A Further Examination for Inventory Models with Two Trade Credits

Feiyu Zhang, Jinyuan Liu

Abstract—**We studied a recently published paper to point its incompleteness and then we presented a patch work. The recently published paper pointed out several questionable results of a source, and then provided their improvements. However, they did not compared their new findings with already existing results and present a detailed explanation why their new derivations are right. Moreover, the published paper suggested practitioner refer to the solution procedure of the source paper. We will point out that the solution procedure of the source paper is not only incomplete, but also contains possible mistakes to further assist researchers to understand inventory models with two trade credits.**

Index Terms—**Trade credit, Time dependent demand, Deteriorating items, Inventory systems**

I. INTRODUCTION

DURING the passed three decades, there is a trend ot improve published papers to help researchs understand the solution structure of inventory systems that was proposed by Hsieh et al. [1] and Liu and Yang [2]. We followed this research tendency to study a nearly published article to indicate its questionable results and then we provide our revisions. Our works will led a hand to practitioners to realize the solution structure of inventory systems with two trade credits. We previded a short reviewing for this kind of inventory systems. Many scholars explore the research of trapezoidal type demand model for inventory in recent years. Cheng and Wang [3] have suggested inventory model from (i) a trapezoidal type demand, and (ii) a ramp type demand, to a random positive demand. Cheng et al. [4] developed deterioration items an inventory model with trapezoidal type demand rate, assuming for partial backlogging and linearly increasing deterioration rate. Chung [5] and Lin [6] amended some shortcoming in Cheng and Wang [3] separately. Uthayakumar and Raeswari [7] considered deterioration inventory model in which defective items with trapezoidal type demand. Chuang et al. [8] studied optimal inventory and pricing strategies maximizing the net present value. This research is a further study related to the inventory model proposed by Cheng et al. [4]. On the other hand, Lin [9] developed an inventory model with a generalized demand, a generalized deteriorated rate and extend backlogged rate. Zhao [10] explored the inventory model with a trapezoidal type demand in with both the deterioration satisfying the Weibull distributed and partial backlogging are considered. Debata et al [11] studied deterioration inventory model with a quadratic trapezoidal demand under partial backlogging was piecewise quadratic function. Mishra [12] developed an inventory model with controllable deterioration rate by assumed trapezoidal type demand. Wu et al. [13] considered deterioration and trapezoidal type demand inventory model in which purchasing cost and time value of money. Cheng et al. [4] developed economic ordering quantity model with general trapezoidal type rate, time-dependent deteriorating items and partial backlogging. Although their model is functional behaviors of annual total cost to determine the optimal solutions such that their solution procedures have shortcoming from logical viewpoints of mathematics. These shortcoming will influence the implementation of inventory model. Lin et al. [14] studied the sensitivity analysis of Cheng et al [4]. On this research, we will continue this main theme and point out problem in quotations of Cheng et al [4]. Hence, there exist reasons and motivations to president the correct solution procedures to reader. Based on our above brief literature reviewing, researchers can have a glance at this hot study trend.

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Feiyu Zhang is an instructor at the School of General Studies, Weifang University of Science and Technology, Weifang 262799, China (email: zfypromising@163.com).

Jinyuan Liu is an Associate Professor in the School of General Studies, Weifang University of Science and Technology, Weifang 262799, China (email: shgljy@126.com).

II. ASSUMPTIONS AND NOTATION

Our paper is a further discussion of Hsieh et al. [1] and Liu and Yang [2] such that we will adopt the same notation and assumptiona as Hsieh et al. [1] and Liu and Yang [2].

Assumptions

1. The inventory models contains only one kind of product.
2. The replenishment rate is infinite.
3. The product decays with a contious rate of decaying mapping $\theta(t)$, with $0 < \theta(t) \le 1$ and $d\theta(t)/dt \ge 0$ to indicate that$\theta(t)$ is a non-decreasing mapping.
4. $g(t)$ is an extra mapping to simplify the expression, where $g(t)$ is defined as
$$g(t) = \int_0^t \theta(u)\,du. \tag{1.1}$$
5. When T (the replenishment cycle) is longer enough with $M \le T$, where M is the tradt credit offered by the manufacturer, then the balence will be handled at the time is M. The retailer will pay those interest charged by the amount I_k. If T (the replenishment cycle) is shorter than M with $T < M$, then the balence will be handled at the time is M.
6. The retailer collect profits and receive cash after the tradt credit provided by the retailer for the customers, denoted as N, with $N < M$.
7. The cash received after N and before M, by the retailer, is denoted by the amount I_e.

Notation

$f(t)$ is a demand mapping that is a continuous mapping with respect to time under the conditions $df(t)/dt > 0$, and $d^2f/dt^2 > 0$ to indicate that $f(t)$ is increasing and concave up.
A is the set up cost per replenishment.
c is the unit purchasing cost per unit item.
p is the unit selling price, under the restriction, with $p > c$.
h is the holding cost per unit item per unit time.
I_e is the interest received by the retailer after the trade credit, N.
I_k is the interest received by the manufacturer after the tradecredit, M.
M is the trade credit offered by the manufacturer to the retailer.
N is the trade credit offered by the retailer to the customer, under the restriction, $N < M$.
T is the decision variable that denotes the replenishment cycle under the condition $T > 0$.
$AC(T)$ is the objective mapping that expresses the average cost per unit time that contains three objective functions: $AC_1(t)$ for $M \leq T$; $AC_2(t)$ for $N \leq T \leq M$; $AC_3(t)$ for $T \leq N$.
$AC_1(t)$ is the first average total cost where the domain is restricted to the domain with the condition, $M \leq T$.
T_1^* is the local minimum solution of $AC_1(t)$, for the domain $M \leq T$.
$AC_1(T_1^*)$ is the the minimum value of the first objective mapping $AC_1(t)$, under the condition of $M \leq T$, with the local minimum solution T_1^*.
$AC_2(t)$ is the second average total cost where the domain is restricted to $N \leq T \leq M$.
T_2^* is the local minimum solution of $AC_2(t)$, for the domain $N \leq T \leq M$.
$AC_2(T_2^*)$ is the the minimum value of the second objective mapping $AC_2(t)$, under the condition of $N \leq T \leq M$, with the local minimum solution T_2^*.
$AC_3(t)$ is the third average total cost where the domain is restricted to $T \leq N$.
T_3^* is the local minimum solution of $AC_3(t)$, for the domain $T \leq N$.
$AC_3(T_3^*)$ is the the minimum value of the third objective mapping $AC_3(t)$, under the condition of $T \leq N$, with the local minimum solution T_3^*.

III. A Short Review of Discussed Inventory Model

In this section, we will present a very brief review for findings of Hsieh et al. [1] and Liu and Yang [2]. We recall that Hsieh et al. [1] mentioned that for $T \geq M$,

$$AC_1(T) = \frac{1}{T}\Big\{A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt + cI_k\int_M^T e^{-g(t)} \int_t^T e^{g(u)}\, dudt - pI_e + cI_k\int_M^T e^{-g(t)} \int_t^T e^{g(u)}\, f(u)dudt - pI_e \int_N^M (M-t)f(t)dt\Big\}. \quad (3.1)$$

Moreover, for $N \leq T \leq M$, Hsieh et al. [1] asserted that

$$AC_2(T) = \frac{1}{T}\Big\{A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt - pI_e\int_0^T f(t)(M-T)dt - pI_e \int_N^T (T-t)f(t)dt\Big\}. \quad (3.2)$$

On the other hand, for $0 < T \leq N$, Hsieh et al. [1] claimed that

$$AC_3(T) = \frac{1}{T}\Big\{A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt - pI_e \int_0^T f(t)(M-T)dt\Big\}. \quad (3.3)$$

Liu and Yang [2] first pointed out that Hsieh et al. [1] forgot to compute those interest obtained during $[0, N]$ of alreadt sold items. Therefore, Liu and Yang [2] derived different results, such that we list their results in the following.

For $T \geq M$, the objective function, $AC_1(T)$, is derived as follows by Liu and Yang [2],

$$AC_1(T) = \frac{1}{T}\Big\{A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt - pI_e\int_0^N f(t)(M-N)dt + cI_k\int_M^T e^{-g(t)} \int_t^T e^{g(u)}\, f(u)dudt - pI_e \int_N^M (M-t)f(t)dt\Big\}. \quad (3.4)$$

For $0 < T \leq N$, Liu and Yang [2] showed that

$$AC_2(T) = \frac{1}{T}\Big\{A + c\int_0^T \left[e^{g(t)} - 1\right] f(t)dt + h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt - pI_e\int_0^N f(t)(M-N)dt - pI_e \int_N^T (M-t)f(t)dt\Big\}. \quad (3.5)$$

At last, for $0 < T \leq N$, Liu and Yang [2] claimed that their objective function is identical to Hsieh et al. [1] that was already cited in Equation (3.3).

IV. Discussion of the First Derivatives

Liu and Yang [2] claimed that they will improve the first derivatives of Hsieh et al. [1]. However, Liu and Yang [2] only presented their first derivatives for three objective functions. Liu and Yang [2] did not show us those questionable findings of the first derivatives with respect to Hsieh et al. [1]. In this section, we will provide a detailed examination for the first derivatives of Hsieh et al. [1] for three objective functions.

We list the findings of Hsieh et al. [1] as follows.

$$\frac{dAC_1(T)}{dT} = \frac{-A}{T^2} + \frac{c\left[Tf(T)(e^{g(T)}-1) - \int_0^T (e^{g(t)}-1)f(t)dt\right]}{T^2} - h\int_0^T e^{-g(t)} \int_t^T e^{g(u)} f(u)dudt/T^2$$

$$+hTf(T)\int_0^T e^{g(T)-g(t)}\,dt/T^2$$
$$+cI_kTf(T)\int_M^T e^{g(T)-g(t)}dt/T^2$$
$$-cI_k\int_M^T e^{-g(t)}\int_t^T e^{g(u)}\,f(u)dudt/T^2$$
$$+pI_e\int_N^M (M-t)f(t)dt/T^2. \qquad (4.1)$$

On the other hand, based on Equation (3.4), Liu and Yang [2] obtained that

$$\frac{dAC_1(T)}{dT}=\frac{-A}{T^2}+\frac{c\left[Tf(T)\left(e^{g(T)}-1\right)-\int_0^T\left(e^{g(t)}-1\right)f(t)dt\right]}{T^2}$$
$$-h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt/T^2$$
$$+hTf(T)\int_0^T e^{g(T)-g(t)}\,dt/T^2$$
$$+cI_kTf(T)\int_M^T e^{g(T)-g(t)}dt/T^2$$
$$-cI_k\int_M^T e^{-g(t)}\int_t^T e^{g(u)}\,f(u)dudt/T^2$$
$$+pI_e\int_N^M (M-t)f(t)dt/T^2$$
$$+pI_e\int_0^N (M-N)f(t)dt/T^2. \qquad (4.2)$$

The difference of Equations (4.1) and (4.2) is the last term of Equation (4.2) which is an extra term provided by Liu and Yang [2].

We cite the findings of Hsieh et al. [1] as follows.

$$\frac{dAC_2(T)}{dT}=\frac{-A}{T^2}+\frac{c\left[Tf(T)\left(e^{g(T)}-1\right)-\int_0^T\left(e^{g(t)}-1\right)f(t)dt\right]}{T^2}$$
$$-h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt/T^2$$
$$+hTf(T)\int_0^T e^{g(T)-g(t)}\,dt/T^2$$
$$-pI_e(M-T)\left(Tf(T)-\int_0^T f(t)dt\right)/T^2$$
$$+pI_e\left[\int_N^T (T-t)f(t)dt-T\int_0^N f(t)dt\right]/T^2. \qquad (4.3)$$

We must point out that the derivative of $AC_2(T)$ proposed by Hsieh et al. [1] contained questionable results in the third and the fourth row of Equation (4.3).

Based on their objective function of Equation (3.2) mentioned by Hsieh et al. [1], we only concern those questionable results related to pI_e, Hsieh et al. [1] should derive that

$$pI_e\left[\int_N^T (T-t)f(t)dt+(M-T)\int_0^T f(t)dt\right]/T^2$$
$$-pI_e\left[-\int_N^T f(t)dt+(M-T)f(T)\right]/T. \qquad (4.4)$$

Moreover, based on our revised results of Equation (3.2) as Equation (3.5) for $AC_2(T)$ concerning pI_e such that Equation (4.4) should be further improved as

$$pI_e\left[\int_N^T (T-t)f(t)dt+(M-T)\int_0^T f(t)dt\right]/T^2$$
$$-pI_e\left[-\int_N^T f(t)dt+(M-T)f(T)\right]/T$$
$$-pI_eN\int_0^N f(t)dt/T^2. \qquad (4.5)$$

Consequently, the results presented in Liu and Yang [2] becomes reasonable. For completeness, we cite the results of Liu and Yang [2] in the following.

$$\frac{dAC_2(T)}{dT}=\frac{-A}{T^2}+\frac{c\left[Tf(T)\left(e^{g(T)}-1\right)-\int_0^T\left(e^{g(t)}-1\right)f(t)dt\right]}{T^2}$$
$$-h\int_0^T e^{-g(t)}\int_t^T e^{g(u)}f(u)dudt/T^2$$
$$+hTf(T)\int_0^T e^{g(T)-g(t)}\,dt/T^2$$
$$+pI_e\left[\int_N^T (T-t)f(t)dt+(M-T)\int_0^N f(t)dt\right]/T^2$$
$$-pI_e(M-T)\left(Tf(T)-\int_0^N f(t)dt\right)/T^2$$
$$-pI_eN\int_0^N f(t)dt/T^2. \qquad (4.6)$$

Based on the above discussion, we know that $AC_1(T)$ and $AC_2(T)$ of Hsieh et al. [1] contained questionable results such that the comment of Bakker et al. [15] for the rigorously analytic proof of Hsieh et al. [1] is improper.

According to our examination, we point out that several first derivatives proposed by Hsieh et al. [1] with respect to $AC_1(T)$ and $AC_2(T)$ of Hsieh et al. [1]. Consequently, those improvements proposed by Liu and Yang [2] have a strong foundation provided by our detailed examinations of Hsieh et al. [1].

V. Limit Values for the First Derivatives

Hsieh et al. [1] mentioned that

$$\lim_{T\to M^+} d\,AC_1(T)/dT=\lim_{T\to M^-} d\,AC_2(T)/dT. \qquad (5.1)$$

The difference for the first derivative between Hsieh et al. [1] and this paper is related to those terms containing pI_e/T^2. Hence, in the following, we only discuss those results related to pI_e/T^2.

Based on results of Hsieh et al. [1], without revisions of Liu and Yang [2], Hsieh et al. [1] will imply that

$$pI_e\left[\int_N^M (M-t)f(t)dt-M\int_0^N f(t)dt\right]/M^2$$
$$=pI_e\int_N^M (M-t)f(t)dt/M^2. \qquad (5.2)$$

However, we know that Equation (5.2) is false. This detailed explanation help Liu and Yang [2] present their revisions.

Hsieh et al. [1] also mentioned that

$$\lim_{T\to N^+} d\,AC_2(T)/dT=\lim_{T\to N^-} d\,AC_3(T)/dT. \qquad (5.3)$$

We still only discuss results related to pI_e/T^2. Based on the third and the fourth rows of Equation (4.3), Hsieh et al. [1] will find that

$$\frac{pI_e}{N^2}\left[(M-2N)\int_0^N f(t)dt-(M-N)Nf(N)\right]$$
$$=\frac{pI_e}{N^2}\left[(M-N)\int_0^N f(t)dt-(M-N)Nf(N)\right]. \qquad (5.4)$$

However, we know that Equation (5.4) is false to indicate that the findings of Hsieh et al. [1] can not satisfy the requirement of Equation (5.3).

Based on the above examinations, we showed that the assertions of Hsieh et al. [1] with respect to the smooth connected property of the proposed model need revisions. Our discussion provided a strong motivation for Liu and Yang [2] to present a revision for Hsieh et al. [1].

VI. Improvement for Solution Procedure of Hsieh

We cite the following sentence form Liu and Yang [2], "Moreover, Proposition 4 of Hsieh et al. [1], which derived several cases, directly decided the location of the global minimum. To examine Proposition 4 will be an interesting research topic in the future."

For the late discussion, we first recall Proposition 4 of Hsieh et al. [1], in the following.

Proposition 4 of Hsieh et al. [1]. For

$$2A > max\{\Delta_1, \Delta_2\}, \quad (6.1)$$

where Δ_1 and Δ_2 are two abbreviations to simplify the expression, with

$$\Delta_1 = (pI_e - cI_k)M^2 f(M), \quad (6.2)$$

and

$$\Delta_2 = pI_e\big(N^2 f(M) + M^3 (df(M)/dM)\big), \quad (6.3)$$

Hsieh et al. [1] claimed that

(a) If two conditions of $\lim_{T \to N^-} dAC_3(T)/dT < 0$ and $\lim_{T \to M^-} dAC_2(T)/dT < 0$ are satisfied, and then

$$AC(T^*) = AC_1(T_1^*). \quad (6.4)$$

(b) If two conditions of $\lim_{T \to N^-} dAC_3(T)/dT < 0$ and $\lim_{T \to M^-} dAC_2(T)/dT > 0$ are satisfied, and then

$$AC(T^*) = AC_2(T_2^*). \quad (6.5)$$

(c) If two conditions of $\lim_{T \to N^-} dAC_3(T)/dT > 0$ and $\lim_{T \to M^-} dAC_2(T)/dT > 0$ are satisfied, and then

$$AC(T^*) = AC_3(T_3^*). \quad (6.6)$$

On the other hand, we recall that in Propositions 1-3 of Hsieh et al. [1], they also used the extra condition of Equation (6.1) to verify that the uniqueness of critical points (that is, $dAC_i(T)/dT = 0$ cannot have two solutions).

However, Hsieh et al. [1] did not examine the existence of $dAC_i(T)/dT = 0$ (that is, $dAC_i(T)/dT = 0$ at least have one solution),which is a local minimum solution.

We may provide the following example, with $f(x) = 1/x$, for $x > 0$.

We know that $df(x)/dx = -1/x^2$, for $x > 0$, and $d^2f(x)/dx^2 = 2/x^3$, for $x > 0$, such that $d^2f(x)/dx^2 = 2/x^3 > 0$ is verified and then the solution of $df(x)/dx = 0$ is at most one.

However, $-1/x^2 = 0$ did not have solution for $x > 0$.

In Hsieh et al. [1], finding a common condition of Equation (6.1) to guarantee that $dAC_i(T)/dT$ are increasing functions for $i = 1,2,3$, is a great result. However, without discussing the solutions for $dAC_i(T)/dT = 0$, the increasing property of $dAC_i(T)/dT$ (that is the convex property of $AC_i(T)$), sometimes did not imply the existence of the critical solution.

In Proposition 4 of Hsieh et al. [1], they derive three cases where the location of the global minimum solution can be decided. In the following, we will show that Proposition 4 of Hsieh et al. [1] is far from completion.

To provide a complete describe of the piecewise connected three objective function, $AC_3(T)$, for $0 < T \le N$, $AC_2(T)$, for $N \le T \le M$, and $AC_1(T)$, for $M \le T < \infty$, we need to consider (at least) the following eight items:

$$\text{(i) } \lim_{T \to N^-} dAC_3(T)/dT > 0, \quad (6.7)$$

$$\text{(ii) } \lim_{T \to N^-} dAC_3(T)/dT < 0, \quad (6.8)$$

$$\text{(iii) } \lim_{T \to N^+} dAC_2(T)/dT > 0, \quad (6.9)$$

$$\text{(iv) } \lim_{T \to N^+} dAC_2(T)/dT < 0, \quad (6.10)$$

$$\text{(v) } \lim_{T \to M^-} dAC_2(T)/dT > 0, \quad (6.11)$$

$$\text{(vi) } \lim_{T \to M^-} dAC_2(T)/dT < 0, \quad (6.12)$$

$$\text{(vii) } \lim_{T \to M^+} dAC_1(T)/dT > 0, \quad (6.13)$$

and

$$\text{(viii) } \lim_{T \to M^+} dAC_1(T)/dT < 0. \quad (6.14)$$

Depending on the sign of (a) $\lim_{T \to N^-} dAC_3(T)/dT$, (b) $\lim_{T \to N^+} dAC_2(T)/dT$, (c) $\lim_{T \to M^-} dAC_2(T)/dT$, and (d) $\lim_{T \to M^+} dAC_1(T)/dT$, there are (about) 16 different graphs for Hsieh et al. [1] which is predicted by $2^4 = 16$.

Remark. We have to point out that there are several cases that are influences each other.

For example, if

$$\lim_{T \to M^-} dAC_2(T)/dT < 0, \quad (6.15)$$

then

$$\lim_{T \to N^+} dAC_2(T)/dT > 0, \quad (6.16)$$

will not happen.

Similarly, if

$$\lim_{T \to N^+} dAC_2(T)/dT > 0, \quad (6.17)$$

then

$$\lim_{T \to M^-} dAC_2(T)/dT < 0, \quad (6.18)$$

will not happen.

In Proposition 4 of Hsieh et al. [1], there are only three results which is too little cases of those possible findings.

Now, we begin to examine Proposition 4 of Hsieh et al. [1].

We agree their result of (a) with (ii) $\lim_{T \to N^-} dAC_3(T)/dT < 0$, and (vi) $\lim_{T \to M^-} dAC_2(T)/dT < 0$, then the global minimum occurs at $AC_1(T)$.

Their result of (b) with (ii) $\lim_{T \to N^-} dAC_3(T)/dT < 0$, and (v) $\lim_{T \to M^-} dAC_2(T)/dT > 0$, then Hsieh et al. [1], claimed that the global minimum occurs at $AC_2(T)$. The above argument is questionable.

We must point out that (v) $\lim_{T \to M^-} dAC_2(T)/dT > 0$ should be replaced by (vii) $\lim_{T \to M^+} dAC_1(T)/dT > 0$.

Their result of (c) with (i) $\lim_{T \to N^-} dAC_3(T)/dT > 0$, and (v) $\lim_{T \to M^-} dAC_2(T)/dT > 0$, then Hsieh et al. [1], claimed that the global minimum occurs at $AC_3(T)$. The above argument is questionable.

We believe that (i) $\lim_{T \to N^-} dAC_3(T)/dT > 0$ should be replaced by (iii) $\lim_{T \to N^+} dAC_2(T)/dT > 0$.

We believe that (v) $\lim_{T \to M^-} dAC_2(T)/dT > 0$ should be replaced by (vii) $\lim_{T \to M^+} dAC_1(T)/dT > 0$.

Hence, we can claim that the solution procedure based on Proposition 4 of Hsieh et al. [4] is incomplete such that we cannot use the solution procedure based on Proposition 4 of Hsieh et al. [4] to locate the optimal solution.

We recall Example 1 that was proposed by Hsieh et al. [1]. $AC_3(T)$ has minimum at $T_3^* = 0.0289$, $AC_2(T)$ has minimum at $T_2^* = 0.1233$, and $AC_1(T)$ has minimum at $T_1^* = 0.1834$.

We point out that

$$T_3^* = 0.0289 < (15/365) \sim 0.0411 < T_2^* = 0.1233 \sim (45/365) = 0.1233 < T_1^* = 0.1834. \quad (6.19)$$

Hence, two convex functions, $AC_3(T)$ and $AC_1(T)$ have interior minimum point, respectively.

This example indicates that

$$\text{(i) } \lim_{T \to N^-} dAC_3(T)/dT > 0, \quad (6.20)$$

$$\text{(iv) } \lim_{T \to N^+} dAC_2(T)/dT < 0, \quad (6.21)$$

$$\text{(v) } \lim_{T \to M^-} dAC_2(T)/dT < 0, \quad (6.22)$$

and

$$\text{(viii) } \lim_{T \to M^+} dAC_1(T)/dT < 0, \quad (6.23)$$

which shows that Proposition 4 of Hsieh et al. [1] is incomplete.

Based on our examination, we show that Proposition 4 of Hsieh et al. [1] is far from completion.

VII. Directions for Future Research

We provide a short list of recently published papers to help practitioners find possible hot topics for their further developments. For computing economic order quantities, Chiang et al. [16] obtained a simple procedure. To realize appliance of mass spring models, Surya and Ganesan [17] examined interval symmetric matrix with respect to axis theory. Under the combustion regulation for service channel, Bai et al. [18] got ventilation and sealing simulation test models. Through crisp symmetric matrix, with respect to eigenvalues and eigenvectors, Singh et al. [19] derived novel interior bounds. According to vague slope optimization and Lagrange reduction, An et al. [20] employed time schedule for fast velocity transit system. Based on vague Nonlinear control models, Luo et al. [21] considered arranged presentation management related to activated occurrences. With respect to circle polynomial, Jummannavar et al. [22] examined Emden and Lane formula with numerical methods. Related to space classification, Song et al. [23] derived narrow piece algorithm with decipher regulator. Based on cross dimension function and strict measurement mapping, Kusuma [24] acquired a novel triple vibrated approach. With respect to entrance smart section, Fu et al. [25] considered online exterior shortcoming recognition procedure with deep learning. To learn information expansion and punctual modification, Wang et al. [26] studied biomedical text to revise medicine interactive withdrawal. According to intense sensation method, Tong et al. [27] gained estimation procedure for virtual community view proliferation. Owing to situation tracking and phase prediction, Solakha, and Salmah [28] found out symmetric relationship. According to neural system, Wang et al. [29] constructed a new wind velocity estimation approach with respect to a brief time period. Referring to prime ideal in Gamma rings and fuzzy or vague ideal, Durgadevi and Devarasan [30] developed new findings. Based on our literature reviewing, practitioners can find interesting research current and hot topics for their future examinations.

VIII. A Related Problem

We examine a recently published article to provide improvements. Wang et al. [31] studied Zhao [32] to provide an abstract approach to deal with inventory models with trapezoidal type demand.

Zhao [32] claimed that $te^{-\delta t}$ is an increasing function, that is, for $0 < t < T$, and then Wang et al. [31] accepted the assertion of Zhao [32]. In this section, we will execute a detailed examination for the assertion of Zhao [32], and then we will point out that sometimes $te^{-\delta t}$ is not an increasing function, that is, for the domain, $0 < t < T$. Consequently, we will present a patch work for Wang et al. [31] and Zhao [32].

We derive that

$$dte^{-\delta t}/dt = (1-\delta t)e^{-\delta t} > 0, \tag{8.1}$$

then this condition can be simplified as

$$1-\delta T > 0, \tag{8.2}$$

to derive an extra condition of T as

$$T < 1/\delta. \tag{8.3}$$

Now, we recall the numerical example in Zhao [32] to find that in the three Examples of Zhao [32], $T = 12$. On the other hand, the values of the parameter, δ, equals to 0.04, 0.02, and 0.2, respectively. On the other hand, from the data of Zhao [32], he run a sensitivity analysis of δ, with the following values: 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, and 0.08.

Based on our above observations, there are twelve values of δ, except one of them as $\delta = 0.2$ that did not satisfy the inequality of Equation (8.3). The rest eleven values of δ all satisfy the inequality of Equation (8.3).

According to our discussion, it shows that the third example of Zhao [32] deserved a detailed examination for the assertion of Equation (8.1).

For example 3 of Zhao [32], $\delta = 0.2$ and $T = 12$ such that Equation (8.2) is invalid. Consequently, the inequality of Equation (8.1) only valid for $0 < t < 5$. On the other hand, when $5 < t < T = 12$, the inequality of Equation (8.1) is failed.

We recall that the purpose of assuming $te^{-\delta t}$ being an increasing function is to obtain the inequality of Equation (8.2) to support his assertion of $h(t_1)$ is an increasing function.

Zhao [32] defined $h(t_1)$ as follows,

$$h(t_1) = c_1\left(e^{\alpha t_1^{\beta}} - 1\right) + c_2\int_0^{t_1} e^{\alpha\left(t_1^{\beta}-t^{\beta}\right)}dt$$
$$+c_3(t_1-T)e^{\delta(t_1-T)} + c_4\left[e^{\delta(t_1-T)} - 1\right]. \tag{8.4}$$

Zhao [32] claimed that

$$h(0) < 0, \tag{8.5}$$

and

$$h(T) > 0. \tag{8.6}$$

Zhao [32] wanted that $h(t_1)$ is an increasing function such that $h(t_1) = 0$ has a unique solution, denoted as t_1^* that satisfy $h(t_1^*) = 0$. Consequently, t_1^* is the optimal solution for the inventory system.

To verify that $h(t_1)$ is an increasing function, Zhao [32] tried to show that for $0 < t_1 < T$,

$$\frac{d}{dt_1}h(t_1) > 0. \tag{8.7}$$

Based on Equation (8.4), Zhao [32] derived that

$$\frac{d}{dt_1}h(t_1) = \alpha\beta t_1^{\beta-1}\left[c_1 e^{\alpha t_1^{\beta}} + c_2\int_0^{t_1} e^{\alpha\left(t_1^{\beta}-t^{\beta}\right)}dt\right],$$
$$+\{c_3[1+\delta(t_1-T)] + c_4\delta\}e^{\delta(t_1-T)} + c_2. \tag{8.8}$$

We find that Zhao [32] only considered that for $0 < t_1 < T$,

$$1+\delta(t_1-T) > 0. \tag{8.9}$$

Consequently, he tried to claim that $0 < t_1 < T$,

$$1-\delta T > 0, \tag{8.10}$$

as we cited in Equation (8.2).

There are other positive term in Equation (8.8) that are overlooked by Zhao [32]. In the following, we will provide our patchwork for example 3 of Zhao [32] to prove that the assertion of Equation (8.7) is still valid, under the condition that Equation (8.9) is failed.

We recall that $c_2 = 10$, $c_3 = 12$, $c_4 = 8$, $\delta = 0.2$, and $T = 12$, where Zhao [32] already showed that $t_1^* = 2.3235$. Our goal is to verify that for $t_1^* < t_1 < 12$

$$\frac{d}{dt_1}h(t_1) > 0. \tag{8.11}$$

Owing to $\alpha\beta t_1^{\beta-1}\left[c_1 e^{\alpha t_1^{\beta}} + c_2\int_0^{t_1} e^{\alpha\left(t_1^{\beta}-t^{\beta}\right)}dt\right]$ is a positive term, and

$$c_2 > c_2 e^{\delta(t_1-T)}, \tag{8.12}$$

we will prove that for $t_1^* < t_1 < 12$

$$\{c_3[1+\delta(t_1-T)] + c_4\delta\} + c_2 > 0. \tag{8.13}$$

We evaluate that

$$c_2 + c_3 + c_4\delta + c_3\delta t_1 > 10 + 12 + 8(0.2) + 2.4(2.3),$$
$$= 29.12 > 28.8 = 12(0.2)12 = c_3\delta T. \tag{8.14}$$

According to Equation (8.14), we show that our assertion of Equation (8.11) is valid under the restriction, $t_1^* < t_1 < 12$, and then the assertion of $h(t_1)$ being an increasing function is still valid after our patchwork.
Based on our above discussion, we presented patchworks not only for Zhao [32] but also Wang et al. [31].

IX. A FURTHER EXAMINATION

We will study inventory systems that have been exanined by Wu and Ouyang [33], Tung et al. [34], and Wu et al. [35] under stochastic demand and defective items. Tung et al. [34] pointed out the solution algorithm proposed by Wu and Ouyang [33], sometimes, cannot generate a convergent sequence such that the iterative method of Wu and Ouyang [33] is not a robust solution approach. Recently, Wu et al. [35] provided a novel solution process to find the optimal solution. We will provide another investigation of Wu and Ouyang [33], Tung et al. [34], and Wu et al. [35] with resoect to inventory systems with defective products and stochastic demand.

It would be nice to have an example where it doesn't converge with respect to iterative algorithms. Consequently, we provide a hypothetical example to explain our point. We assume that there are two relations governing x and y,

$$y = x - 1, \tag{9.1}$$

and

$$x = (2/y). \tag{9.2}$$

Our goal is to obtain the pair of positive solution with

$$x = 1, \tag{9.3}$$

and

$$y = 2. \tag{9.4}$$

Referring to Equations (9.1) and (9.2), we develop the following iterative algorithm:

$$y_{n+1} = x_n - 1, \tag{9.5}$$

and.

$$x_{n+1} = 2/y_{n+1}. \tag{9.6}$$

If we take $x_1 = 0$, by iterative method, then it yields $y_1 = -1$, $x_2 = -2$, $y_2 = -3$, $x_3 = -2/3$, $y_3 = -5/3$, $x_3 = -6/5$, $y_4 = -11/5$, …, eventually, two sequences will converge to their limit, respectively.

$$\lim_{n\to\infty} x_n = -1, \tag{9.7}$$

and

$$\lim_{n\to\infty} y_n = -2. \tag{9.8}$$

However, the findings of Equations (9.7) and (9.8) do not match our requirement of positive solutions. We demonstrate that by iterative approach, sometimes, the convergent sequence did not offer an acceptable solution.
On the other hand, for the seconds example, we take $x_1 = 3$, by iterative method, then it yields $y_1 = 2$, $x_2 = 1$, $y_2 = 0$. However, x_3 can not be derived. Hence, both $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ do not exist.

Our above discussion provide a technical patchwork for Wu and Ouyang [33], Tung et al. [34], and Wu et al. [35] such that it our two examples will help researchers aware that the iterative method had severe convergent problems.

We point out that it is certainly true that

$$\sqrt{\alpha_1 + \alpha_2} < \alpha_4/\alpha_3(1 + \beta), \tag{9.9}$$

for these particular values of β and L. But it is not always true since if we choose $D = 0$, to check Equation (9.9), and then we will face the following dilemma,

$$0 > 0. \tag{9.10}$$

So the paper should include conditions on β and L such that $\sqrt{\alpha_1 + \alpha_2}$ is the upper bound.

We claim that these results of Tung et al. [34] are better than the ones in Wu and Ouyang [33], because those derived costs of Tung et al. [34] is smaller than that of Wu and Ouyang [33], as shown in Table 3 of Tung et al. [34].

We point out that the condition of β as $\beta \in [0,1]$ has been provided in the notation of section 2 of Tung et al. [34]. The length of lead time L defining on $[L_n, L_0]$ has been provided in the assumptions of section 2 of Tung et al. [34]. It is problematically assigned. For the numerical examples in Tung et al. [34], there are $L_n = 3$ and $L_0 = 8$.

It is not mentioned at all in Tung et al. [34] how the values in the last 3 columns of Table 2 are obtained. So there has to be a detailed explanation how these values are determined. Hence, we provide an enhancement in the following.

For given pair of β and i, Q_i is the solution of $f(Q_i) = 0$. From the convex property of $f(Q)$, $f(\sqrt{\alpha_1}) < 0$, and $f(\sqrt{\alpha_1 + \alpha_2}) > 0$, Tung et al. [34] applied the bisection method to locate solution for $f(Q) = 0$ in the fourth column of Table 2. From equation (4.7) in Tung et al. [34], we plug into Q_i to derive k_i in the fifth column of Table 2. Tung et al. [34] plug Q_i, k_i, and L_i into equation (3.1) to find $EAC^u(Q_i, k_i, L_i)$, in the sixth column of Table 2.

Table 4 of Tung et al. [34] has to be constructed in a different way, because it does not look good in the current format. In the following, we provide a reasonable explanation. In Table 2 of Tung et al. [34], they listed the computation results of (Q_n) for $n = 1,2, \dots ,9$ to the sixth decimal place. In Table 4, for Q_n, Tung et al. [34] denoted to the second decimal place. For k_n, Tung et al. [34] changed the notation to the fourth decimal place, as shown below.
Our reason is explained as follows.If Tung et al. [34] still use the second decimal place, for the consistency of expression, and then Table 4 will become its formation as follows. As shown in the following table 2,

$$k_4 = k_5 = k_6 = 3.15, \tag{9.11}$$

but Q_4, Q_5, and Q_6 are different, which may seem illogical to ordinary readers.
However, taking the fourth decimal place for the expression of k_n will not lead to this problem.

X. SOME RELATED COMMENTS

Wu and Ouyang [33] developed two inventory models with defective items. In the first model, the lead time demand follows a normal distribution, and in the second model, it is distribution-free. In Tung et al. [34], they derived an analytical approach to verify the uniqueness of the optimal solution for the second model in Wu and Ouyang [33]. After examining the content of Tung et al. [34], we recommend that Tung et al. [34] can be revised thoroughly according to the following comments. The writing style and presentation of the abstract and the introduction section of Tung et al. [34] were not easy to read. The English needs some improvement in Tung et al. [34]. We present a detailed derivation for their Equation (4.6) in the following for ordinary practitioners.

$$\sqrt{1+k^2}/k = [(1-\beta)\alpha_3 Q + \alpha_4]/[\alpha_4 - (1+\beta)\alpha_3 Q], \tag{10.1}$$

$$\left[\sqrt{1+k^2}/k\right]^2 - 1 = \{[(1-\beta)\alpha_3 Q + \alpha_4]/[\alpha_4 - (1+\beta)\alpha_3 Q]\}^2 - 1, \tag{10.2}$$

and

$$[(1-\beta)\alpha_3 Q + \alpha_4]^2 - [\alpha_4 - (1+\beta)\alpha_3 Q]^2 = \{[(1-\beta)\alpha_3 Q + \alpha_4] - [\alpha_4 - (1+\beta)\alpha_3 Q]\} \times \{[(1-\beta)\alpha_3 Q + \alpha_4] + [\alpha_4 - (1+\beta)\alpha_3 Q]\}$$

$$= (2\alpha_3 Q)[2(\alpha_4 - \beta\alpha_3 Q)]$$
$$= 4\alpha_3 Q(\alpha_4 - \beta\alpha_3 Q). \quad (10.3)$$

XI. Conclusion

We provide revisions for a fourthcoming article to point out that some questionable results of Hsieh et al. [1] that did not implicitly mention in Liu and Yang [2]. Our paper will help researchers realize the meaning of smoothly connected property for inventory models with piecewisely constructed in several sub-domains. We point out a problem in the derivation of the first derivative being an increasing function in Zhao [32] and Wang et al. [31]. Moreover, we procide our numerical checking to patch the research gap.

Table 1. Reproduction of Table 4 of Wu and Ouyang [33].

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
Q_n	315.62	194.45	181.25	179.90	179.75	179.74
k_n	2.2916	3.0248	3.1386	3.1510	3.1523	3.1524

Table 2. For Consistency of expression, all in the second decimal place.

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
Q_n	315.62	194.45	181.25	179.90	179.75	179.74
k_n	2.29	3.02	3.14	3.15	3.15	3.15

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Feiyu Zhang received his B. S. degree in Mathematics and Applied Mathematics from Shandong University of Technology, Zibo, China, in 2020, and the M. S. degree in Operational Research and Cybernetics from the China University of Petroleum (East China), Qingdao, China, in 2023. The main research directions are Digital Economy, Communication Engineering, Electronic Information, and Inventory Systems.

Jinyuan Liu received his Master's degree from the School of Mathematics and Systems Science, Shandong University of Science and Technology in 2009. He is an Associate Professor at the School of General Studies of Weifang University of Science and Technology. The main research directions are Pattern Recognition, Lanchester's Model, Fuzzy Set Theorem, Isolate Points, Inventory Models and Analytical Hierarchy Process.