *clique distance* $d(l_1, l_2)$ is the length of the clique path from l_1 to l_2. Further $d(l_1, l_2)$ has the following properties.

1) $d(l_1, l_2) \geq 0$ and $d(l_1, l_2) = 0$ if and only if $l_1 = l_2$
2) $d(l_1, l_2) = d(l_2, l_1)$
3) $d(l_1, l_2) \leq d(l_1, l_3) + d(l_3, l_2)$

Using clique distance we can define the central tendences of a clique, like clique radius and clique diameter of a

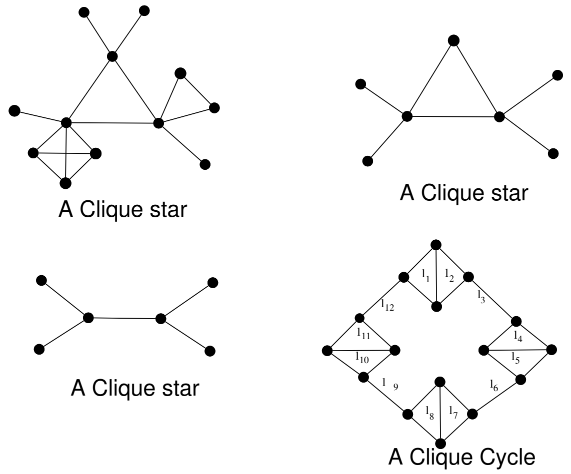

Fig. 4. Different types of clique star and a clique cycle

clique in a graph. The $clique-eccentricity$ of a clique $l \in K(G)$ is defined as $e(l) = \max_{l_1, l_2 \in K(G)} \{d(l_1, l_2)\}$. Then $clique-diameter$ $d_c(G) = \max_{l \in K(G)} \{e(l)\}$ and $clique-radius$ $r_c(G) = \min_{l \in K(G)} \{e(l)\}$. The set of all cliques with minimum eccentricity is called $clique-center$ $C_c(G)$ of the graph G.

Proposition V.1. *For any graph G, $C_{mG}(G) \cong G$ if and only if G has no unicliqual vertex.*

Proof: If there is no unicliqual vertex then every vertex is polycliqual vertex. ■

Proposition V.2. *For any graph G,*

$$r_b(G) \le r_c(G)$$

Proof: As every cutvertex is a polycliqual vertex, we have $e(h) \le e(l)$. ■

Proposition V.3. *For any graph G,*

$$d(B_G(G)) \le d(K_G(G)) \le d(G)$$

Proof: Diametrical path is a subset of diametrical clique path which is a subset of diametrical block path. Any block path contains $m-1$ blocks, any clique path contains $l-1$ cliques and any path contains n-1 edges. We observe that number of blocks in the block path is less than or equal to number of cliques in clique path which is less than or equal to number of edges in a path. Therefore $m-1 \le l-1 \le e-1$. Which implies $d_b \le d_c \le d$. ■

Proposition V.4. *For any graph G,*

$$C_G(G) \subseteq C_{mG}(G) \subseteq P_G(G)$$

Proof: We observe that $C(G) \subseteq C_m(G) \subseteq V(G)$. If u and w are clique adjacent implies u and w are vv-adjacent. Therefore $C_{mG}(G) \subseteq P_G(G)$. If c_1 and c_2 are two adjacent cutvertices implies c_1 and c_2 are cutvertices of a block b. Which implies c_1 and c_2 are cutvertices of a clique which is a subgraph of b. Which implies c_1 and c_2 are clique adjacent. ■

Proposition V.5. *For any graph G,*

$$d_b(G) \le d_c(G)$$

Further equality holds iff G is a block graph.

Proof: Let $b_1, c_2, b_3, c_4, b_5, \ldots, c_s, b_{s+1}$ be a diametrical block path with $d_b = \frac{s}{2}$. Inside this diametrical block path there is a clique path $k_1, u_2, k_3, u_4, k_5, \ldots, u_{t_1} = c_2, k_{t_1+1}, u_{t_1+2}, \ldots, u_{t_2} = c_3, k_{t_2+1}, u_{t_2+2}, \ldots, u_{t_s} = c_s, k_{t_{s+1}}, u_{t_{s+2}}, \ldots, u_{t_{s+h}}, k_{t_{s+h+1}}$ with clique distance $= \frac{t_{s+h}}{2} = \frac{s+h}{2}$ since $t_s = s$. Therefore $d_b = \frac{s}{2} < \frac{s+h}{2} \le d_c$. ■

Note V.1. *For any block graph G, block diameter and clique diameter are identical.*

Proposition V.6. *For any graph G, clique diameter and diameter are incomparable.*

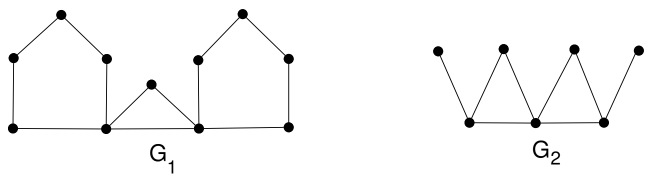

Fig. 5. A Graph G

Example V.1. *For the graph G_1 of Fig. 5, clique diameter $d_c = 6$ where as diameter $d = 5$. For G_2 of Fig. 5, clique diameter $d_c = 3$ where as diameter $d = 4$. Thus clique diameter and diameter are incomparable.*

Corollary V.6.1. *For any block graph,*

$$d_c = d - 1$$

Let $\Delta_{s_n}(G), \delta_{s_n}(G)$ denote the maximum and minimum signature of G respectively. A graph G is said to be k-signature regular if $s_n(v) = k$ for every $v \in V$. Any signature regular graph need not be regular. But if every edge is contained in a triangle, then every signature regular graph is also regular. For example, the graph obtained by removing the edges joining antipodal vertices from K_6 is 4-signature regular and 4-regular graph. We (refer [10]) obtained bounds for n-covering number n_0 (neighborhood number) of G in terms of maximum strength $\Delta_s(G)$.

Proposition V.7. *For any graph G,*

$$\frac{q}{\Delta_s} \le n_0(G) \le q - \Delta_s + 1 \quad (1)$$

Further, these bounds are sharp.

The argument for Proposition (V.7) leads us to identify a corresponding lower bound for the *open full domination number*, as defined by Brigham et al. [1]. In a graph G, a vertex v is said to openly dominate the subgraph $\langle N(v) \rangle$, which consists of its open neighborhood $N(v)$. A collection of vertices S is classified as a *full open dominating set* if it ensures that every edge in G is included in $\langle N(v) \rangle$ for at least one vertex $v \in S$. The minimum size of such a set is known as the *full open domination number* $\gamma_{FO}(G)$. A graph G is guaranteed to have a full open dominating set if it contains no isolated vertices and if each edge of G is part of a triangle. Therefore, we derive a lower bound for the full open domination number in relation to the maximum signature $\Delta_{s_n}(G)$.

Proposition V.8. *For any graph G in which every edge lies in a triangle,*

$$\frac{q}{\Delta_{s_n}(G)} \leq \gamma_{FO}(G) \tag{2}$$

Proof: Given that a vertex v can openly dominate a maximum of $\Delta_{s_n}(G)$ edges, it follows that to dominate all edges of G, a minimum of $\frac{q}{\Delta_{s_n}(G)}$ vertices is required. Therefore, we conclude that $\gamma_{FO}(G) \geq \frac{q}{\Delta_{s_n}(G)}$. ■

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