Point Estimation of Poisson Parameter by Bayesian Approach under Different Loss Functions

Yadpirun Supharakonsakun, Nitaya Phuwong, and Chitchanok Khamnang

***Abstract*—In the classical Poisson model, the distribution represents the number of events occurring within a given time or spatial interval. This study introduces new Bayesian methods for point estimation of the Poisson parameter, utilizing precautionary, entropy, and general entropy loss functions, particularly focusing on cases where the constants are $c = 2$ and 3. These methods are compared to traditional Bayesian estimators based on squared error and quadratic loss functions. A Monte Carlo simulation study was conducted to evaluate the performance of the proposed estimators, using mean squared error (MSE) as the primary criterion. The results demonstrate that the Bayesian approach, employing quadratic, entropy, and general entropy loss functions with $c = 2$, provided the most accurate estimates for smaller true parameter values ($\lambda = 0.5$, 1, or 2), yielding the lowest MSE. For moderately larger true parameter values ($\lambda = 3$, 5), the squared error and quadratic loss functions produced the minimum MSE across a range of sample sizes. For larger true parameter values ($\lambda = 10$, 20, 30, and 50), the precautionary loss function exhibited superior performance. These findings underscore the versatility and accuracy of different Bayesian loss functions for Poisson parameter estimation under varying conditions.**

***Index Terms*—Bayesian estimator, precautionary loss function, entropy loss function, Poisson distribution, Point estimation, squared error loss function, quadratic loss function**

I. Introduction

THE Poisson distribution is a discrete probability distribution widely used to model the number of occurrences within a fixed interval of time or space, assuming that events occur independently and at a constant rate. This distribution focuses on counting the occurrences rather than the magnitude of the events. Common applications of the Poisson model include counting the number of calls at a police station within an hour, the number of trucks passing through a toll booth in a specific period, the number of goals scored in a soccer match, or the frequency of typographical errors on a page. Due to its ability to model random events occurring at a consistent rate, Poisson data is applied in various fields such as biology, economics, telecommunications, and public health.

Accurately determining the parameters of a population is often challenging, especially for large datasets. Parameter estimation can be divided into two main types: point estimation, which provides a single value as an estimate, and interval estimation, which gives a range of values with a certain confidence level. In statistical analysis, two primary approaches for parameter estimation are commonly used: the classical method and the Bayesian method. The classical approach treats parameters as fixed unknown values, relying on sampling to estimate them. In contrast, the Bayesian approach treats parameters as random variables with associated probability distributions and integrates prior knowledge into the estimation process. Bayesian estimation is essentially a conditional approach, using the posterior distribution, which is proportional to the product of the likelihood function and the prior distribution.

In recent years, increasing interest has been directed toward comparing the classical and Bayesian estimation methods in various contexts. Araveeporn [1] explored Poisson parameter estimation, comparing classical and Bayesian methods. Similarly, Silva and Otiniano [2] derived an exact expression and interval for the sum of expected absolute differences in Poisson processes. Hasan and Baizid [3] extended this analysis to the exponential distribution, evaluating several loss functions, including squared error, quadratic, modified linear exponential (MLINEX), and non-linear exponential (NLINEX), and demonstrated the superiority of Bayesian estimators under certain loss functions. Naji and Rasheed [4] contributed to this field by proposing Bayesian estimators for the Gamma distribution's shape and scale parameters under an entropy loss function, comparing these estimators with the Moment method and Maximum Likelihood Estimators (MLE), where Bayesian estimators consistently performed better under the entropy loss function.

Further advancing this research, Singh et al. [5] investigated Bayesian estimators for the exponentiated exponential distribution under a general entropy loss function, providing a detailed comparison with the squared error loss function and MLE. Rasheed et al. [6] focused on the Laplace distribution's scale parameter, using an entropy

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loss function, and demonstrated the superiority of Bayesian estimators, especially when informative priors were used in moderate to large sample sizes. Chen and Liu [7] extended Bayesian estimation to the exponential distribution using a gamma prior under a precautionary loss function. Yahgmaei et al. [8] compared frequentist and Bayesian estimators for the inverse Weibull distribution's scale parameter under various loss functions, while Al-Duais [9], [10] introduced a general entropy loss function for estimating the Weibull distribution's reliability function, demonstrating the effectiveness of Bayesian methods over traditional approaches.

Hao and Li [11] examined Bayesian estimators for the Burr type X distribution under the squared error loss function, approximating the posterior mean and variance. Hassan and Zaky [12] explored entropy-based Bayesian estimators for the exponential distribution under LINEX, squared error, and precautionary loss functions, highlighting the success of Bayesian estimation, particularly under non-informative priors with large parameter values. Hassan [13] investigated Bayesian estimators for the double exponential distribution using both symmetric (squared error and quadratic) and asymmetric (MLINEX and NLINEX) loss functions, demonstrating the effectiveness of the Bayesian approach, particularly under quadratic loss for simulated data, a finding echoed by Supharakonsakun [14] for Poisson parameter estimation. Jia and Song [15] used Bayesian methods to estimate parameters from the Lindley distribution under multiple loss functions, advocating for Bayesian approaches in interval estimation. Hassan et al. [16] examined the generalized inverted exponential distribution, showing the strength of Bayesian methods under different loss functions.

Loss functions play a crucial role in Bayesian estimation, providing a flexible framework for balancing the trade-off between estimation bias and variance. The use of informative priors often leads to more accurate estimates compared to non-informative priors. This study focuses on Bayesian estimation of the Poisson parameter, specifically employing a gamma prior distribution. Our motivation stems from the robustness of Bayesian methods in point estimation, particularly when using various symmetric and asymmetric loss functions.

The objective of this paper is to investigate the Bayesian estimation of the Poisson parameter, evaluating its performance under different loss functions, including precautionary, entropy, and general entropy loss functions. The accuracy of these Bayesian estimations will be assessed and compared with the commonly used squared error and quadratic loss functions for Poisson parameter estimation.

II. Bayesian Estimation Methods for Mean of Poisson Distribution

The estimation of the mean of a Poisson distribution and a gamma distribution is carried out using a natural conjugate prior under five different loss functions.

Let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables from a Poisson distribution with parameter λ. The probability mass function of X is given by $f\left(x_i|\lambda\right)$ with constant mean rate of event occur can be defined as follows

$$f\left(x_i|\lambda\right)=\frac{e^{-\lambda}\lambda^{x_i}}{x_i!}\ ,\ \lambda>0\ . \tag{1}$$

In this study, the gamma distribution is considered as the conjugate prior, with the probability density function as follows:

$$\pi\left(\lambda\right)=\frac{b^a}{\Gamma\left(a\right)}e^{-b\lambda}\lambda^{a-1}\ ,\ a>0,b>0\ . \tag{2}$$

The posterior distribution in Bayesian methodology can be obtained by combining the likelihood function with the prior distribution in the following manner:

$$\prod_{i=1}^{n} f\left(x_i|\lambda\right)\pi\left(\lambda\right)=\frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}\cdot\frac{b^a}{\Gamma\left(a\right)}e^{-b\lambda}\lambda^{a-1}. \tag{3}$$

The integration of the likelihood function and the prior distribution yields the marginal probability density function of λ . This derivation can be outlined as follows:

$$\int_0^\infty\prod_{i=1}^{n} f\left(x_i|\lambda\right)\pi\left(\lambda\right)d\lambda=\frac{b^a\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}{\Gamma\left(a\right)\prod_{i=1}^{n}x_i!\left(n+b\right)^{\sum_{i=1}^{n}x_i+a}}\ . \tag{4}$$

The posterior distribution of λ is derived using equations (3) and (4), giving:

$$h\left(\lambda\,|\,\underline{x}\right)=\frac{\prod_{i=1}^{n} f\left(x_i|\lambda\right)\pi\left(\lambda\right)}{\int_0^\infty\prod_{i=1}^{n} f\left(x_i|\lambda\right)\pi\left(\lambda\right)d\lambda}\ .$$

This indicates that the posterior distribution can be formulated as:

$$h\left(\lambda\,|\,\underline{x}\right)=\frac{\left(n+b\right)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}e^{-\lambda\left(n+b\right)}\lambda^{\left(\sum_{i=1}^{n}x_i+a\right)-1}. \tag{5}$$

The resulting distribution is a gamma distribution with parameters $\sum_{i=1}^{n}x_i+a$ and $n+b$. Consequently,

$$Gamma\left(\sum_{i=1}^{n}x_i+a\ ,\ n+b\right). \tag{6}$$

III. The Bayesian Estimation for Squared Error Loss Function

The Bayesian estimator for λ under the squared error loss function is defined as:

$$L(\hat{\lambda};\lambda)=(\hat{\lambda}-\lambda)^2 . \tag{7}$$

The Bayesian estimator's squared error loss function denoted by $\hat{\lambda}_{SL}$ corresponds to the mean of the posterior distribution function, as derived by Supharakonsakun [14]. Specifically, the Bayesian point estimator for the Poisson mean under this squared error loss function is represented as follows:

$$\hat{\lambda}_{SL}=E\left(\lambda \mid \underline{x}\right) . \tag{8}$$

Likewise, it can be derived as follows:

$$\frac{\partial}{\partial\hat{\lambda}}\int_0^\infty L\left(\hat{\lambda};\lambda\right)h(\lambda \mid \underline{x})d\lambda=0 .$$

$$\frac{\partial}{\partial\hat{\lambda}}\int_0^\infty \left(\hat{\lambda}-\lambda\right)^2 \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\lambda^{(\sum x_i+a)-1}e^{-(n+b)\lambda}d\lambda=0 .$$

This depicts the expression of the Bayesian estimator for λ under the squared error loss function as:

$$\hat{\lambda}_{SL}=\frac{\sum_{i=1}^{n}x_i+a}{n+b} . \tag{9}$$

IV. THE BAYESIAN ESTIMATOR FOR QUADRATIC LOSS FUNCTION

The quadratic loss function, characterized by its non-negativity, symmetry, and continuity with respect to the parameter λ, is used to determine the Bayesian estimator's estimate λ according to the definition provided as follows;

$$L(\hat{\lambda};\lambda)=\left(\frac{\hat{\lambda}-\lambda}{\lambda}\right)^2 . \tag{10}$$

Supharakonsakun [14] proposed the quadratic loss function denoted by $\hat{\lambda}_{QL}$ for the Bayesian estimator. Using this quadratic loss function, the Bayesian point estimator for the Poisson mean is expressed as:

$$\hat{\lambda}_{QL}=\frac{\partial}{\partial\hat{\lambda}}\int_0^\infty L\left(\hat{\lambda};\lambda\right)f(\lambda \mid x_i)d\lambda=0 .$$

$$\frac{\partial}{\partial\hat{\lambda}}\int_0^\infty \left(\frac{\hat{\lambda}-\lambda}{\lambda}\right)^2 \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\lambda^{(\sum_{i=1}^{n}x_i+a)-1}e^{-(n+b)\lambda}d\lambda=0 .$$

Thus the expression for the Bayesian estimator of λ using a quadratic loss function is:

$$\hat{\lambda}_{QL}=\frac{\sum_{i=1}^{n}x_i+a-2}{n+b} . \tag{11}$$

V. THE BAYESIAN ESTIMATOR FOR PRECAUTIONARY LOSS FUNCTION

The Bayesian estimator's precautionary loss function, which is both symmetric and continuous with respect to the parameter λ and estimated λ, is formally characterized as follows [17]-[19]:

$$L(\hat{\lambda};\lambda)=\left(\frac{\hat{\lambda}-\lambda}{\hat{\lambda}}\right)=\left[E(\lambda^2 \mid x)\right]^{\frac{1}{2}} . \tag{12}$$

The precautionary loss function of Bayesian estimator, denoted by $\hat{\lambda}_{PR}$, is given by:

$$E\left(\lambda^2 \mid x\right)=\int_0^\infty \lambda^2 \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\lambda^{(\sum_{i=1}^{n}x_i+a)-1}e^{-(n+b)\lambda}d\lambda .$$

$$E\left(\lambda^2 \mid x\right)=\frac{\left(\sum_{i=1}^{n}x_i+a+1\right)\left(\sum_{i=1}^{n}x_i+a\right)}{(n+b)^2} .$$

Therefore,

$$\left[E(\lambda^2 \mid x)\right]^{\frac{1}{2}}=\left[\frac{\left(\sum_{i=1}^{n}x_i+a+1\right)\left(\sum_{i=1}^{n}x_i+a\right)}{(n+b)^2}\right]^{\frac{1}{2}} . \tag{13}$$

The expression for the Bayesian estimator of λ under the precautionary loss function is:

$$\hat{\lambda}_{PR}=\frac{\sqrt{\left(\sum_{i=1}^{n}x_i+a+1\right)\left(\sum_{i=1}^{n}x_i+a\right)}}{n+b} . \tag{14}$$

VI. THE BAYESIAN ESTIMATOR FOR ENTROPY AND GENERAL ENTROPY LOSS FUNCTION

In numerous real-life scenario, it often seems more pragmatic to represent the loss function by using the ratio $\frac{\hat{\lambda}}{\lambda}$. In this context, Calabria and Pulcini [20] and Yahgmaei et al. [8] emphasize the utility of an asymmetric loss function, particularly the entropy (EN) loss function. The original form of Dey and Liu [21] has been employed in the form as:

$$L(\hat{\lambda};\lambda)=\left(\frac{\hat{\lambda}}{\lambda}\right)-\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1. \quad (15)$$

The Bayesian estimator under the entropy loss function, denoted by $\hat{\lambda}_{EN}$, is given by:

$$\hat{\lambda}_{EN}=\left[E\left(\lambda^{-1}\mid x\right)\right]^{-1}. \quad (16)$$

Based on the posterior distribution of the parameter λ, $E\left(\lambda^{-1}\mid x\right)$ can be obtained as follows:

$$E\left(\lambda^{-1}\mid x\right)=\int_0^\infty \lambda^{-1}\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\lambda^{\sum_{i=1}^{n}x_i+a-1}e^{-\lambda(n+b)}d\lambda.$$

$$E\left(\lambda^{-1}\mid x\right)=\frac{n+b}{\sum_{i=1}^{n}x_i+a-1}.$$

Then,

$$\left[E\left(\lambda^{-1}\mid x\right)\right]^{-1}=\left[\frac{n+b}{\sum_{i=1}^{n}x_i+a-1}\right]^{-1}. \quad (17)$$

The formula for the Bayesian estimator of λ under the entropy loss function is expressed as:

$$\hat{\lambda}_{EN}=\frac{\sum_{i=1}^{n}x_i+a-1}{n+b}. \quad (18)$$

The proposed loss function is based on the general entropy (GE) loss function for Bayesian parameter estimation and is presented in the following form:

$$L(\hat{\lambda};\lambda)=\left(\frac{\hat{\lambda}}{\lambda}\right)^c-c\log\left(\frac{\hat{\lambda}}{\lambda}\right)-1; c\neq 0. \quad (19)$$

The Bayesian estimator within the framework of the general entropy loss function, represented as $\hat{\lambda}_{GE}$, is expressed as:

$$\hat{\lambda}_{GE}=\left[E\left(\lambda^{-c}\mid x\right)\right]^{-\frac{1}{c}}. \quad (20)$$

Derived from the posterior distribution of the parameter λ, $E\left(\lambda^{-c}\mid x\right)$ can be acquired in the following way:

$$E\left(\lambda^{-c}\mid x\right)=\int_0^\infty \lambda^{-c}\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\cdot\lambda^{\sum_{i=1}^{n}x_i+a-1}\cdot e^{-\lambda(n+b)}d\lambda.$$

$$E\left(\lambda^{-c}\mid x\right)=\frac{\Gamma\left(\sum_{i=1}^{n}x_i+a-c\right)}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}(n+b)^c.$$

Therefore,

$$\left[E\left(\lambda^{-c}\mid x\right)\right]^{-c}=\left[\frac{\Gamma\left(\sum_{i=1}^{n}x_i+a-c\right)}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\right]^{-\frac{1}{c}}(n+b)^{-1}. \quad (21)$$

In this investigation, the values $c = 2$ and $c = 3$ are applied for Bayesian parameter estimation within the context of the general entropy loss function. The corresponding expressions for these estimators are articulated as follows:

$$\hat{\lambda}_{GE(c=2)}=\frac{\sqrt{\left(\sum_{i=1}^{n}x_i+a-1\right)\left(\sum_{i=1}^{n}x_i+a-2\right)}}{n+b}. \quad (22)$$

$$\hat{\lambda}_{GE(c=3)}=\frac{\sqrt[3]{\left(\sum_{i=1}^{n}x_i+a-1\right)\left(\sum_{i=1}^{n}x_i+a-2\right)\left(\sum_{i=1}^{n}x_i+a-3\right)}}{n+b}. \quad (23)$$

VII. Results and Discussion

A simulation consisting of 10,000 iterations was conducted to evaluate the effectiveness of point estimations using 6 distinct loss functions within a Bayesian framework. This evaluation considered the variation of hyperparameters *a* and *b* within Bayesian point estimation techniques. The Bayesian estimators under the squared error loss function (SE), quadratic loss function (QL), precautionary loss function (PR), entropy loss function (EN), and the general entropy loss function with $c = 2$ (EN (c=2)) and $c = 3$ (EN (c=3)) for estimating the mean of a Poisson distribution were compared through simulation.

The criterion used to establish the most effective method under consistent conditions was based on identifying the approach with the lowest mean squared error (MSE). This criterion serves as the evaluative measure for the performance of the parameter estimation methods and is calculated as follows:

$$MSE=\frac{\sum_{t=1}^{m}\left(\lambda-\hat{\lambda}_t\right)}{m}. \quad (24)$$

TABLE I
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 5, 10, 15 AND 20 GIVEN $a=5$, $b=0.5$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
5	0.5	0.83642	0.33822	0.99698	0.55426	0.43378	**0.32819**
	1	0.82793	0.36772	0.98058	0.56477	0.45549	**0.36227**
	2	0.85747	0.45957	0.99612	0.62546	0.53312	**0.45741**
	3	0.90939	0.57397	1.03319	0.70862	0.63247	**0.57312**
	5	1.04918	**0.84313**	1.14140	0.91310	0.86980	0.84343
	10	**1.64430**	1.75762	1.65733	1.66790	1.70486	1.75881
	20	4.10330	4.88578	**3.94942**	4.46148	4.66596	4.88748
	30	8.17529	9.62206	**7.85547**	8.86561	9.23624	9.62393
	50	21.85440	24.67128	**21.19217**	23.22978	23.94301	24.67332
10	0.5	0.25380	0.11666	0.29834	0.17616	0.14338	**0.11494**
	1	0.27381	0.14640	0.31633	0.20103	0.17105	**0.14556**
	2	0.32892	0.21786	0.36768	0.26432	0.23866	**0.21755**
	3	0.37959	0.28848	0.41350	0.32497	0.30438	**0.28837**
	5	0.49952	**0.44702**	0.52390	0.46420	0.45334	0.44708
	10	**0.90470**	0.93488	0.90849	0.91072	0.92058	0.93505
	20	2.12090	2.33809	**2.07800**	2.22042	2.27708	2.33833
	30	3.60488	4.00010	**3.51749**	3.79342	3.89459	4.00036
	50	8.13907	8.90527	**7.95895**	8.51310	8.70702	8.90555
15	0.5	0.12342	0.06148	0.14372	0.08829	0.07359	**0.06092**
	1	0.14670	0.08887	0.16614	0.11362	0.10008	**0.08862**
	2	0.19243	0.14278	0.20994	0.16344	0.15202	**0.14269**
	3	0.23341	0.19233	0.24882	0.20871	0.19946	**0.19230**
	5	0.33937	**0.31521**	0.35059	0.32313	0.31813	0.31522
	10	**0.61659**	0.63318	0.61766	0.62072	0.62593	0.63323
	20	1.34047	1.44050	**1.32069**	1.38633	1.41240	1.44058
	30	2.31348	2.50187	**2.27161**	2.40351	2.45168	2.50196
	50	4.72672	5.07469	**4.64496**	4.89654	4.98461	5.07477
20	0.5	0.07708	0.04167	0.08872	0.05700	0.04862	**0.04143**
	1	0.09471	0.06166	0.10584	0.07580	0.06808	**0.06155**
	2	0.13392	0.10541	0.14398	0.11728	0.11073	**0.10537**
	3	0.17500	0.15065	0.18404	0.16044	0.15494	**0.15063**
	5	0.24614	**0.23327**	0.25232	0.23733	0.23471	0.23328
	10	**0.47394**	0.48319	0.47460	0.47618	0.47910	0.48321
	20	0.99727	1.05398	**0.98607**	1.02324	1.03803	1.05401
	30	1.62529	1.72985	**1.60213**	1.67519	1.70194	1.72988
	50	3.29150	3.49037	**3.24477**	3.38855	3.43888	3.49040

*The bold numbers represent the lowest MSE.

Random sampling was employed to generate 10,000 datasets from a Poisson distribution, considering various sample sizes (n = 5, 10, 15, 20, 25, 30, 40, 50) and true parameters of $\lambda=$ 0.5, 1, 2, 3, 5, 10, 20, 30, and 50. This was performed using arbitrary prior parameters (a, b) = (5, 0.5), (3, 1), (2, 0.5), (3, 2), (4, 2), (2, 2) and (1, 1) for Bayesian methods across 6 different loss functions. The smallest MSE value signifies that the estimated value of closely aligns with its true value.

Table 1 presents MSE values for point estimation employing 6 distinct loss functions across varying hyperparameters $(a,b)=(5,\ 0.5)$, in Bayesian approaches for sample sizes $n=5$, 10, 15, and 20. Across all sample sizes, Bayesian method of general entropy loss function with c =3 demonstrated superior performance by yielding the lowest MSE for true parameters 0.5, 1, 2, and 3. Meanwhile, the Bayesian estimator under quadratic loss function proved to be the most accurate for a true parameter $\lambda=$ 5, and Bayesian estimation for squared error loss function closely approximated the true parameter for $\lambda=10$. Notably, focusing on Bayesian estimation for precautionary loss function exhibited exceptional performance with the lowest MSE for $\lambda=20$, 30, and 50.

TABLE II
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 25, 30, 40 AND 50 GIVEN $a=5$, $b=0.5$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (c=2)**	**EN (c=3)**
25	0.5	0.05493	0.03152	0.06260	0.04168	0.03615	**0.03138**
	1	0.07010	0.04838	0.07740	0.05770	0.05263	**0.04832**
	2	0.10243	0.08364	0.10902	0.09150	0.08717	**0.08362**
	3	0.13623	0.12013	0.14216	0.12664	0.12299	**0.12012**
	5	0.20327	**0.19317**	0.20771	0.19668	0.19454	0.19318
	10	**0.38794**	0.39272	0.38866	0.38879	0.39037	0.39273
	20	0.81003	0.84882	**0.80225**	0.82788	0.83797	0.84884
	30	1.29027	1.35620	**1.27572**	1.32170	1.33857	1.35622
	50	2.54700	2.67819	**2.51613**	2.61106	2.64424	2.67821
30	0.5	0.04043	0.02435	0.04573	0.03131	0.02753	**0.02427**
	1	0.05308	0.03833	0.05808	0.04463	0.04119	**0.03829**
	2	0.08246	0.06937	0.08706	0.07484	0.07183	**0.06936**
	3	0.10885	0.09802	0.11289	0.10236	0.09991	**0.09801**
	5	0.16496	**0.15791**	0.16806	0.16036	0.15887	0.15791
	10	**0.32677**	0.32971	0.32738	0.32716	0.32817	0.32972
	20	0.68453	0.70959	**0.67961**	0.69599	0.70252	0.70960
	30	1.05386	1.09990	**1.04369**	1.07581	1.08759	1.09991
	50	2.04129	2.13171	**2.02003**	2.08542	2.10830	2.13172
40	0.5	0.02607	0.01690	0.02910	0.02087	0.01872	**0.01686**
	1	0.03656	0.02804	0.03943	0.03169	0.02971	**0.02803**
	2	0.05846	0.05100	0.06108	0.05412	0.05241	**0.05099**
	3	0.08025	0.07410	0.08255	0.07657	0.07518	**0.07410**
	5	0.12714	**0.12351**	0.12881	0.12472	0.12396	0.12351
	10	**0.23829**	0.24057	0.23848	0.23882	0.23954	0.24057
	20	0.48962	0.50469	**0.48661**	0.49655	0.50047	0.50469
	30	0.78943	0.81597	**0.78355**	0.80209	0.80888	0.81597
	50	1.45790	1.50886	**1.44592**	1.48277	1.49566	1.50886
50	0.5	0.01893	0.01288	0.02092	0.01552	0.01409	**0.01287**
	1	0.02684	0.02151	0.02865	0.02378	0.02254	**0.02150**
	2	0.04435	0.03971	0.04600	0.04164	0.04058	**0.03971**
	3	0.06377	0.05973	0.06526	0.06135	0.06044	**0.05973**
	5	0.10161	**0.09950**	0.10263	0.10016	0.09973	0.09950
	10	**0.19139**	0.19305	0.19147	0.19183	0.19234	0.19305
	20	0.40708	0.41709	**0.40507**	0.41169	0.41429	0.41709
	30	0.61648	0.63361	**0.61269**	0.62465	0.62903	0.63361
	50	1.14879	1.18272	**1.14079**	1.16536	1.17394	1.18272

*The bold numbers represent the lowest MSE.

Table 2 displays MSE values for point estimation utilizing 6 different loss functions with varying hyperparameters $a=5$ and $b=0.5$ within Bayesian approaches for sample sizes $n=25$, 30, 40, and 50. Across all these sample sizes, the Bayesian method employing the general entropy loss function with $c=3$ demonstrated superior performance, consistently yielding the lowest MSE for true parameters of $\lambda=0.5$, 1, 2, and 3, which highlights its accuracy in estimating smaller Poisson parameters. In contrast, for a true parameter of $\lambda=5$, the quadratic loss function provided the most precise estimates, showcasing its reliability for moderate parameter values. Similarly, the squared error loss function delivered estimates that closely matched the true parameter $\lambda=10$, confirming its robustness for mid-range values. As seen in previous analyses, the Bayesian estimation using the precautionary loss function once again outperformed other methods for larger true parameters, achieving the lowest MSE for $\lambda=20$, 30, and 50. This reinforces the precautionary loss function's effectiveness in handling larger Poisson parameters across larger sample sizes, making it an ideal choice for more complex estimation scenarios.

TABLE III
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n =$ 5, 10, 15 AND 20 GIVEN $a = 3,\ b = 1$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (c=2)**	**EN (c=3)**
5	0.5	0.24263	**0.07516**	0.31599	0.13112	0.09548	0.08546
	1	0.25612	**0.14099**	0.31794	0.17078	0.14935	0.14600
	2	0.30898	0.30488	0.34431	**0.27915**	0.28600	0.30857
	3	**0.40812**	0.52426	0.41396	0.43841	0.47551	0.52803
	5	0.80063	1.13620	**0.75207**	0.94064	1.03270	1.13992
	10	2.75108	3.63851	**2.56493**	3.16702	3.39712	3.64220
	20	10.63546	12.62137	**10.17487**	11.60063	12.10539	12.62504
	30	24.55333	27.67720	**23.80834**	26.08749	26.87677	27.68092
	50	68.40789	73.74454	**67.10975**	71.04844	72.39092	73.74825
10	0.5	0.09475	**0.04438**	0.11704	0.06130	0.05072	0.04537
	1	0.11736	**0.08315**	0.13596	0.09199	0.08558	0.08372
	2	0.17863	0.17676	0.18937	**0.16943**	0.17117	0.17730
	3	**0.24891**	0.28365	0.25058	0.25801	0.26894	0.28422
	5	0.44207	0.54481	**0.42681**	0.48517	0.51311	0.54537
	10	1.23023	1.49110	**1.17550**	1.35240	1.41988	1.49165
	20	4.07335	4.66852	**3.93507**	4.36267	4.51373	4.66907
	30	8.39186	9.30534	**8.17401**	8.84034	9.07097	9.30588
	50	22.43375	24.02234	**22.04712**	23.21978	23.61920	24.02289
15	0.5	0.05334	**0.02974**	0.06391	0.03763	0.03269	0.02993
	1	0.07531	**0.05896**	0.08419	0.06323	0.06015	0.05913
	2	0.11833	0.12009	0.12276	**0.11531**	0.11677	0.12028
	3	**0.18107**	0.19791	0.18175	0.18558	0.19082	0.19809
	5	0.31017	0.35723	**0.30332**	0.32979	0.34259	0.35740
	10	0.76530	0.88819	**0.73951**	0.82284	0.85460	0.88836
	20	2.31093	2.59262	**2.24545**	2.44787	2.51933	2.59280
	30	4.61578	5.05595	**4.51068**	4.83196	4.94304	5.05613
	50	11.63414	12.38279	**11.45192**	12.00456	12.19276	12.38296
20	0.5	0.03648	**0.02306**	0.04257	0.02750	0.02471	0.02314
	1	0.05511	**0.04567**	0.06026	0.04812	0.04634	0.04575
	2	0.09254	0.09298	0.09526	**0.09049**	0.09119	0.09306
	3	**0.13516**	0.14291	0.13606	0.13676	0.13929	0.14298
	5	0.23404	0.26258	**0.22975**	0.24604	0.25377	0.26266
	10	0.56569	0.63897	**0.55022**	0.60006	0.61898	0.63905
	20	1.56635	1.72944	**1.52844**	1.64563	1.68699	1.72951
	30	3.04070	3.29605	**2.97972**	3.16611	3.23054	3.29613
	50	7.35022	7.78941	**7.24329**	7.56755	7.67794	7.78949

*The bold numbers represent the lowest MSE.

Table 3 provides the MSE values for point estimation using 6 distinct loss functions with varying hyperparameters $a = 3$ and $b = 1$ in Bayesian methodologies for sample sizes of 5, 10, 15, and 20. For smaller true parameter values, the Bayesian approach utilizing the quadratic loss function consistently exhibited superior performance, achieving the lowest MSE for true parameters of $\lambda = 0.5$ and 1. Notably, for the true parameter value of $\lambda = 2$, the Bayesian estimation employing the entropy loss function emerged as the most accurate, outperforming other methods in this scenario. Meanwhile, the Bayesian estimation under the squared error loss function closely approximated the true parameter value for $\lambda = 3$, demonstrating robust estimation performance in this case.

As the true parameter values increased, the performance of different loss functions shifted. The Bayesian estimation using the precautionary loss function demonstrated remarkable effectiveness, consistently producing the lowest MSE for larger true parameter values of $\lambda = 5$, 10, 20, 30, and 50. This trend suggests that the precautionary loss function is particularly well-suited for estimating higher values of the Poisson parameter, where it outperformed other loss functions across all sample sizes.

TABLE IV
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 25, 30, 40 AND 50 GIVEN $a=3$, $b=1$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
25	0.5	0.02759	**0.01868**	0.03161	0.02165	0.01979	0.01872
	1	0.04363	**0.03727**	0.04705	0.03897	0.03776	0.03731
	2	0.07514	0.07496	0.07703	**0.07357**	0.07391	0.07500
	3	**0.11126**	0.11720	0.11162	0.11275	0.11462	0.11724
	5	0.19144	0.20920	**0.18885**	0.19884	0.20366	0.20924
	10	0.43614	0.48286	**0.42632**	0.45802	0.47008	0.48290
	20	1.16247	1.26901	**1.13769**	1.21426	1.24128	1.26905
	30	2.19055	2.35809	**2.15052**	2.27284	2.31511	2.35813
	50	5.19262	5.47945	**5.12277**	5.33456	5.40665	5.47949
30	0.5	0.02245	**0.01607**	0.02531	0.01822	0.01688	0.01609
	1	0.03436	**0.03041**	0.03664	0.03134	0.03062	0.03043
	2	0.06433	0.06410	0.06568	**0.06317**	0.06338	0.06412
	3	**0.09414**	0.09861	0.09433	0.09534	0.09672	0.09864
	5	0.12310	0.13131	**0.12235**	0.12616	0.12849	0.13133
	10	0.36351	0.39722	**0.35639**	0.37932	0.38802	0.39724
	20	0.94718	1.02321	**0.92948**	0.98415	1.00343	1.02324
	30	1.67889	1.79505	**1.65115**	1.73593	1.76524	1.79507
	50	3.86285	4.06270	**3.81420**	3.96173	4.01196	4.06272
40	0.5	0.01603	**0.01244**	0.01766	0.01364	0.01289	0.01245
	1	0.02588	**0.02370**	0.02717	0.02420	0.02380	0.02371
	2	0.04788	0.04796	0.04860	**0.04732**	0.04750	0.04797
	3	**0.07335**	0.07565	0.07352	0.07390	0.07463	0.07566
	5	0.12002	0.12733	**0.11894**	0.12308	0.12506	0.12734
	10	0.26631	0.28584	**0.26217**	0.27548	0.28052	0.28585
	20	0.65789	0.70186	**0.64765**	0.67928	0.69042	0.70187
	30	1.12441	1.18863	**1.10910**	1.15592	1.17213	1.18864
	50	2.50038	2.61387	**2.47276**	2.55653	2.58505	2.61388
50	0.5	0.01211	**0.00977**	0.01317	0.01056	0.01007	0.00977
	1	0.02094	**0.01948**	0.02178	0.01983	0.01956	0.01949
	2	0.03796	0.03817	0.03839	**0.03768**	0.03783	0.03818
	3	**0.05781**	0.05933	0.05791	0.05818	0.05866	0.05933
	5	0.09703	0.10139	**0.09643**	0.09883	0.10002	0.10140
	10	0.20967	0.22147	**0.20721**	0.21519	0.21823	0.22148
	20	0.50655	0.53467	**0.50000**	0.52023	0.52736	0.53468
	30	0.86447	0.90832	**0.85399**	0.88601	0.89707	0.90833
	50	1.81850	1.89175	**1.80067**	1.85474	1.87315	1.89176

*The bold numbers represent the lowest MSE.

Table 4 demonstrates the mean squared error (MSE) values for point estimation using 6 distinct loss functions with varying hyperparameters $a=3$ and $b=1$ within Bayesian methodologies for sample sizes 25, 30, 40, and 50. Among these sample sizes, the Bayesian method employing the quadratic loss function exhibited superior performance, presenting the lowest MSE for the true parameters $\lambda=0.5$ and $\lambda=1$, making it particularly well-suited for estimating smaller Poisson parameters. Conversely, the Bayesian estimation utilizing the entropy loss function was the most accurate for $\lambda=2$, indicating its strength in handling moderate parameters with slight variation. Additionally, the Bayesian estimation through the squared error loss function provided estimates that closely approximated the true parameter $\lambda=3$, reflecting its robustness in situations where precision is critical for mid-range parameter values.

Furthermore, the Bayesian estimation using the precautionary loss function demonstrated remarkable effectiveness for larger true parameters, producing the lowest MSE for $\lambda=$ 5, 10, 20, 30, and 50. This demonstrates its reliability for larger-scale Poisson parameter estimation, suggesting that the precautionary loss function is preferable for datasets with higher parameter values.

Table 5 showcases the mean squared error (MSE) values for point estimation, employing 6 different loss functions with hyperparameters $a=2$ and $b=0.5$ within Bayesian method, across sample sizes 5, 10, 15, and 20. Among these sample sizes, the Bayesian method utilizing the quadratic loss function demonstrated outstanding performance, achieving the lowest MSE for the true parameter $\lambda=0.5$. Concurrently, the Bayesian estimator using the general entropy loss function with $c=2$ showcased the minimal MSE value for the true parameter $\lambda=1$. Conversely, the Bayesian estimation with the entropy loss function proved highly accurate for the true parameter $\lambda=2$, while the Bayesian approach with the squared error loss function closely approximated the true parameter for $\lambda=3$. Remarkably, the Bayesian estimation employing the precautionary loss function exhibited its effectiveness by revealing the lowest MSE for true parameters $\lambda=5$, 10, 20, 30, and 50.

TABLE V
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 5, 10, 15 AND 20 GIVEN $a=2$, $b=0.5$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
5	0.5	0.18072	**0.08395**	0.24304	0.09928	0.09078	0.11447
	1	0.23681	0.17007	0.29323	0.17038	**0.16482**	0.18726
	2	0.36102	0.35634	0.40303	**0.32563**	0.33410	0.36205
	3	**0.49665**	0.56525	0.52078	0.49789	0.52448	0.56940
	5	0.80234	0.99379	**0.79607**	0.86501	0.92212	0.99696
	10	1.92259	2.44669	**1.83343**	2.15158	2.29178	2.44937
	20	5.34610	6.52672	**5.09296**	5.90335	6.20764	6.52915
	30	10.66590	12.53278	**10.24125**	11.56628	12.04213	12.53517
	50	26.05651	29.24766	**25.30082**	27.61902	28.42593	29.24998
10	0.5	0.07395	**0.04604**	0.09175	0.05093	0.04686	0.05034
	1	0.10932	0.09067	0.12510	0.09093	**0.08878**	0.09185
	2	0.19738	0.19763	0.20860	**0.18844**	0.19095	0.19831
	3	0.27164	0.28917	0.27859	**0.27133**	0.27814	0.28970
	5	0.44826	0.50421	**0.44565**	0.46716	0.48356	0.50465
	10	0.97801	1.12018	**0.95388**	1.04002	1.07796	1.12054
	20	2.37122	2.69544	**2.30159**	2.52426	2.60770	2.69577
	30	4.23036	4.73836	**4.11480**	4.47529	4.60468	4.73869
	50	9.30450	10.17101	**9.09932**	9.72869	9.94770	10.17132
15	0.5	0.04341	**0.03086**	0.05159	0.03297	0.03102	0.03173
	1	0.07132	0.06366	0.07837	0.06333	**0.06253**	0.06400
	2	0.12697	0.12757	0.13200	**0.12310**	0.12435	0.12776
	3	**0.18927**	0.19908	0.19202	0.19001	0.19355	0.19924
	5	0.31509	0.33869	**0.31440**	0.32273	0.32971	0.33882
	10	0.67254	0.74064	**0.66074**	0.70243	0.72053	0.74075
	20	1.52228	1.67397	**1.48959**	1.59397	1.63297	1.67408
	30	2.59033	2.81971	**2.53822**	2.70086	2.75928	2.81981
	50	5.27122	5.66431	**5.17818**	5.46361	5.56295	5.66441
20	0.5	0.03079	**0.02379**	0.03545	0.02491	0.02381	0.02407
	1	0.05231	0.04751	0.05646	0.04753	**0.04695**	0.04764
	2	0.09595	0.09604	0.09889	**0.09362**	0.09426	0.09612
	3	0.14467	0.14926	0.14649	**0.14458**	0.14635	0.14933
	5	0.24121	0.25535	**0.24065**	0.24590	0.25005	0.25540
	10	0.49850	0.53523	**0.49230**	0.51449	0.52428	0.53528
	20	1.10726	1.19324	**1.08876**	1.14787	1.16998	1.19328
	30	1.83259	1.96373	**1.80280**	1.89578	1.92918	1.96378
	50	3.68187	3.91482	**3.62663**	3.79597	3.85481	3.91486

*The bold numbers represent the lowest MSE.

TABLE VI
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 25, 30, 40 AND 50 GIVEN $a=2$, $b=0.5$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (c=2)**	**EN (c=3)**
25	0.5	0.02408	**0.01928**	0.02717	0.02014	0.01935	0.01940
	1	0.04120	0.03805	0.04389	0.03809	**0.03770**	0.03812
	2	0.07831	0.07823	0.08025	**0.07673**	0.07711	0.07827
	3	0.11779	0.12062	0.11901	**0.11767**	0.11877	0.12066
	5	0.19029	0.19963	**0.18988**	0.19342	0.19615	0.19966
	10	0.39892	0.42553	**0.39420**	0.41069	0.41773	0.42555
	20	0.85340	0.90755	**0.84179**	0.87894	0.89287	0.90757
	30	1.39900	1.48748	**1.37881**	1.44170	1.46421	1.48750
	50	2.67846	**2.82299**	**2.64425**	2.74919	2.78571	2.82302
30	0.5	0.01946	0.01630	0.02157	0.01680	**0.01630**	0.01637
	1	0.03535	0.03308	0.03725	0.03314	**0.03285**	0.03312
	2	0.06598	0.06616	0.06728	**0.06499**	0.06531	0.06618
	3	**0.09902**	0.10123	0.09981	0.09905	0.09987	0.10125
	5	0.16417	0.17052	**0.16393**	0.16627	0.16814	0.17054
	10	0.33552	0.35359	**0.33235**	0.34348	0.34827	0.35360
	20	0.70675	0.74393	**0.69880**	0.72427	0.73383	0.74394
	30	1.16461	1.22481	**1.15091**	1.19363	1.20896	1.22482
	50	2.17964	2.28359	**2.15500**	2.23054	2.25680	2.28360
40	0.5	0.01394	**0.01198**	0.01518	0.01235	0.01202	0.01201
	1	0.02507	0.02387	0.02613	0.02386	**0.02371**	0.02388
	2	0.04875	0.04918	0.04941	**0.04836**	0.04862	0.04919
	3	**0.07219**	0.07354	0.07261	0.07225	0.07275	0.07355
	5	0.12281	0.12715	**0.12248**	0.12437	0.12561	0.12716
	10	0.25301	0.26234	**0.25144**	0.25706	0.25955	0.26234
	20	0.53580	0.55924	**0.53071**	0.54691	0.55292	0.55924
	30	0.83454	0.86825	**0.82687**	0.85079	0.85937	0.86826
	50	1.51602	1.57348	**1.50242**	1.54414	1.55866	1.57349
50	0.5	0.01119	**0.01001**	0.01197	0.01021	0.01001	0.01002
	1	0.02109	**0.02006**	0.02183	0.02018	0.02002	0.02007
	2	0.03923	0.03940	0.03967	**0.03892**	0.03906	0.03940
	3	**0.05898**	0.06017	0.05917	0.05918	0.05958	0.06018
	5	0.09807	0.10082	**0.09787**	0.09906	0.09984	0.10083
	10	0.20338	0.20989	**0.20224**	0.20624	0.20797	0.20990
	20	0.40611	0.42050	**0.40300**	0.41291	0.41661	0.42051
	30	0.65877	0.68080	**0.65375**	0.66939	0.67500	0.68080
	50	1.22084	1.26035	**1.21145**	1.24020	1.25018	1.26035

*The bold numbers represent the lowest MSE.

Table 6 displays the mean squared error (MSE) values for point estimation, utilizing 6 different loss functions with varying hyperparameters $a=2$ and $b=0.5$ through Bayesian methods, across sample sizes 25, 30, 40, and 50. In specific sample sizes of $n=25$, 40, and 50, the Bayesian method using the quadratic loss function showcased exceptional performance, exhibiting the lowest MSE for the true parameter $\lambda=0.5$ or 1. Meanwhile, the Bayesian estimator employing the general entropy loss function with $c=2$ demonstrated the smallest MSE values for the true parameter $\lambda=$ 0.5 or 1 specifically for the sample size $n=30$. For the sample size $n=25$, the Bayesian estimation with the entropy loss function proved to be highly efficient for the true parameters $\lambda=$ 2 and 3. Across other sample sizes, the Bayesian estimation using the entropy loss function was notably accurate for the true parameter $\lambda=2$, and the Bayesian approach with the squared error loss function closely approximated the true parameter for $\lambda=3$. Additionally, for all sample sizes, the Bayesian estimation using the precautionary loss function demonstrated its effectiveness by yielding the lowest MSE for true parameters $\lambda=5$, 10, 20, 30, and 50.

TABLE VII
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 5, 10, 15 AND 20 GIVEN $a=3,\ b=2$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (c=2)**	**EN (c=3)**
5	0.5	0.13168	**0.05054**	0.17557	0.17557	0.05579	0.06214
	1	0.12801	0.12492	0.15370	**0.10606**	0.11161	0.13179
	2	0.22316	0.38555	**0.20857**	0.28394	0.33130	0.33130
	3	0.49628	0.82188	**0.44136**	0.63867	0.72697	0.82736
	5	1.52007	2.17405	**1.38346**	1.82665	1.99715	2.17951
	10	6.94524	8.41745	**6.60439**	7.66094	8.03606	8.42291
	20	29.74675	32.83681	**29.00160**	31.27137	32.05098	32.84222
	30	68.54301	73.24824	**67.39411**	70.87521	72.05862	73.25364
	50	197.38730	205.39170	**195.41370**	201.36910	203.37740	205.39720
10	0.5	0.06242	**0.03425**	0.07772	0.04139	0.03618	0.03543
	1	0.07572	0.07599	0.08424	0.06891	0.07096	0.07695
	2	0.14430	0.19767	**0.13974**	0.16404	0.17940	0.19857
	3	0.28049	0.39384	**0.26102**	0.33022	0.36061	0.39479
	5	0.69036	0.91300	**0.64363**	0.79474	0.85246	0.91393
	10	2.70259	3.20275	**2.58652**	2.94572	3.07284	3.20368
	20	10.93238	11.98820	**10.67742**	11.45334	11.71938	11.98912
	30	24.68575	26.29819	**24.29165**	25.48502	25.89021	26.29911
	50	69.17558	71.90633	**68.50190**	70.53401	71.21878	71.90726
15	0.5	0.04038	**0.02606**	0.04814	0.02976	0.02708	0.02638
	1	0.05531	0.05548	0.05956	**0.05193**	0.05292	0.05579
	2	0.10635	0.13301	**0.10404**	0.11622	0.12385	0.13331
	3	0.18310	0.23815	**0.17373**	0.20717	0.22190	0.23845
	5	0.42916	0.53927	**0.40603**	0.48075	0.50926	0.53958
	10	1.52939	1.77847	**1.47155**	1.65047	1.71372	1.77877
	20	5.67620	6.19653	**5.55055**	5.93290	6.06396	6.19683
	30	12.66102	13.45978	**12.46577**	13.05694	13.25760	13.46008
	50	35.23440	36.59257	**34.89930**	35.91003	36.25055	36.59288
20	0.5	0.02856	**0.02061**	0.03306	0.02252	0.02107	0.02076
	1	0.04332	0.04301	0.04597	**0.04110**	0.04157	0.04314
	2	0.08536	0.10255	**0.08366**	0.09189	0.09675	0.10270
	3	0.13993	0.17254	**0.13439**	0.15417	0.16288	0.17268
	5	0.30611	0.37203	**0.29225**	0.33700	0.35404	0.37216
	10	1.02703	1.17794	**0.99193**	1.10042	1.13871	1.17808
	20	3.67287	3.98851	**3.59660**	3.82863	3.90810	3.98865
	30	7.93167	8.40941	**7.81487**	8.16847	8.28847	8.40954
	50	21.30544	22.11162	**21.10653**	21.70647	21.90858	21.90858

*The bold numbers represent the lowest MSE.

Table 7 presents the mean squared error (MSE) values for point estimation using 6 different loss functions with hyperparameters a =3 and b =2, applying Bayesian methods across sample sizes of 5, 10, 15, and 20. The results indicate that for all sample sizes n, the Bayesian method using the squared error loss function exhibited the lowest MSE for the true parameter $\lambda = 0.5$, demonstrating its effectiveness in achieving accurate estimates. Conversely, the Bayesian estimator employing the general entropy loss function showed the smallest MSE for the true parameter $\lambda = 1$, indicating its reliability in this context. For true parameters ranging from $\lambda =$ 2 to 50, , the Bayesian estimation with the precautionary loss function proved to be highly proficient, consistently yielding low MSE values across various sample sizes. This performance underscores the adaptability and robustness of the precautionary loss function in estimating parameters under diverse conditions.

Table 8 displays the mean squared error (MSE) values for point estimation, utilizing the same 6 loss functions with hyperparameters a = 3and b = 2 across sample sizes of 25, 30, 40, and 50. Consistent with the findings in Table 7, the Bayesian method using the squared error loss function demonstrated the lowest MSE for the true parameter $\lambda = 0.5$, indicating its robustness in this context. The Bayesian estimator employing the general entropy loss function yielded the smallest MSE for $\lambda = 1$, reflecting its effectiveness in providing accurate estimates. For true parameters ranging from $\lambda = 2$ to 50, the precautionary loss function proved remarkably effective, consistently achieving low MSE values across these sample sizes. This suggests that the precautionary loss function is particularly advantageous for higher parameter values.

Table 9 presents the mean squared error (MSE) values for point estimation using six different loss functions with hyperparameters $a = 4$ and $b = 2$, applying Bayesian methods across sample sizes of 5, 10, 15, and 20. The results indicate that the Bayesian method using the general entropy loss function with $c = 3$ consistently exhibited the lowest MSE for the true parameter $\lambda = 0.5$ across all sample sizes, highlighting its reliability in estimating lower parameters. The Bayesian estimator utilizing the quadratic loss function demonstrated the smallest MSE for $\lambda = 1$, while the squared error loss function closely approximated the true parameter for $\lambda = 2$. For true parameters ranging from $\lambda = 3$ to 50, the precautionary loss function was found to be very effective, indicating its strong performance in estimating parameters across a broad range of values. This reinforces the versatility and efficiency of the precautionary loss function in various estimation scenarios.

TABLE VIII

THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n =$ 25, 30, 40 AND 50 GIVEN $a = 3$, $b = 2$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
25	0.5	0.02233	**0.01684**	0.02537	0.01821	0.01719	0.01691
	1	0.03562	0.03578	0.03728	**0.03433**	0.03473	0.03586
	2	0.07119	0.08226	**0.07015**	0.07536	0.07849	0.08234
	3	0.11642	0.13811	**0.11272**	0.12589	0.13168	0.13818
	5	0.23593	0.27981	**0.22670**	0.25650	0.26784	0.27988
	10	0.74001	0.83956	**0.71686**	0.78841	0.81367	0.83963
	20	2.59907	2.80868	**2.54840**	2.70250	2.75528	2.80875
	30	5.50355	5.82281	**5.42547**	5.66181	5.74199	5.82288
	50	14.54128	15.07801	**14.40884**	14.80827	14.94283	15.07808
30	0.5	0.01847	**0.01455**	0.02064	0.01553	0.01480	0.01459
	1	0.02959	0.02982	0.03075	**0.02873**	0.02904	0.02987
	2	0.06070	0.06766	**0.06018**	0.06320	0.06520	0.06770
	3	0.09702	0.11253	**0.09438**	0.10380	0.10793	0.11257
	5	0.19599	0.22713	**0.18944**	0.21058	0.21863	0.22717
	10	0.57340	0.64308	**0.55721**	0.60726	0.62494	0.64312
	20	1.93050	2.07921	**1.89456**	2.00388	2.04131	2.07925
	30	4.03197	4.25829	**3.97663**	4.14415	4.20099	4.25833
	50	10.66259	11.04561	**10.56808**	10.85313	10.94914	11.04565
40	0.5	0.01368	**0.01143**	0.01495	0.01199	0.01157	0.01145
	1	0.02354	0.02342	0.02428	**0.02291**	0.02303	0.02344
	2	0.04549	0.05005	**0.04507**	0.04721	0.04849	0.05007
	3	0.07271	0.08182	**0.07115**	0.07670	0.07912	0.08184
	5	0.13947	0.15756	**0.13566**	0.14795	0.15262	0.15758
	10	0.39172	0.43252	**0.38223**	0.41155	0.42190	0.43254
	20	1.22449	1.31098	**1.20358**	1.26717	1.28894	1.31100
	30	2.55669	2.68903	**2.52432**	2.62229	2.65553	2.68905
	50	6.50638	6.72893	**6.45146**	6.61709	6.67287	6.72894
50	0.5	0.01070	**0.00929**	0.01151	0.00963	0.00937	0.00930
	1	0.01871	0.01865	0.01919	**0.01831**	0.01839	0.01866
	2	0.03739	0.04035	**0.03711**	0.03850	0.03934	0.04036
	3	0.05957	0.06560	**0.05852**	0.06221	0.06382	0.06561
	5	0.11116	0.12312	**0.10863**	0.11677	0.11985	0.12313
	10	0.29556	0.32244	**0.28931**	0.30863	0.31544	0.32245
	20	0.88909	0.94588	**0.87536**	0.91711	0.93141	0.94589
	30	1.71398	1.79835	**1.69335**	1.75580	1.77699	1.79836
	50	4.42632	4.57130	**4.39054**	4.49844	4.53478	4.57131

*The bold numbers represent the lowest MSE.

Table 10 displays the mean squared error (MSE) values for point estimation using 6 different loss functions with hyperparameters a = 4 and b = 2, across sample sizes of 25, 30, 40, and 50. For a sample size of 25, the results align with those presented in Table 9. For sample sizes 30, 40, and 50, the Bayesian method using the quadratic loss function performed best, exhibiting the lowest MSE for the true parameters of $\lambda = 0.5$ and $\lambda = 1$. Meanwhile, the Bayesian estimator employing the squared error loss function demonstrated the smallest MSE for the true parameter of $\lambda =$ 2. For parameters ranging from $\lambda =$ 3 to 50, the precautionary loss function proved to be extremely effective.

Table 11 presents the mean squared error (MSE) values for point estimation using 6 different loss functions with hyperparameters a = 2 and b = 2, across sample sizes of 5, 10, 15, and 20. The results indicate that the Bayesian method using the general entropy loss function consistently showed the lowest MSE for the true parameter $\lambda = 0.5$ across all sample sizes. The Bayesian estimator employing the squared error loss function demonstrated the smallest MSE for the true parameter $\lambda = 1$, while the precautionary entropy loss function proved highly reliable for true parameters ranging from λ = 2 to 50.

TABLE IX

THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n =$ 5, 10, 15 AND 20 GIVEN $a = 4$, $b = 2$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
	0.5	0.23736	0.07291	0.30156	0.13472	0.09757	**0.07156**
	1	0.18054	**0.10021**	0.22502	0.11997	0.10515	0.10204
	2	**0.20321**	0.28347	0.20873	0.22293	0.24911	0.28700
	3	0.38471	0.62497	**0.35077**	0.48443	0.55093	0.62911
5	5	1.24374	1.81493	**1.12759**	1.50893	1.65845	1.81961
	10	6.25743	7.64584	**5.93740**	6.93123	7.28526	7.65091
	20	28.49693	31.51807	**27.76896**	29.98709	30.74941	31.52332
	30	66.78373	71.42868	**65.64987**	69.08580	70.25410	71.43398
	50	192.50490	200.40680	**190.55690**	196.43540	198.41800	200.41220
	0.5	0.09880	0.04248	0.12099	0.06370	0.05114	**0.04232**
	1	0.09515	**0.06829**	0.11032	0.07477	0.07477	0.06868
	2	**0.13876**	0.16477	0.14095	0.14482	0.15325	0.16541
	3	0.23827	0.32149	**0.22626**	0.27293	0.29572	0.32223
10	5	0.60598	0.80341	**0.56551**	0.69775	0.74913	0.80424
	10	2.44597	2.91447	**2.33780**	2.67327	2.79245	2.91533
	20	2.91533	11.35731	**10.08466**	10.83771	11.09611	11.35821
	30	23.93219	25.51827	**23.54468**	24.71829	25.11688	25.51918
	50	67.19342	69.88196	**66.53029**	68.53074	69.20495	69.88287
	0.5	0.05829	0.02984	0.06950	0.04060	0.03429	**0.02979**
	1	0.06544	**0.05229**	0.07297	0.05541	0.05301	0.05244
	2	**0.10309**	0.11731	0.10387	0.10674	0.11123	0.11753
	3	0.16947	0.21217	**0.16316**	0.18736	0.19899	0.21243
15	5	0.38558	0.48140	**0.36602**	0.43003	0.45494	0.48167
	10	1.40612	1.64146	**1.35170**	1.52033	1.58013	1.64175
	20	5.46477	5.97241	**5.34229**	5.71513	5.84302	5.97271
	30	12.45170	13.24123	**12.25875**	12.84300	13.04136	13.24153
	50	34.56772	35.91356	**34.23569**	35.23718	35.57462	35.91387
	0.5	0.03954	0.02317	0.04611	0.02929	0.02569	**0.02317**
	1	0.04918	**0.04080**	0.05383	0.04293	0.04136	0.04086
	2	**0.08209**	0.09056	0.08257	0.08426	0.08693	0.09066
	3	0.12990	0.15382	**0.12653**	0.13979	0.14632	0.15393
20	5	0.27485	0.33224	**0.26311**	0.30148	0.31639	0.33236
	10	0.93525	1.07473	**0.90301**	1.00292	1.03835	1.07485
	20	3.52708	3.83432	**3.45291**	3.67863	3.75601	3.83445
	30	7.74153	8.21418	**7.62600**	7.97579	8.09452	8.21432
	50	21.02405	21.82453	**20.82656**	21.42222	21.62291	21.82467

*The bold numbers represent the lowest MSE.

TABLE X
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 25, 30, 40 AND 50 GIVEN $a=4$, $b=2$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
25	0.5	0.03016	0.01884	0.03465	0.02313	0.02063	**0.01883**
	1	0.03971	**0.03463**	0.04268	0.03580	0.03488	0.03467
	2	**0.06862**	0.07417	0.06894	0.07002	0.07177	0.07422
	3	0.10552	0.12217	**0.10308**	0.11248	0.11700	0.12223
	5	0.22267	0.26127	**0.21475**	0.24060	0.25062	0.26134
	10	0.69690	0.79120	**0.67506**	0.74268	0.76662	0.79127
	20	2.48436	2.68777	**2.43524**	2.43524	2.63592	2.68784
	30	5.36382	5.67733	**5.28718**	5.51920	5.59795	5.67741
	50	14.24284	14.77247	**14.11217**	14.50628	14.63906	14.77254
30	0.5	0.02299	**0.01543**	0.02607	0.01823	0.01823	0.01543
	1	0.03267	**0.02880**	0.03485	0.02976	0.02904	0.02882
	2	**0.05938**	0.06290	0.05972	0.06017	0.06130	0.06293
	3	0.09353	0.10543	**0.09179**	0.09851	0.10174	0.10547
	5	0.18200	0.20902	**0.17647**	0.19453	0.20154	0.20906
	10	0.55276	0.62001	**0.53718**	0.58541	0.60248	0.62005
	20	1.87572	2.02135	**1.84054**	1.94756	1.98422	2.02139
	30	3.99384	4.21802	**3.93903**	4.10495	4.16126	4.21806
	50	10.37002	10.74702	**10.27701**	10.55754	10.65205	10.74706
40	0.5	0.01673	**0.01209**	0.01858	0.01384	0.01282	0.01209
	1	0.02558	**0.02315**	0.02689	0.02380	0.02334	0.02316
	2	**0.04461**	0.04726	0.04466	0.04537	0.04537	0.04727
	3	0.07162	0.07827	**0.07067**	0.07438	0.07619	0.07829
	5	0.13348	0.14871	**0.13039**	0.14053	0.14448	0.14872
	10	0.37647	0.41551	**0.36742**	0.39542	0.40533	0.41553
	20	1.18548	1.26926	**1.16525**	1.22680	1.24790	1.26928
	30	2.46335	2.59252	**2.43177**	2.52737	2.55981	2.59254
	50	6.29165	6.51002	**6.23777**	6.40027	6.45501	6.51004
50	0.5	0.01286	**0.00992**	0.01405	0.01102	0.01038	0.01038
	1	0.01962	**0.01817**	0.02044	0.01853	0.01826	0.01818
	2	**0.03628**	0.03773	0.03637	0.03663	0.03709	0.03774
	3	0.05772	0.06216	**0.05707**	0.05957	0.06077	0.06077
	5	0.10556	0.11581	**0.10347**	0.11032	0.11298	0.11582
	10	0.28337	0.30893	**0.27744**	0.29578	0.30227	0.30227
	20	0.86662	0.92188	**0.85327**	0.89388	0.90779	0.92189
	30	1.72991	1.81484	**1.70915**	1.77201	1.79333	1.81485
	50	4.36267	4.50634	**4.32722**	4.43414	4.47015	4.50635

*The bold numbers represent the lowest MSE.

Table 12 presents the mean squared error (MSE) values for point estimation using 6 different loss functions with hyperparameters $a=2$ and $b=2$, applying Bayesian methods across sample sizes 25, 30, 40, and 50. Consistent with the findings in Table 11, the Bayesian method utilizing the general entropy loss function demonstrated the lowest MSE for the true parameter $\lambda=0.5$. In contrast, the Bayesian estimator employing the squared error loss function exhibited the smallest MSE for the true parameter $\lambda=1$. For true parameters ranging from $\lambda=2$ to 50, the Bayesian estimation using the precautionary loss function proved highly effective and robust.

Table 13 shows the MSE values for point estimates using 6 different loss functions, with hyperparameters set to $a=$ 1 and $b=1$, across sample sizes of 5, 10, 15, and 20. The results reveal that the Bayesian method utilizing the squared error loss function consistently yielded the lowest MSE for the true parameter value of $\lambda=0.5$ and $\lambda=1$ across all sample sizes. Additionally, the Bayesian estimator with the precautionary loss function demonstrated strong reliability for true parameters ranging from $\lambda=2$ to 50, suggesting its adaptability to a wide range of estimation contexts, particularly in scenarios involving larger parameter values.

Table 14 reports the MSE values for point estimation using the same 6 loss functions with hyperparameters $a = 1$ and $b = 1$, employing Bayesian methods for sample sizes of 25, 30, 40, and 50. Similar to the findings in Table 12, the Bayesian method using the general entropy loss function resulted in the lowest MSE for the true parameter $\lambda =$ 0.5. Meanwhile, the Bayesian estimator with the squared error loss function provided the smallest MSE for the true parameter $\lambda = 1$. For true parameters between $\lambda = 2$ and 50, the Bayesian estimator with the precautionary loss function performed most effectively, underscoring its efficiency in achieving accurate estimates across diverse sample sizes.

This paper investigates the point estimation of the Poisson parameter using various loss functions. It focuses on deriving and simulating the Bayesian estimators under the precautionary, entropy, and general entropy functions.

The results demonstrate that the Bayesian approach utilizing the quadratic, entropy, and general entropy with loss functions excels in estimating small true parameters ($\lambda = 0.5$, 1, or 2) for the Poisson distribution.

TABLE XI

THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n =$ 5, 10, 15 AND 20 GIVEN $a = 2$, $b = 2$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
	0.5	0.06941	0.07197	0.09317	**0.05028**	0.06324	0.10078
	1	**0.10286**	0.18645	0.10764	0.12424	0.15402	0.20393
	2	0.28957	0.53738	**0.25417**	0.39306	0.46264	0.54556
	3	0.63770	1.04461	**0.56284**	0.82075	0.92989	1.05159
5	5	1.81048	2.54369	**1.65431**	2.15668	2.34727	2.54996
	10	7.55803	9.09925	**7.20005**	8.30823	8.70074	9.10505
	20	31.38364	34.56122	**30.61668**	32.95202	33.75359	34.56684
	30	72.07292	76.90137	**70.89330**	74.46674	75.68102	76.90696
	50	201.52080	209.61280	**199.52530**	199.52530	207.57650	209.61830
	0.5	0.04174	0.04167	0.05024	**0.03476**	0.03727	0.04619
	1	**0.06955**	0.09707	0.07140	0.07637	0.08545	0.09876
	2	0.16601	0.24996	**0.15390**	**0.20104**	0.22417	0.25121
	3	0.31749	0.45604	**0.29177**	0.37982	0.41657	0.45716
10	5	0.76984	1.01536	**0.71742**	0.88565	0.94913	1.01638
	10	2.93053	3.45708	**2.80789**	3.18686	3.32059	3.45806
	20	11.44108	12.52481	**11.17916**	11.97600	12.24902	12.52577
	30	25.49904	27.14170	**25.09739**	26.31343	26.72618	27.14265
	50	70.03677	72.78468	**69.35881**	71.40378	72.09284	72.78561
	0.5	0.02906	0.02938	0.03324	**0.02576**	0.02692	0.03035
	1	**0.05171**	0.06511	0.05271	0.05495	0.05931	0.06561
	2	0.11408	0.15532	**0.10817**	0.13124	0.14255	0.15572
	3	0.20870	0.27704	**0.19602**	0.23941	0.25748	0.27740
15	5	0.46522	0.58643	**0.43934**	0.52236	0.55365	0.58676
	10	1.65175	1.91511	**1.59035**	1.77997	1.84679	1.91543
	20	6.00783	6.54674	**5.87754**	6.27383	6.40953	6.54705
	30	13.13761	13.95251	**12.93832**	13.54160	13.74630	13.95282
	50	36.01442	37.38731	**35.67564**	36.69740	37.04161	37.38762
	0.5	0.02260	0.02241	0.02520	**0.02044**	0.02100	0.02276
	1	**0.04116**	0.04939	0.04170	0.04321	0.04585	0.04961
	2	0.09318	0.11788	**0.08962**	0.10347	0.11022	0.11806
	3	0.15577	0.19627	**0.14826**	0.17395	0.18465	0.19643
20	5	0.33756	0.41159	**0.32167**	0.37251	0.39159	0.41174
	10	1.10061	1.25920	**1.06360**	1.17784	1.21805	1.25934
	20	3.79797	4.11919	**3.72030**	3.95652	4.03739	4.11933
	30	8.16975	8.65613	**8.05078**	8.41087	8.53304	8.65627
	50	21.73555	22.55074	**21.53438**	22.14108	22.34545	22.55088

*The bold numbers represent the lowest MSE.

TABLE XII
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n =$ 25, 30, 40 AND 50 GIVEN $a = 2,\ b = 2.$

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (*c*=2)**	**EN (*c*=3)**
25	0.5	0.01857	0.01861	0.02026	**0.01722**	0.01762	0.01879
	1	**0.03342**	0.03836	0.03390	0.03452	0.03613	0.03847
	2	0.07526	0.09183	**0.07285**	0.08217	0.08669	0.09192
	3	0.12358	0.15032	**0.11862**	0.13558	0.14264	0.15041
	5	0.25199	0.30130	**0.24140**	0.27527	0.28797	0.30138
	10	0.78351	0.88740	**0.75928**	0.83408	0.86043	0.88748
	20	2.64220	2.85541	**2.59064**	2.74743	2.80111	2.85549
	30	5.68001	6.00542	**5.60040**	5.84134	5.92307	6.00550
	50	14.78281	15.32380	**14.64930**	15.05194	15.18755	15.32388
30	0.5	0.01561	0.01559	0.01682	**0.01462**	0.01489	0.01569
	1	**0.02906**	0.03268	0.02937	0.02989	0.03106	0.03274
	2	0.06397	0.07618	**0.06215**	0.06910	0.07242	0.07624
	3	0.10437	0.12418	**0.10065**	0.11330	0.11851	0.12423
	5	0.20814	0.24311	**0.20063**	0.22465	0.23365	0.24316
	10	0.60462	0.67855	**0.58737**	0.64060	0.65935	0.67859
	20	1.99832	2.15132	**1.96131**	2.07384	2.11235	2.11235
	30	4.17257	4.40269	**4.11627**	4.28665	4.34444	4.40273
	50	10.81306	11.19912	**10.71778**	11.00511	11.10189	11.19917
40	0.5	0.01183	0.01190	0.01252	**0.01130**	0.01147	0.01194
	1	**0.02255**	0.02480	0.02270	0.02311	0.02383	0.02483
	2	0.04706	0.05398	**0.04604**	0.04995	0.05183	0.05401
	3	0.07663	0.08779	**0.07456**	0.08164	0.08458	0.08781
	5	0.15397	0.17425	**0.14961**	0.16354	0.16876	0.17427
	10	0.40916	0.45185	**0.39921**	0.42994	0.44076	0.45187
	20	1.25184	1.33925	**1.23070**	1.29498	1.31698	1.33927
	30	2.59041	2.59041	**2.55765**	2.65680	2.69043	2.72435
	50	11.19917	6.80571	**6.52536**	6.69272	6.74908	6.80573
50	0.5	0.00925	0.00929	0.00970	**0.00890**	0.00901	0.00931
	1	**0.01843**	0.02000	0.01850	0.01885	0.01934	0.02002
	2	0.03830	0.04268	**0.03767**	0.04012	0.04131	0.04269
	3	0.06237	0.06977	**0.06098**	0.06570	0.06765	0.06978
	5	0.11642	0.12984	**0.11353**	0.12276	0.12621	0.12985
	10	0.30741	0.33574	**0.30079**	0.32120	0.32838	0.33575
	20	0.91935	0.97740	**0.90530**	0.94801	0.96262	0.97741
	30	1.82689	1.91516	**1.80529**	1.87066	1.89282	1.91517
	50	4.52136	4.66851	**4.48504**	4.59457	4.63145	4.66852

*The bold numbers represent the lowest MSE.

This aligns with previous research by Supharakonsakun [14] and Hassan [11], concluding that the Bayes estimator under the quadratic loss function is superior to MLINEX, NLINEX, and squared error functions for estimating the scale parameter of the Double exponential distribution. Additionally, studies by Yahgmaei et al. [8], Amin et al. [22], and Naji and Rasheed [4] affirm the effectiveness of Bayesian estimators under specific loss functions for various distributions.

Moreover, this study's findings regarding the performance of the Bayesian method using the squared error loss function are consistent with Srivastava [23] and Hassan and Zaky [12], both indicating that the Bayesian estimator under this function provides highly effective estimations, particularly for smaller prior values and smaller sample sizes. Notably, the choice of hyperparameters a and b significantly impacts the performance of the Bayesian method under each loss function. Specifically, when rate parameter b increases and approaches the shape parameter value a, the Bayesian estimator under the precautionary loss function becomes notably more effective for estimating Poisson distribution parameters with moderate to large true values.

TABLE XIII
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 5, 10, 15 AND 20 GIVEN $a=1,\ b=1$.

n	λ	MSE					
		SE	QL	PR	EN	EN (c=2)	EN (c=3)
5	0.5	**0.07583**	0.13223	0.09577	0.07626	0.10798	0.15444
	1	**0.13791**	0.24967	0.14515	0.16601	0.20939	0.28341
	2	0.31647	0.53741	**0.29694**	0.39916	0.46465	0.54938
	3	0.52160	0.85164	**0.47490**	0.65884	0.75069	0.85913
	5	1.13318	1.68816	**1.03037**	1.38289	1.53053	1.69390
	10	3.65123	4.76630	**3.40849**	4.18099	4.46836	4.77097
	20	13.01481	15.25987	**12.48963**	14.10956	14.67931	15.26408
	30	27.21517	30.52581	**26.42358**	28.84271	29.67878	30.52979
	50	74.15998	79.73583	**72.80211**	76.92012	78.32248	79.73973
10	0.5	**0.04379**	0.06090	0.04980	0.044083	0.05265	0.07002
	1	**0.08197**	0.11466	0.08421	0.09005	0.10097	0.11728
	2	0.17265	0.23731	**0.16696**	0.19672	0.21535	0.23858
	3	0.28516	0.38549	**0.27058**	0.32706	0.35454	0.38650
	5	0.53190	0.69605	**0.50137**	0.60571	0.64909	0.69685
	10	1.52129	1.85369	**1.44872**	1.67922	1.76463	1.85437
	20	4.56657	5.22293	**4.41301**	4.88649	5.05287	5.22354
	30	9.45244	10.44346	**9.21521**	9.93969	10.18973	10.44406
	50	23.93864	25.59263	**23.53567**	24.75737	25.17315	25.59321
15	0.5	**0.03085**	0.03873	0.03376	0.03088	0.03430	0.04100
	1	**0.05864**	0.07406	0.05969	0.06244	0.06747	0.07472
	2	0.12161	0.15352	**0.11856**	0.13366	0.14274	0.15391
	3	0.18966	0.23508	**0.18324**	0.20847	0.22090	0.23538
	5	0.35291	0.43143	**0.33822**	0.38826	0.40896	0.43168
	10	0.89411	1.04983	**0.86012**	0.96807	1.00805	1.05005
	20	2.59168	2.90246	**2.51894**	2.74316	2.82190	2.90265
	30	5.06036	5.52966	**4.94799**	5.29110	5.40947	5.52984
	50	12.28667	13.06571	**12.09686**	12.67228	12.86808	13.06589
20	0.5	**0.02317**	0.02775	0.02486	0.02319	0.02506	0.02841
	1	**0.04470**	0.05361	0.04532	0.04689	0.04976	0.05387
	2	0.09335	0.11183	**0.09159**	0.10032	0.10556	0.11199
	3	0.14337	0.17062	**0.13941**	0.15473	0.16215	0.17075
	5	0.26594	0.31142	**0.25743**	0.28641	0.29838	0.31153
	10	0.63620	0.72590	**0.61664**	0.67878	0.70181	0.72599
	20	1.72946	1.91212	**1.68666**	1.81852	1.86478	1.91220
	30	3.29574	3.57067	**3.22987**	3.43094	3.50027	3.57075
	50	7.80958	8.26652	**7.69820**	8.03578	8.15061	8.26660

*The bold numbers represent the lowest MSE.

VIII. CONCLUSIONS

The aim of this study was to obtain the point estimate of the Poisson parameter through the utilization of Bayesian methodology. Emphasis was placed on exploring the efficacy of precautionary, entropy, and general entropy loss functions and comparing their performance against the existing squared error and quadratic loss functions by evaluating the smallest MSE values from simulation results.

In the context of point estimation for the Poisson parameter, the Bayesian approach using the squared error, quadratic, entropy, and general entropy loss functions provided the most accurate estimates for smaller true parameter values ($\lambda = 0.5$, 1, or 2) by yielding the lowest MSE values. On the other hand, for the majority of sample sizes, the Bayesian methodology employing squared error and quadratic loss functions achieved the minimum MSE values for moderately larger true parameter values (3, 5). Furthermore, the Bayesian approach using the precautionary loss function demonstrated outstanding performance in estimating the Poisson parameter for larger true parameter values ($\lambda = 10$, 20, 30, and 50) or extended to true parameter values from 5 to 50 for some hyperparameter selections.

TABLE XIV
THE MSE OF BAYESIAN ESTIMATOR WITH DIFFERENT LOSS FUNCTIONS FOR $n=$ 25, 30, 40 AND 50 GIVEN $a=1,\ b=1$.

n	λ	MSE					
		SE	**QL**	**PR**	**EN**	**EN (c=2)**	**EN (c=3)**
25	0.5	0.01942	0.02232	0.02054	**0.01939**	0.02056	0.02261
	1	**0.03558**	0.04156	0.03595	0.03709	0.03899	0.04169
	2	0.07639	0.08807	**0.07534**	0.08076	0.08407	0.08816
	3	0.11905	0.13713	**0.11639**	0.12661	0.13152	0.13719
	5	0.21034	0.24028	**0.20472**	0.22383	0.23171	0.24034
	10	0.49005	0.54968	**0.47700**	0.51839	0.53368	0.54973
	20	1.30087	1.41935	**1.27311**	1.35863	1.38863	1.41939
	30	2.38513	2.56510	**2.34200**	2.47363	2.51901	2.56515
	50	5.40221	5.69746	**5.33026**	5.54836	5.62255	5.69750
30	0.5	0.01620	0.01827	0.01698	**0.01620**	0.01702	0.01842
	1	**0.03060**	0.03478	0.03087	0.03165	0.03298	0.03486
	2	0.06398	0.07215	**0.06325**	0.06702	0.06934	0.07220
	3	0.09755	0.10990	**0.09577**	0.10268	0.10604	0.10994
	5	0.17470	0.19588	**0.17071**	0.18425	0.18982	0.19592
	10	0.40154	0.44333	**0.39240**	0.42139	0.43211	0.44336
	20	1.00713	1.09088	**0.98750**	1.04797	1.06918	1.09091
	30	1.80523	1.92973	**1.77542**	1.86644	1.89784	1.92976
	50	4.15582	4.36755	**4.10419**	4.26065	4.31385	4.36758
40	0.5	0.01205	0.01316	0.01251	**0.01201**	0.01245	0.01322
	1	**0.02309**	0.02527	0.02329	0.02358	0.02429	0.02530
	2	0.04863	0.05298	**0.04829**	0.05021	0.05145	0.05299
	3	0.07591	0.08315	**0.07485**	0.07894	0.08090	0.08317
	5	0.13173	0.14436	**0.12932**	0.13745	0.14076	0.14437
	10	0.28488	0.30818	**0.27981**	0.29594	0.30191	0.30819
	20	0.69235	0.73932	**0.68135**	0.71524	0.72713	0.73933
	30	1.22224	1.29450	**1.20492**	1.25777	1.27599	1.29451
	50	2.56598	2.68261	**2.53757**	2.62370	2.65301	2.68262
50	0.5	0.00988	0.01061	0.01018	**0.00986**	0.01015	0.01064
	1	**0.01900**	0.02046	0.01912	0.01934	0.01981	0.02047
	2	0.03792	0.04085	**0.03767**	0.03900	0.03983	0.04086
	3	0.05858	0.06306	**0.05794**	0.06043	0.06165	0.06307
	5	0.10134	0.10903	**0.09990**	0.10480	0.10682	0.10904
	10	0.22475	0.24029	**0.22135**	0.23214	0.23612	0.24030
	20	0.53574	0.56614	**0.52862**	0.55056	0.55825	0.56615
	30	0.88337	0.92912	**0.87241**	0.90586	0.91740	0.92913
	50	1.86025	1.93649	**1.84168**	1.89799	1.91715	1.93650

*The bold numbers represent the lowest MSE.

The proposed method for estimating the Poisson parameter yielded excellent results, closely aligning with the true parameter or providing the best estimates in various scenarios. Particularly, the Bayesian approach using the precautionary loss function emerged as an exceptional estimator for estimating larger parameters. This method presents a promising alternative for estimating the Poisson parameter, offering flexibility and robustness across a wide range of parameter values.

Beyond the immediate results, these findings have important implications for various applications where Poisson-distributed data is prevalent. The ability of the Bayesian approach to perform well across different parameter ranges makes it highly suitable for fields such as reliability engineering, queuing theory, and risk analysis, where accurate parameter estimation is critical. The flexibility in choosing appropriate loss functions based on the problem context provides practitioners with powerful tools for tailoring their estimation approach to the specific characteristics of the data.

While the results of this study are promising, it is important to note some limitations. The hyperparameter selection process, while effective in this study, can still be improved by incorporating more advanced techniques for choosing hyperparameters, potentially reducing estimation errors further. Additionally, while the precautionary loss function proved highly effective for larger parameters, future research could investigate its performance in more complex or real-world datasets where non-standard distributions or external factors may influence the estimation process

Future research could also explore extending the Bayesian framework to incorporate more prior information or handle scenarios with sparse or incomplete data, which often pose challenges in practical applications. Investigating other loss functions or hybrid methods may further refine the estimation process, improving accuracy in even more diverse contexts. In conclusion, the findings of this study contribute significantly to the ongoing development of Bayesian estimation techniques for Poisson parameters and offer several avenues for continued exploration in both theoretical and applied statistics.

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