The Fuzzy Sumudu Decomposition Method for Solving Differential Equations in an Imprecise Context

B. Divya and K. Ganesan

Abstract—**This paper presents the fuzzy Sumudu decomposition technique (FSDM) as a novel method for dealing with differential equations that contain uncertainty. The FSDM method allows for the direct solution of fuzzy differential equations, eliminating the need to convert them into their classical counterparts, which is a need in conventional methodologies. This is achieved by analysing the equations using a novel fuzzy arithmetic approach. By employing FSDM, we generate solutions using infinite series that gradually converge towards the exact solutions for the specific problems being studied. This approach offers a reliable and powerful tool for dealing with differential equations in a fuzzy setting, enabling us to understand and handle the inherent uncertainty inside the system. An illustrative numerical example is presented to showcase the potential and practicality of the FSDM. This case serves as an illustration of how the technique can be employed to tackle real-world difficulties. This illustration showcases the efficacy of the FSDM (Fuzzy Set Differential Method) in effectively solving fuzzy differential equations and generating accurate results. This study contributes to the subject of differential equations by introducing the fuzzy Sumudu decomposition methodology as an alternative strategy for dealing with uncertainty. The FSDM approach offers a novel and efficient technique for resolving fuzzy differential equations through the redefinition of the issue and the use of fuzzy arithmetic. This is illustrated with a numerical example.**

Index Terms—**Fuzzy numbers, Fuzzy differential equations, Fuzzy Sumudu transform, Fuzzy decomposition method.**

I. Introduction

The whole of the actual world is complicated, and it has been discovered that the complexity results from uncertainty in the form of ambiguity. When one does not have comprehensive information on the variables and parameters of a real-world problem, fuzzy set theory may be used to simply describe the uncertainties that are present in the situation. Lofti Zadeh's [1] fuzzy set theory can handle these kinds of situations. Differential equations have the potential to be utilised in the simulation of any and all occurrences and conditions that may be found in the real world. In the most recent year, a great deal of focus has been placed on researching various aspects of fuzzy differential equations. For the purpose of resolving FDEs with fuzzy beginning values, Abdul Rahman and Ahmad came up with the idea of the FST. This was done in accordance with the highly generalised differentiability notion. After going through some of the core ideas and features of FST, they proceeded to show the proposed approaches on a number of numerical instances. In addition to this, they utilised FST in order to solve fuzzy partial differential equations. Next, they applied FST to fuzzy fractional differential equations, and finally, they utilised FST on fuzzy Volterra integral equations. In addition to that, they achieved this by explaining the application of utilising FST to solve linear FDEs with fuzzy constant coefficients. In other words, they talked about using FST to solve linear FDEs. The FST was utilised by Haydar [2] to solve nth-order FDEs with ambiguous beginning values. The Sumudu transform method was used by Sahni et al. [3] to study an ODEs of a mechanical vibration system with fuzzified beginning values under a strongly generalised differentiability condition. Sumudu Transform (a type of Laplace Transform) was invented by Watugala [4] with significant benefits over existing integral transforms, notably the 'unity' aspect that might give appropriateness while solving differential equations. Khan [27] progressive and studied Sumudu transform for the solution of linear differential models with uncertainty.

Abdul Rahman and Ahmad [5], [6], [7], [8], [9], [10], [11] discussed a few of the fundamental properties and theories of FST also they demonstrated the proposed methods on various numerical examples. Also they used FST to solve FPDEs then, fuzzy Volterra integral equations(FVIEs) and fuzzy fractional differential equations(FFDEs). In addition to this, they explained a system of linear fuzzy differential equations(LFDEs) based on the FST that had fuzzy constant coefficients. Jafari and Razvarz [23] The solutions of FDEs are approximated by utilizing the FST method. Significant theorems are recommended in order to describe the properties of FST. Razvarz, Jafari and Yu, [32] approximated the solutions of FDEs by utilizing the FST method. Here, the uncertainties are in the sense of Z-numbers. Important theorems are laid down to illustrate the properties of FST. The theoretical analysis and simulation outcomes shows that this new technique is efficient to estimate the solutions of FDEs. (Jafari, Razvarz, Gegov, Paul and Keshtkar) [22],[23],[24] approximated the solutions to FDEs that were obtained by the use of the fuzzy sumudu transform (FST) approach. The unpredictability shown here appears in the form of fuzzy numbers and Z-numbers. There have been some key theorems presented in order to better illustrate the characteristics of FST. It is contrasted with the Max-Min Euler technique as well as the Average Euler approach.

Asiru [14] studied the Sumudu transform's general features as well as the Sumudu transform of specific functions.

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B. Divya is a Research Scholar in Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Tamilnadu, Kattankulathur, Chengalpattu - 603203, India. (e-mail: db3976@srmist.edu.in).

K. Ganesan is a Professor in Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Chengalpattu, Tamilnadu - 603203, India. (Corresponding author; phone: +91 94444 51344; e-mail: ganesank@srmist.edu.in).

The impact of moving the parameter t in f (t) by on the Sumudu transform F (u) is discovered for every function f (t) with corresponding Sumudu transform F (u). The impact of multiplying any function f (t) by a power of t and dividing any function f (t) by t on the Sumudu transform F (u) is also obtained. The Sumudu transform is simply calculated for any periodic function f (t) with periodicity $T > 0$. Abel's integral equation, an integro-differential equation, a dynamic system with delayed time signals, and a differential dynamic system are used as examples.

Georgieva [21] solved the partial Volterra fuzzy integro-differential equations (PVFIDEs) with convolution kernel under Hukuhara differentiability by applying the double fuzzy Sumudu transform technique. These findings are well-known to develop the PVFIDEs solution. Chalco-Cano and Roman-Flores [19], investigated the class of FDEs where the dynamics are formed by a continuous fuzzy mapping using Zadeh's extension method. Then, along with several illustrated examples, a fuzzy solution for this class of FDEs are demonstrated. They provided additional qualities, and demonstrate how those properties relate to various interpretations. Belgacem [16]analyzed Deeper Sumudu properties, connections and new results are presented. In 2003, Belgacem et al [17] presented the fundamental properties of Sumudu transform. Mazandarani [28] presented a paper on chronological survey on FDEs of integer and fractional orders. Moreover, some of the proposed FDEs applications and methods for solving them are investigated. Chalco-Cano [18] extended the idea of H-differentiability was used to study FDEs. This approach expands the class of differentiable fuzzy mappings using lateral Hukuhara derivatives. Also, examples and comparisons with various FDE approaches are presented. Bede et al [15]introduced and studied generalized concepts of differentiability (of any order n in N). The existence of the solutions to FDEs including generalised differentiability is examined, and a formula of the sort of Newton-Leibnitz is produced. Khan et al [13] designed Sumudu transform for the solution of linear differential models with uncertainty. For this purpose, Sumudu transform is coupled with fuzzy theory and all its fundamental properties are determined in fuzzy sense. At last, to demonstrate the accuracy of this approach, FST is employed to some examples of LFDE considered under generalized H-differentiability. Kaleva [26] studied the initial condition of the linear first-order fuzzy differential dynamical systems was characterised by a fuzzy number.

II. Preliminaries

This section contains vital definitions and characteristics for fuzzy functions, fuzzy numbers, and fuzzy sumudu transformation. Real and fuzzy numbers are identified in this work as $\mathbb{R}$ and $F(\mathbb{R})$ respectively.

Definition 1. *[29] A mapping from the set of all real numbers* $\mathbb{R}$ *to* $[0,1]$ *is considered to be a fuzzy number* $\tilde{a}$ *if it satisfied the following conditions:*

- $\tilde{a}$ *is upper semi continuous, for every* $\tilde{a} \in F(\mathbb{R})$,
- $\tilde{a}$ *is fuzzy convex, i.e.,* $\tilde{a}(\lambda x + (1-\lambda)y) \geq min\{\tilde{a}(x), \tilde{a}(y)\}$ $\forall x, y \in \mathbb{R}$ *, for every* $\tilde{a} \in F(\mathbb{R})$*, and* $\lambda \in [0,1]$,
- $\tilde{a}$ *is normal, for every* $\tilde{a} \in F(\mathbb{R})$*, i.e.,* $\exists(x \in \mathbb{R})$ *such that* $\tilde{a}(x) = 1$,
- *Sup*($\tilde{a}$) *is bounded in real line.*

Definition 2. *A triangular fuzzy number* $\tilde{a}$ *is a fuzzy number on R whose membership function has the following requirments*

$$\tilde{a}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & for\ a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & for\ a_2 \leq x \leq a_3 \\ 0, & otherwise. \end{cases}$$

We denote this triangular fuzzy number as $\tilde{a} = (a_1, a_2, a_3)$.

Definition 3. *[29] An ordered pair* $[\underline{a}, \overline{a}]$ *is the parametric form of an arbitrary fuzzy number* $\tilde{a}$*. There are of functions* $\underline{a}$ *and* $\overline{a}$*, that satisfy the following conditions:*

- $\underline{a}$ *is a bounded monotonic increasing function and left continuous in [0,1],*
- $\overline{a}$ *is a bounded monotonic increasing function and left continuous in [0,1],*
- $\underline{a} \leq \overline{a}$

Definition 4. *[29] For a fuzzy number* $\tilde{a} = (\underline{a}, \overline{a})$*, the number* $a_0 = \left(\dfrac{\underline{a}(1) + \overline{a}(1)}{2}\right)$*, is called the location index number and* $a_* = (a_0 - \underline{a})$, $a^* = (\overline{a} - a_0)$ *are called the left and right fuzziness index functions respectively. Hence every fuzzy number* $\tilde{a} = (\underline{a}, \overline{a})$ *can also be expressed as* $\tilde{a} = (a_0, a_*, a^*)$.

A collection of all fuzzy numbers on real $\mathbb{R}$ *is referred by* $F(\mathbb{R})$.

Definition 5. *[29]**(Arithmetic Operations)***
For arbitrary fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ *and* $\tilde{b} = (b_0, b_*, b^*)$*, the four arithmetical operations are defined by*

$$\begin{aligned} \tilde{a} * \tilde{b} &= (a_0 * b_0, a_* \vee b_*, a^* \vee b^*) \\ &= (a_0 * b_0, \max\{a_*, b_*\}, \max\{a^*, b^*\}), \end{aligned}$$

where $a * b$ *is either of* $\tilde{a} + \tilde{b}$, $\tilde{a} - \tilde{b}$, $\tilde{a}.\tilde{b}$ *and* $\tilde{a}/\tilde{b}$ *is possible only when* $b_0 \neq 0$.

The fuzziness index functions follows the lattice rule, which is the least upper bound in the lattice L. On the other hand, the location index number following the existing usual arithmetic.

Definition 6. *[29]**(Comparing Fuzzy Numbers)***
An efficient method for comparing fuzzy numbers is to make use of a ranking function that is determined by the graded means of the fuzzy numbers. For a triangular fuzzy number $\tilde{a} = (a_0, a_*, a^*)$*, we define*

$$\mathcal{R}(\tilde{a}) = \left(\frac{a^* + 4a_0 - a_*}{4}\right).$$

For any two triangular fuzzy numbers $\tilde{a} = (a_0, a_*, a^*)$ *and* $\tilde{b} = (b_0, b_*, b^*)$ *in* $F(\mathcal{R})$*, we have*

- $\mathcal{R}(\tilde{a}) > \mathcal{R}(\tilde{b}) \Leftrightarrow \tilde{a} \succ \tilde{b}$
- $\mathcal{R}(\tilde{a}) < \mathcal{R}(\tilde{b}) \Leftrightarrow \tilde{a} \prec \tilde{b}$
- $\mathcal{R}(\tilde{a}) = \mathcal{R}(\tilde{b}) \Leftrightarrow \tilde{a} \approx \tilde{b}$

If $\mathcal{R}(\tilde{a}) \geq \mathcal{R}(\tilde{0})$*, then the TFN* $\tilde{a} = (a_0, a_*, a^*)$ *is said to be non-negative and is referred by* $\tilde{a} \succeq \tilde{0}$.

Definition 7. *[29] For arbitrary FNs* $\tilde{a} = (\underline{a}, \overline{a})$ *and* $\tilde{b} = (\underline{b}, \overline{b})$*, the measure*

$$\mathcal{D}(\tilde{a}, \tilde{b}) = \sup_{0 \leq r \leq 1} \max \left\{ | \underline{a}(r) - \underline{b}(r) |, | \overline{a}(r) - \overline{b}(r) | \right\}$$

is a fuzzy metric which defines the distance between $\tilde{a}$ and $\tilde{b}$. Such metric is comparable to the one employed by Puri and Ralescu [31] and Kaleva [25].

Definition 8. *A fuzzy-valued function $\tilde{f} : \mathbb{R} \rightarrow F(\mathbb{R})$ is said to be continuous on $\mathbb{R}$ for every single $x_0 \in \mathbb{R}$ and every $\epsilon > 0$, there exist $\delta > 0$ such that if $\mid x - x_0 \mid < \delta$, then $D(\tilde{f}(p), \tilde{f}(x_0)) < \epsilon$.*

Definition 9. *[29] Let $f : [a, b] \rightarrow F(\mathbb{R})$ be a fuzzy-valued function and its derivative exist. Then, $f^{'}(x_0)$ of f at the point x_0 is defined by*

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta p}$$

Definition 10. ***(Definite integral)***
[29] Let $f : [a, b] \rightarrow F(\mathbb{R})$. For any partition $P = t_0, t_1, ..., t_n$ of $[a, b]$ and for arbitrary $\zeta : t_{i-1} \leq \zeta_i \leq t_i, 1 \leq i \leq n$ let

$$R_p = \sum_{i=1}^{n} f(\zeta_i)(t_i - t_{i-1})$$

The definite integral of f(t) over [a,b] is

$$\int f(t)dt = lim R_p, \quad \max_{1 \leq i \leq n} \mid t_i - t_{i-1} \mid \rightarrow 0$$

provided that this limit exists in the metric D.

III. THEOREM

[29] Let $\tilde{f}(x)$ be integrable, then

$$\int_a^b \tilde{f}(t)dt = \Big(\int_a^b f_0(t)dt, \sup_{t \in [m,n]} (f(t))_*, \sup_{t \in [m,n]} (f(t))^* \Big).$$

IV. Fuzzy Sumudu Transformation

Let $\tilde{f}(t)$ be a continuous fuzzy valued function. Assume that $\tilde{f}(ut)e^{-t}$ be an improper fuzzy Riemann integrable on $[0, \infty)$, then $\int_0^\infty \tilde{f}(ut)e^{-t}dt$ is called fuzzy Sumudu transform of $\tilde{f}(t)$ and it is represented by,

$$G(u) = S[\tilde{f}(t)] = \int_0^\infty \tilde{f}(ut)e^{-t}dt, \ (u \in [-\tau, \tau]) \qquad (1)$$

which are represented in fuzzy parametric form as

$$\tilde{f}(t, r) = (f_0(t), f_*(t), f^*(t)).$$

Then $$\int_0^\infty \tilde{f}(ut)e^{-t}dt = \Big(\int_0^\infty f_0(t)e^{-t}dt, \max\{f_*(t)\}e^{-t}dt, \max\{f^*(t)\}e^{-t}dt \Big)$$

and also

$$S[\tilde{f}(t)] = S[f_0(t)], \max\{f_*(t)\}, \max\{f^*(t)\}$$

V. THEOREM

[20] If $\tilde{G}(u)$ is fuzzy Sumudu transform of $\tilde{x}(t)$, then the fuzzy Sumudu transform of the derivatives with integer order is given by

$$S\left[\frac{d^n\tilde{x}(t)}{dt^n}\right] = u^{-n}\left[\tilde{G}(u) - \sum_{k=1}^{n-1} u^k \frac{d^n\tilde{x}(t)}{dt^n} \mid_{t=0}\right] \qquad (2)$$

VI. Fuzzy Sumudu Decomposition Method

A crisp differential equation that has been converted to a fuzzy differential equation

$$\frac{d\tilde{x}}{dt} + \tilde{\mathcal{R}}(\tilde{x}) = \tilde{f}(t) \qquad (3)$$

with beginning condition

$$\tilde{x}(0) = \tilde{x}_0 = [x_0, x_*, x^*]$$

where $\tilde{x} = \tilde{x}(t)$, $\mathcal{R}$ is a linear bounded operator and $\tilde{f}(t)$ is a continuous fuzzy valued function. By employing FST on both sides of equation 3, we will have

$$S\Big[\frac{d\tilde{x}}{dt}\Big] + S[\tilde{\mathcal{R}}(\tilde{x})] = S[\tilde{f}(t)] \qquad (4)$$

From theorem,

$$\frac{S[\tilde{x}(t)] - \tilde{x}_0}{u} + S[\tilde{\mathcal{R}}(\tilde{x})] = S[\tilde{f}(t)] \qquad (5)$$

Finally we get,

$$S[\tilde{x}(t)] = \tilde{x}_0 - uS[\tilde{\mathcal{R}}(\tilde{x})] + uS[\tilde{f}(t)]. \qquad (6)$$

Hence,

$$\tilde{x}(t) = \sum_{n=0}^{\infty} \tilde{x}_n(t) \qquad (7)$$

By applying 7 to 6, we get

$$S\Big[\sum_{n=0}^{\infty} \tilde{x}(t)\Big] = \tilde{x}_0 - uS\Big[\tilde{\mathcal{R}} \sum_{n=0}^{\infty} \tilde{x}(t)\Big] \qquad (8)$$

Comparing the coefficients of x in equation 8 yields,

$$S[\tilde{x}_0] = \tilde{x}_0 + uS[\tilde{f}(t)]$$
$$S[\tilde{x}_1] = -uS[\tilde{\mathcal{R}}\tilde{x}_0(t)]$$
$$S[\tilde{x}_2] = -uS[\tilde{\mathcal{R}}\tilde{x}_1(t)]$$

In general, we have

$$S[\tilde{x}_n] = -uS[\tilde{\mathcal{R}}_{n-1}\tilde{x}(t)] \qquad (9)$$

Then, in equation 9, we apply inverse FST to get ,

$$\begin{aligned} \tilde{x}_0 &= S^{-1}[\tilde{x}_0 + S^{-1}[uS(\tilde{f}(t))]] \\ \tilde{x}_n &= -S^{-1}[uS[\tilde{\mathcal{R}}\tilde{x}_{n-1}(t)]] \end{aligned} \qquad (10)$$

VII. Numerical examples

A. Example 1

Consider the following FDE taken from crisp DEs in [30]: $\tilde{x}^{'}(t) - \tilde{x}(t/2) = 0$, with fuzzy initial condition $\tilde{x}(0) = (0, 1 - \lambda, 1 - \lambda)$
Making use of FST we have,

$$S[\tilde{x}^{'}(t) - \tilde{x}(t/2)] = S[0]$$
$$S[\tilde{x}^{'}(t)] = S[\tilde{x}(t/2)]$$

Now using first derivative theorem of FST and figuring out for $S[\tilde{x}(t)]$, therefore we have,

$$S[\tilde{x}(t)] = \tilde{x}(0) + uS[\tilde{x}(t/2)] \qquad (11)$$

Applying inverse FST in 19 we have,

$$\tilde{x}(t) = S^{-1}[\tilde{x}(0)] + S^{-1}\Big[uS[\tilde{x}(t/2)]\Big]$$

$$\tilde{x}_0(t) = S^{-1}[\tilde{x}(0)] = (0, 1-\lambda, 1-\lambda)$$
$$\tilde{x}_0(t/2) = (0, 1-\lambda, 1-\lambda) \tag{12}$$
$$\tilde{x}_{n+1}(t) = S^{-1}\Big[uS[\tilde{x}_n(t/2)]\Big]$$

Whenever $n = 0$,

$$\tilde{x}_{0+1}(t) = S^{-1}\Big[uS[\tilde{x}_0(t/2)]\Big]$$
$$\tilde{x}_1(t) = S^{-1}\Big[uS[(0, 1-\lambda, 1-\lambda)]\Big] \tag{13}$$
$$\tilde{x}_1(t) = S^{-1}[u(0, 1-\lambda, 1-\lambda)]$$
$$\tilde{x}_1(t) = (0, 1-\lambda, 1-\lambda)t$$

In addition, we have

$$\tilde{x}_1(t/2) = (0, 1-\lambda, 1-\lambda)t/2 \tag{14}$$

Whenever $n = 1$,

$$\tilde{x}_2(t) = S^{-1}\Big[uS[\tilde{x}_1(t/2)]\Big]$$
$$= S^{-1}\Big[uS[(0, 1-\lambda, 1-\lambda)(t/2)]\Big]$$
$$= S^{-1}\Big[\frac{(0, 1-\lambda, 1-\lambda)u^2}{2}\Big] \tag{15}$$
$$\tilde{x}_2(t) = \frac{(0, 1-\lambda, 1-\lambda)t^2}{4}$$
$$\tilde{x}_2(t/2) = \frac{(0, 1-\lambda, 1-\lambda)t^2}{16}$$

Likewise, for $n = 2$

$$\tilde{x}_3(t) = S^{-1}\Big[uS[\tilde{x}_2(t/2)]\Big]$$
$$= S^{-1}\Big[uS[(0, 1-\lambda, 1-\lambda)(t^2/16)]\Big]$$
$$= S^{-1}\Big[\frac{(0, 1-\lambda, 1-\lambda)u^3}{8}\Big] \tag{16}$$
$$= S^{-1}\Big[\frac{(0, 1-\lambda, 1-\lambda)u^3}{8}\Big]$$
$$\tilde{x}_3(t) = \frac{(0, 1-\lambda, 1-\lambda)t^3}{48}$$

The formula for the fuzzy series is as follows:

$$\begin{aligned}\tilde{x}(t) &= \tilde{x}_0(t) + \tilde{x}_1(t) + \tilde{x}_2(t) + \tilde{x}_3(t) + ... \\ &= (0, 1-\lambda, 1-\lambda) + (0, 1-\lambda, 1-\lambda)t \\ &+ \frac{(0, 1-\lambda, 1-\lambda)t^2}{4} + \frac{(0, 1-\lambda, 1-\lambda)t^3}{48} + ...\end{aligned} \tag{17}$$

Changing all the terms interms of parametric form

$$\begin{aligned}\tilde{x}(t) &= \tilde{x}_0(t) + \tilde{x}_1(t) + \tilde{x}_2(t) + \tilde{x}_3(t) + ... \\ &= (1-\lambda, 0, \lambda-1) + (1-\lambda, 0, \lambda-1)t \\ &+ \frac{(1-\lambda, 0, \lambda-1)t^2}{4} + \frac{(1-\lambda, 0, \lambda-1)t^3}{48} + ... \\ \tilde{x}(t) &= (1-\lambda, 0, \lambda-1)\Big(1 + t + \frac{t^2}{4} + \frac{t^3}{48} + ...\Big)\end{aligned} \tag{18}$$

Figure 1 shows the solution achieved in this example graphically. It consists of the first four terms of $\tilde{x}(t)$.

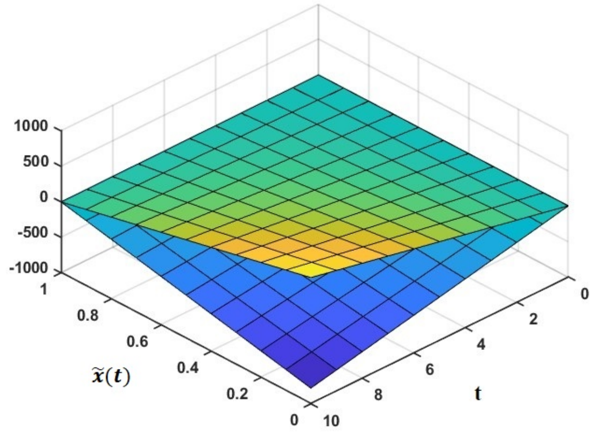

Fig. 1. The solution of $\tilde{x}(t)$

B. Example 2

Consider the following FDE taken from crisp DEs in [12]: $\tilde{x}'(t) = 1 - 2\tilde{x}^2(t/2) = 0$, with fuzzy initial condition $\tilde{x}(0) = (0, 1-\lambda, 1-\lambda)$
Making use of FST we have,

$$S[\tilde{x}'(t)] = S[1 - 2\tilde{x}^2(t/2)]$$

Now using first derivative theorem of FST and figuring out for $S[\tilde{x}(t)]$, therefore we have,

$$S[\tilde{x}(t)] = \tilde{x}(0) + u - uS[2\tilde{x}^2(t/2)] \tag{19}$$

Applying inverse FST in 19 we have,

$$\tilde{x}(t) = S^{-1}[\tilde{x}(0)] + u - 2S^{-1}\Big[uS[\tilde{x}^2(t/2)]\Big]$$
$$\tilde{x}_0(t) = S^{-1}[\tilde{x}(0)] = (0, 1-\lambda, 1-\lambda)$$
$$\tilde{x}_0(t/2) = (0, 1-\lambda, 1-\lambda) \tag{20}$$
$$\tilde{x}_{n+1}(t) = S^{-1}\Big[uS[\tilde{x}_n^2(t/2)]\Big]$$

From FSDM,

$$A_0 = x_0^2(1/2) = (0, 1-\lambda, 1-\lambda)^2 + t^2/4 + 2(0-\lambda, 1-\lambda)(t/2)$$
$$A_1 = 2x_0(t/2)x_1(t/2) \tag{21}$$

Whenever $n = 0$,

$$\tilde{x}_1(t) = -2[(0, 1-\lambda, 1-\lambda)^2 t + t^3/12 + (0, 1-\lambda, 1-\lambda)(t^2/2)$$
$$\tilde{x}_1(t/2) = -2\Big[\frac{(0, 1-\lambda, 1-\lambda)^2}{2}t + t^3/96 + (0, 1-\lambda, 1-\lambda)t^2/8\Big] \tag{22}$$

Whenever $n = 1$,

$$\begin{aligned}\tilde{x}_2(t) = 8\Big[&(0, 1-\lambda, 1-\lambda)^3 t^2/4 + 3(0, 1-\lambda, 1-\lambda)^2 t^3/24 \\ &+ 7(0, 1-\lambda, 1-\lambda)t^4/384 + t^5/960\Big]\end{aligned} \tag{23}$$

The formula for the fuzzy series is as follows:

$$\begin{aligned}\tilde{x}(t) &= \tilde{x}_0(t) + \tilde{x}_1(t) + \tilde{x}_2(t) + \tilde{x}_3(t) + ... \\ &= (0, 1-\lambda, 1-\lambda) + t - 2[(0, 1-\lambda, 1-\lambda)^2 t + t^3/12 \\ &+ (0, 1-\lambda, 1-\lambda)t^2/2] + 8(0, 1-\lambda, 1-\lambda)^3 t^2/4 \\ &+ 3(0, 1-\lambda, 1-\lambda)^2 t^3/24 + 7(0, 1-\lambda, 1-\lambda)t^4/3 \\ &+ t^5/960] + ...\end{aligned} \tag{24}$$

Changing all the terms interms of parametric form

$$\tilde{x}(t) = \tilde{x}_0(t) + \tilde{x}_1(t) + \tilde{x}_2(t) + \tilde{x}_3(t) + ...$$
$$= (1-\lambda, 0, \lambda-1) + t - 2[(0, 1-\lambda, 0, \lambda-1)^2 t + t^3/12$$
$$+ (1-\lambda, 0, \lambda-1)t^2/2] + 8(0, 1-\lambda, 0, \lambda-1)^3 t^2/4$$
$$+ 3(1-\lambda, 0, \lambda-1)^2 t^3/24 + 7(0, 1-\lambda, 0, \lambda-1)t^4/3$$
$$+ t^5/960] + ...$$
$$\tilde{x}(t) = (1-\lambda, 0, \lambda-1)\Big[1 + 2t^2 + 4/3t^3 + 11/48t^4$$
$$+ 1/240t^5\Big] \tag{25}$$

VIII. Conclusion

This work examines the application of the fuzzy sumudu decomposition method for solving differential equations that have fuzzy beginning conditions. We successfully solved the equations by using the parametric form of triangular fuzzy numbers, which includes the location index function, left fuzziness index function, and right fuzziness index function.

An important benefit of our approach is that we did not have to convert the fuzzy differential equation into its corresponding crisp version. Alternatively, we implemented a novel arithmetic operation to the fuzzy decomposition approach, enabling us to directly derive the fuzzy answer. This obviates the necessity for supplementary procedures and streamlines the process of solution.

In order to showcase the efficacy of the suggested approach, we presented a numerical illustration inside a fuzzy context. Through a comparison of the results derived from the fuzzy solution and the crisp solution, we were able to clearly demonstrate the opposing consequences. This comparison further emphasises the advantages of employing fuzzy methods in resolving differential equations with fuzzy initial conditions.

In summary, our research enhances the progress in solving fuzzy differential equations by presenting the fuzzy sumudu decomposition method and demonstrating its efficacy through practical illustrations. Through the act of redefining the problem and utilising inventive methods, we create opportunities for improved examination and comprehension within the realm of fuzzy mathematics.

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