Queueing Inventory System with Impatient Customers and Working Vacations

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***Abstract*—A queueing inventory system with impatient customers and working vacations is studied in this article. Customers have two kinds of impatient behaviors when the inventory is zero and the server is on a working vacation. The stability condition, steady-state probability vectors and the performance measures are obtained by quasi-birth-and-birth process theory and Neuts-Rao truncation method. The cost function is established, and the system constrained optimization problem is solved by a genetic algorithm. Finally, the impact of the parameters on performance measures, optimal strategy and minimum cost are illustrated by numerical example analysis.**

***Index Terms*—queueing inventory system; impatient customers; working vacations; Neuts-Rao truncation; cost function.**

I. Introduction

WITH the development of fierce competition in the market business, more and more companies have changed their business model to one that can provide products and services to customers at the same time. Therefore, it is vital for companies to design a new business model that can satisfy customers' two kinds of needs and can reduce company's operating costs also.

Krishnamoorthy and Viswanath [1] investigated a production inventory model. They used the (s, S) production policy. Then they analyzed the system stability condition and conducted numerical experiments to analyze the impact of s and S on the system's out of inventory time and the expected rate of loss to customers. Subsequently, Krishnamoorthy and Viswanath [2] studied the optimal values of s and S and obtain analytical expressions for the optimal values. Beak and Moon [3] investigated an M/M/1 model of an attach production inventory and sales losses. The system had both external replenishment and internal production strategies: the external replenishment used the (r, Q) replenishment strategy, the internal production strategy obeyed a Poisson arrival process. They obtained a continuous joint distribution for both the queue length and the current inventory, and finally performed a numerical analysis to derive the optimal cost. Later, Beak and Moon [4] built on this foundation and investigated the M/M/c production service inventory system with sales loss. Gayon et al. [5] investigated a production-inventory system with product returns and two disposal options. They established an optimal control strategy with three threshold parameters. The numerical results show that the two disposal options are complementary. Kocer and Ozkar [6] studied a production inventory model which have priority customer and a server. They built a five-dimensional Markov chain to give the steady-state conditions and steady-state probabilities. Through numerical experiments, they obtained the optimal inventory and minimum cost.

In the classical server vacation strategy, server does not provide any service in the vacation period. However, in real life, this vacation often leads to customer dissatisfaction, so the working vacation policy has attracted the attention of many scholars. The concept of working vacation was proposed by Servi and Finn [7]. They investigated an M/M/1 working vacation queueing system, derived the steady-state performance measures, then applied them to the performance analysis of communication networks. Since then, many scholars have conducted in-depth studies on working vacation queueing systems. Majid et al. [8] considered an M/M/1 queueing model with working vacation policy. Random decomposition expressions for queue length and wait time were obtained. Manikandan and Nair [9] investigated a system containing working vacations and vacation interruptions. They got the system stability condition, and analyzed busy period. Li and Li [10] studied a retrial queue with working vacation, orbit search and balking system. Majid [11] investigated a system with the variable working vacations policy. He used the constant form of the function and degenerate hyper geometric function to get the steady-state probability, and derived the performance measures, which is in contrast to the general multiple working vacations. This model sets a maximum frequency of K of consecutive working vacations, and the server keeps idle after reaching the maximum frequency. Karthick and Suvitha [12] considered three servers with different service rates, and given the stationary conditions and boundary probability vectors of the model. Lv et al. [13] took into account the different customer arrival rates, and derived the conditional stochastic decomposition framework for queue length and waiting time.

Impatience is the most prominent feature of the queueing model. The waiting customers are always frustrated and dissatisfied. Therefore, customers either decide not to enter the queue when they arrive, or join a queue, but lose patience after some time and choose to leave. That leads to a loss of customer base. Some scholars have studied impatient customers in the early days.

Perry and Stadje [14] studied inventory systems that have impatient customers and perishable products, obtaining the steady-state probability distributions and cost function of system. Altman and Yechiali [15] investigated three queueing systems with single-server and multi-server . They focused

Manuscript received March 21, 2024; revised November 1, 2024.

This work was supported by the National Nature Science Foundation of China (No. 71971189).

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on customer impatience behavior when servers are on vacation. They also compared the percentage of customers abandoning under a single and multiple vacations, yielded that the proportion of customers renunciation is smaller under a single vacation system. Later, Altman and Yechiali [16] extended the impatience phenomenon investigated in the above studies to an infinite server system. Benjaafar et al. [17] researched a production inventory system with impatient customers. They used two thresholds to describe the optimal strategy: the production level base inventory, which determines when it will be produced, and the entry threshold, which determines when to accept orders. Hamadi et al. [18] researched a perishable product inventory system, and solved the optimal control of service in a service facility with impatient customers based on the (s, Q) replenishment policy. They obtained the optimal service rate through the Markov decision process and linear programming algorithm. Melikov et al. [19] studied a queueing inventory system with impatient customers, considering the cases of finite and infinite captains, and proposed exact and approximate algorithms for the model. Koroliuk et al. [20] studied a perishable queueing inventory system with impatient customers and server vacations, and the replenishment policy obeys a two-level strategy. Shajin and Krishnamoorthy [21] conducted research on a queueing inventory system involving impatient customers as well as reservations, cancellations, and overbookings. They obtained the stability condition and the associated performance measures. In addition, they gave a specific case where all distributions are exponential, getting solutions in the form of asymptotic products of the system. Zhang et al. [22] studied a queueing inventory system with impatient customers and mixed sales. And they made use of the Bright-Taylor truncated approximation for attaining the steady-state probability, and carried out an analysis of the impact of the parameters on the performance measures, the optimal policy as well as the optimal cost. Mathew et al. [23] studied a production inventory system with server failures and impatient customers. By performance measures, they deduced the distribution of five important performance characteristics. Among them, customer impatience is generated by server maintenance. In this model, each inventory product has to go through k production stages, and the production time of each production stage follows the phase distribution.

Based on the above, a production queueing inventory system with multiple working vacations and impatient customers is investigated. We mainly consider two kinds of situations of customers' impatience. The first one, assuming that the inventory is zero, the customers will become impatient, if the customer not served in the impatience time, and then the customer will leave the system at any time; the second one, if the inventory is not zero, during a working vacation, the customers will become impatient, if the customer not served in the impatience time, the customer will leave the system at any time. Both impatience times follow an exponential distribution.

The rest of the paper is organized as follows: The system model is described in section 2. In Section 3, we get the steady-state conditions and the steady-state probabilities' matrix geometry solution by using the Neuts-Rao truncation method. Performance measures and cost function are given in Section 4. Section 5 is numerical analysis. The conclusion and future prospects of the paper are presented in Section 6.

II. Model Description

We research a production queueing inventory system with impatient customers and working vocations, and the model is described as follows:

1) The arrival of customers obey a Poisson process with rate λ. A queue is formed when the number of customers is more than one. Each customer needs only one product after serviced. Therefore, the system inventory will reduce one unit after serving a customer.
2) The system has one server. During the regular busy period, the service time follows an exponential distribution with the parameter μ_b. The service rule is First-Come-First-Served(FCFS). If the number of customers is zero, the server begins a working vacation. During the working vacation period, the service time follows an exponential distribution with the parameter $\mu_v\,(\mu_v < \mu_b)$. The system adopts multiple working vacations policy: If there are still customers in the system after a working vacation, the server will immediately start a regular busy period. Otherwise, the server starts another working vacation. The vacation time obeys an exponential distribution with a parameter θ.
3) The system has only one equipment for providing products and adopts (s, S) product supply strategy: The production equipment starts producing when the system inventory level drops to s. And equipment produces one by one. A finished product will be sent to the system immediately. The production equipment stops when the inventory level reaches S. We hypothesize that the production time of a single unit product conforms to the exponential distribution with the parameter η.
4) When the inventory is zero, the customer waiting in the system will become impatient, and the impatient waiting time T_1 follows the exponential distribution with the parameter ξ_1. Furthermore, if the inventory is not zero but in a working vacation period, the customer becomes impatient also, and the waiting time T_2 follows an exponential distribution with parameter ξ_2.
5) During a working vacation, the customer impatient waiting time changes between T_2 and T_1 according to the inventory level is zero or not.
6) Let us consider that the inter-arrival times, service durations, vacation periods, impatience durations, and production times are all mutually independent.

III. Steady Performance Analysis

III.i State Processes of the System

Define $\Psi(t) = \{N(t), I(t), J(t), C(t), t \geq 0\}$ is the state process of system, where $N(t)$ means the number of customers in the system at time t, $I(t)$ means the inventory level in the system at time t, $J(t)$ means the state of server in the system at time t. $J(t)$=0 means that the server is in a working vocation at time t. $J(t)$=1 means that the server is in a regular busy period at time t. $C(t)$ denotes the state of production equipment in the system at time t. Let $C(t)$=0 denote that the production equipment is stop at time t. Let

$C(t)$=1 denote that the production equipment is working at time t. The state space of the system is $\Omega = \Omega_1 \bigcup \Omega_2$, where,
$\Omega_1 = \{(0,i,0,1), 0 \le i \le S-1\} \bigcup \{(0,i,0,0), s+1 \le i \le S\}$, $\Omega_2 = \{(n,i,j,1), n \ge 1, 0 \le i \le S-1, j = 0,1\} \bigcup \{(n,i,j,0), n \ge 1, s+1 \le i \le S, j = 0,1\}$.
$\Psi(t)$ is a Markov process on the state space, and its infinitesimal generator is as follows:

$$Q = \begin{pmatrix} A_0 & C_0 & & & & & \\ B_1 & A_1 & C & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & B_n & A_n & C & & \\ & & & B_{n+1} & A_{n+1} & C & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where A_0 is a $(2S-s) \times (2S-s)$ dimensional matrix, C_0 is a $(2S-s) \times (4S-2s)$ dimensional matrix, B_1 is a matrix of order $(4S-2s) \times (2S-s)$, the rest are $(4S-2s) \times (4S-2s)$ order matrices. And they are given by

$$A_0 = \begin{pmatrix} -d & \eta & 0 & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ 0 & d & \eta & \cdots & 0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d & D_3 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & D_0 & D_2 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_0 & D_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & D_1 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} \lambda & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda & 0 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} E_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ F_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & F_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & F_0 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & F_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F_0 & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

$C = \lambda I$,

$$A_n = \begin{pmatrix} G_{n0} & H_0 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G_{n1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & G_{n1} & H_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & G_{n1} & \mathbf{0} & H_0 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & G_{n2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & G_{n1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & H_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & G_{n2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & G_{n1} & H_0 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & G_{n2} \end{pmatrix},$$

$n \ge 1$,

$$B_n = \begin{pmatrix} E_{n1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ F_1 & E_{n2} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & F_1 & E_{n2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & F_1 & \mathbf{0} & E_{n2} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F_1 & \mathbf{0} & E_{n2} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & E_{n2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & E_{n2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_1 & \mathbf{0} & E_{n2} \end{pmatrix},$$

$n \ge 2$,
where I is an identity matrix of the order (4S-2s),
$d = -(\lambda + \eta)$,
$g_1 = -(\lambda + \eta + \theta + n\xi_1)$,
$g_2 = -(\lambda + \eta + n\xi_1)$,
$g_{n1} = -[\lambda + \eta + \theta + (n-1)\xi_2 + \mu_\nu]$,
$g_{n2} = -(\lambda + \eta + \mu_b)$,

$$D_0 = \begin{array}{c} \\ 0,j,0,1 \\ 0,j+1,0,0 \end{array} \begin{array}{c} \begin{array}{cc} 0,j,0,1 & 0,j+1,0,0 \end{array} \\ \begin{pmatrix} d & 0 \\ 0 & -\lambda \end{pmatrix} \end{array},$$

$s \le j \le S-2$,

$$D_1 = \begin{array}{c} \\ 0,S-1,0,1 \\ 0,S,0,0 \end{array} \begin{array}{c} \begin{array}{cc} 0,S-1,0,1 & 0,S,0,0 \end{array} \\ \begin{pmatrix} d & \eta \\ 0 & -\lambda \end{pmatrix} \end{array},$$

$$D_2 = \begin{array}{c} \\ 0,j,0,1 \\ 0,j+1,0,0 \end{array} \begin{array}{c} \begin{array}{cc} 0,j+1,0,1 & 0,j+2,0,0 \end{array} \\ \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix} \end{array},$$

$s \le j \le S-2$,

$$D_3 = \begin{array}{c} \\ 0,s-1,0,1 \end{array} \begin{array}{c} \begin{array}{cc} 0,s,0,1 & 0,s+1,0,0 \end{array} \\ \begin{pmatrix} \eta & 0 \end{pmatrix} \end{array},$$

$$E_0 = \begin{array}{c} \\ 1,0,0,1 \\ 1,0,1,0 \end{array} \begin{array}{c} 0,0,0,1 \\ \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} \end{array},$$

$$F_0 = \begin{array}{c} \\ 1,s+1,0,0 \\ 1,s+1,1,0 \end{array} \begin{array}{c} 0,s,0,1 \\ \begin{pmatrix} \mu_\nu \\ \mu_b \end{pmatrix} \end{array},$$

$$F_0 = \begin{array}{c} \\ 1,j,0,c \\ 1,j,1,c \end{array} \begin{array}{c} 0,j-1,0,c \\ \begin{pmatrix} \mu_\nu \\ \mu_b \end{pmatrix} \end{array}, c = 0,1.$$

If $c = 0$, then $s+2 \le j \le S$; if $c = 1$, then $1 \le j \le S-1$.

$$\begin{array}{cc} & \begin{array}{cc} n,j+1,0,1 & n,j+1,1,1 \end{array}\\ H_0 = \begin{array}{c} n,j,0,1\\ n,j,1,1 \end{array} & \begin{pmatrix} \eta & 0\\ 0 & \eta \end{pmatrix} \end{array},$$

$0 \le j \le S-2$,

$$\begin{array}{cc} & \begin{array}{cc} n,S,0,0 & n,S,1,0 \end{array}\\ H_0 = \begin{array}{c} n,S-1,0,1\\ n,S-1,1,1 \end{array} & \begin{pmatrix} \eta & 0\\ 0 & \eta \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} n,0,0,1 & n,0,1,1 \end{array}\\ G_{n0} = \begin{array}{c} n,0,0,1\\ n,0,1,1 \end{array} & \begin{pmatrix} g_1 & \theta\\ 0 & g_2 \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} n,j,0,1 & n,j,1,1 \end{array}\\ G_{n1} = \begin{array}{c} n,j,0,1\\ n,j,1,1 \end{array} & \begin{pmatrix} g_{n1} & \theta\\ 0 & g_{n2} \end{pmatrix} \end{array},$$

$1 \le j \le S-1$,

$$\begin{array}{cc} & \begin{array}{cc} n,j,0,0 & n,j,1,0 \end{array}\\ G_{n2} = \begin{array}{c} n,j,0,0\\ n,j,1,0 \end{array} & \begin{pmatrix} g_{n1}-\eta & \theta\\ 0 & g_{n2}-\eta \end{pmatrix} \end{array},$$

$s+1 \le j \le S$,

$$\begin{array}{cc} & \begin{array}{cc} n-1,s,0,1 & n,s,1,1 \end{array}\\ F_1 = \begin{array}{c} n,s+1,0,0\\ n,s+1,1,0 \end{array} & \begin{pmatrix} \mu_\nu & 0\\ 0 & \mu_b \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} n-1,j-1,0,c & n,j-1,1,c \end{array}\\ F_1 = \begin{array}{c} n,j,0,c\\ n,j,1,c \end{array} & \begin{pmatrix} \mu_\nu & 0\\ 0 & \mu_b \end{pmatrix} \end{array},$$

$c = 0, 1$.

If $c = 0$, then $s+2 \le j \le S$; if $c = 1$, then $1 \le j \le S-1$.

$$\begin{array}{cc} & \begin{array}{cc} n-1,0,0,0 & n,0,1,0 \end{array}\\ E_{n1} = \begin{array}{c} n,0,0,0\\ n,0,1,0 \end{array} & \begin{pmatrix} n\xi_1 & 0\\ 0 & n\xi_1 \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} n-1,j,0,c & n,j,1,c \end{array}\\ E_{n2} = \begin{array}{c} n,j,0,c\\ n,j,1,c \end{array} & \begin{pmatrix} (n-1)\xi_2 & 0\\ 0 & 0 \end{pmatrix} \end{array}, c = 0, 1.$$

If $c = 0$, then $s+1 \le j \le S$; if $c = 1$, then $1 \le j \le S-1$.

III.ii Stability conditions of the system

In this thesis, the Neuts-Rao truncation method [24] is used to solve the steady-state probability vector, assuming that the quasi-birth-and-birth process is no longer changing from a certain level, at which time the infinitesimal generator of the process is

$$Q^* = \begin{pmatrix} A_0 & C_0 & & & & & & \\ B_1 & A_1 & C & & & & & \\ & B_2 & A_2 & C & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & B_N & A_N & C & & \\ & & & & B_N & A_N & C & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}.$$

From the matrix of Q^*, the process $\Psi(t) = \{N(t), I(t), J(t), C(t), t \ge 0\}$ is a quasi-birth-and-birth process. Let $H = A_N + B_N + C$, then we have

$$H = \begin{pmatrix}
G_0 & H_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
F_1 & G_1 & H_0 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & G_1 & H_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_1 & G_1 & \mathbf{0} & H_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F_1 & G_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & F_1 & \mathbf{0} & G_1 & \mathbf{0} & H_0 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & F_1 & \mathbf{0} & G_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & F_1 & \mathbf{0} & G_1 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & G_2 & \mathbf{0} & \mathbf{0}\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & G_1 & H_0\\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & F_1 & \mathbf{0} & G_2
\end{pmatrix},$$

where,

$$\begin{array}{cc} & \begin{array}{cc} 0,0,1 & 0,1,1 \end{array}\\ G_0 = \begin{array}{c} 0,0,1\\ 0,1,1 \end{array} & \begin{pmatrix} -(\eta+\theta) & \theta\\ 0 & -\eta \end{pmatrix} \end{array},$$

$$\begin{array}{cc} & \begin{array}{cc} j,0,1 & j,1,1 \end{array}\\ G_1 = \begin{array}{c} j,0,1\\ j,1,1 \end{array} & \begin{pmatrix} -(\mu_\nu+\eta+\theta) & \theta\\ 0 & -(\mu_b+\eta) \end{pmatrix} \end{array}, 1 \le j \le S-1,$$

$$\begin{array}{cc} & \begin{array}{cc} j,0,0 & j,1,0 \end{array}\\ G_2 = \begin{array}{c} j,0,0\\ j,1,0 \end{array} & \begin{pmatrix} -(\mu_\nu+\theta) & \theta\\ 0 & -(\mu_b) \end{pmatrix} \end{array}, s+1 \le j \le S,$$

F_1 and H_0 are defined above.

Let $\alpha = (\alpha_{01}, \alpha_{11}, \cdots, \alpha_{s1}, \alpha_{s+1,0}, \alpha_{s+1,1}, \cdots, \alpha_{S-1,0}, \alpha_{S-1,1}, \alpha_{S,0})$ be H's steady-state probability vector, where,

$\alpha_{i0} = (\alpha(i,0,0), \alpha(i,1,0)), s+1 \le i < S$,

$\alpha_{i1} = (\alpha(i,0,1), \alpha(i,1,1)), 0 \le i < S-1$.

Then, α satisfies the balance equations

$$\begin{cases} \alpha H = 0, \\ \alpha e = 1, \end{cases} \tag{1}$$

e is a column vector with $(4S-2s) \times (4S-2s)$ dimensional elements all 1.

Theorem 1. If and only if

$$\begin{aligned} \lambda <& (-1)^S \alpha_{S0}[W_S - (F_1 G_2{}^{-1})^{S-s-1} Z_0] \begin{pmatrix} N\xi_1 \\ N\xi_1 \end{pmatrix} \\ &+ \alpha_{S0}[\sum_{i=1}^{S-1} (-1)^i W_i \\ &+ \sum_{i=S-s+1}^{S-1} (-1)^{i-1} (F_1 G_2{}^{-1})^{S-s-1} Z_{S-i} \\ &+ \sum_{i=0}^{S-s-1} (-1)^i (F_1 G_2{}^{-1})^i] \begin{pmatrix} \mu_\nu + (N-1)\xi_2 \\ \mu_b \end{pmatrix}, \end{aligned}$$

the quasi-birth-and-birth process returns normally. Which

$$\begin{aligned} \alpha_{S0} =& [\sum_{i=1}^{S} (-1)^i W_i e_1 \\ &+ \sum_{i=S-s+1}^{S} (-1)^{i-1} (F_1 G_2{}^{-1})^{S-s-1} Z_{S-i} e_1 \\ &+ \sum_{i=0}^{S-s-1} (-1)^i (F_1 G_2{}^{-1})^i e_1]^{-1}, \end{aligned}$$

$$W_i=\begin{cases}\frac{1}{\eta}G_2, i=1,\\ \frac{1}{\eta}G_2G_1, i=2,\\ \frac{1}{\eta}W_{i-1}G_1-\frac{1}{\eta}W_{i-2}F_1, 3\le i\le S,\end{cases}$$

$$Z_j=\begin{cases}\frac{1}{\eta}F_1, j=s-1,\\ \frac{1}{\eta}F_1G_1, j=s-2,\\ \frac{1}{\eta}Z_{j+1}G_1-\frac{1}{\eta}Z_{j+2}F_1, 0\le j\le s-3.\end{cases}$$

Proof According to equation $\alpha H=0$, we can obtain that

$$\alpha_{01}G_0+\alpha_{11}F_1=0, \tag{2}$$

$$\alpha_{i-1,1}H_0+\alpha_{i1}G_1+\alpha_{i+1,1}F_1=0, 1\le i\le s-1, \tag{3}$$

$$\alpha_{s-1,1}H_0+\alpha_{s1}G_1+\alpha_{s+1,1}F_1+\alpha_{s+1,0}F_1=0, \tag{4}$$

$$\alpha_{i0}G_2+\alpha_{i+1,0}F_1=0, s+1\le i\le S-1, \tag{5}$$

$$\alpha_{i-1,1}H_0+\alpha_{i1}G_1+\alpha_{i+1,1}F_1=0, s+1\le i\le S-2, \tag{6}$$

$$\alpha_{S-2,1}H_0+\alpha_{S-1,1}G_1=0, \tag{7}$$

$$\alpha_{S-1,1}H_0+\alpha_{S0}G_2=0. \tag{8}$$

From Eq. (8), we can obtain that

$$\alpha_{S-1}=-\alpha_{S0}\frac{1}{\eta}G_2. \tag{9}$$

Substituting Eq. (9) into Eq. (7), we have

$$\alpha_{S-2}=(-1)^2\alpha_{S0}\frac{1}{\eta^2}G_2G_1. \tag{10}$$

Taking i=S-2, substituting Eq. (9) and Eq. (10) into Eq. (6), we have

$$\alpha_{S-3}=(-1)^3\alpha_{S0}[\frac{1}{\eta}(\frac{1}{\eta^2}G_2G_1)G_1-\frac{1}{\eta}(\frac{1}{\eta}G_2)F_1]. \tag{11}$$

Taking i=S-3, substituting Eq. (10) and Eq. (11) into Eq. (6), we have

$$\begin{aligned}\alpha_{S-4}=&(-1)^4\alpha_{S0}\{\frac{1}{\eta}[\frac{1}{\eta}(\frac{1}{\eta^2}G_2G_1)G_1\\&-\frac{1}{\eta}(\frac{1}{\eta}G_2)F_1]G_1-\frac{1}{\eta}(\frac{1}{\eta^2}G_2G_1)F_1\}.\end{aligned} \tag{12}$$

And so on to obtain

$$\alpha_{S-i,1}=(-1)^i\alpha_{S0}W_i, 1\le i\le S-s, \tag{13}$$

which,

$$W_i=\begin{cases}\frac{1}{\eta}G_2, i=1,\\ \frac{1}{\eta}G_2G_1, i=2,\\ \frac{1}{\eta}W_{i-1}G_1-\frac{1}{\eta}W_{i-2}F_1, 3\le i\le S.\end{cases}$$

From Eq. (5), we can obtain that

$$\alpha_{S-i,0}=(-1)^i\alpha_{S0}(F_1{G_2}^{-1})^i, 0\le i\le S-s-1. \tag{14}$$

From Eq. (4), we can obtain that

$$\begin{aligned}\alpha_{s-1,1}=&(-1)^{S-s+1}\alpha_{S0}(\frac{1}{\eta}W_{S-s}G_1-\frac{1}{\eta}W_{S-s-1}F_1)\\&+(-1)^{S-s}\alpha_{S0}(F_1{G_2}^{-1})^{S-s-1}Z_{s-1},\end{aligned} \tag{15}$$

where $Z_{s-1}=\frac{1}{\eta}F_1$.

Let $W_{S-s+1}=\frac{1}{\eta}W_{S-s}G_1-\frac{1}{\eta}W_{S-s-1}F_1$, taking i = s-1, substituting Eq. (15) and α_{s1} into Eq. (3), we have

$$\begin{aligned}\alpha_{s-2,1}=&(-1)^{S-s+2}\alpha_{S0}(\frac{1}{\eta}W_{S-s+1}G_1-\frac{1}{\eta}W_{S-s}F_1)\\&+(-1)^{S-s+1}\alpha_{S0}(F_1{G_2}^{-1})^{S-s-1}Z_{s-2},\end{aligned} \tag{16}$$

where $Z_{s-2}=\frac{1}{\eta^2}F_1G_1$.

Let $W_{S-s+2}=\frac{1}{\eta}W_{S-s+1}G_1-\frac{1}{\eta}W_{S-s}F_1$, taking i = s-2, substituting Eq. (16) and Eq. (15) into Eq. (3), we have

$$\begin{aligned}\alpha_{s-3,1}=&(-1)^{S-s+3}\alpha_{S0}(\frac{1}{\eta}W_{S-s+2}G_1-\frac{1}{\eta}W_{S-s+1}F_1)\\&+(-1)^{S-s+2}\alpha_{S0}(F_1{G_2}^{-1})^{S-s-1}Z_{s-3},\end{aligned} \tag{17}$$

where $Z_{s-3}=(\frac{1}{\eta}Z_{s-2}G_1-\frac{1}{\eta}Z_{s-1}F_1)$.

Let $W_{S-s+3}=\frac{1}{\eta}W_{S-s+2}G_1-\frac{1}{\eta}W_{S-s+1}F_1$, taking i = s-3, substituting Eq. (17) and Eq. (16) into Eq. (3), we have

$$\begin{aligned}\alpha_{s-4,1}=&(-1)^{S-s+4}\alpha_{S0}(\frac{1}{\eta}W_{S-s+3}G_1-\frac{1}{\eta}W_{S-s+2}F_1)\\&+(-1)^{S-s+3}\alpha_{S0}(F_1{G_2}^{-1})^{S-s-1}Z_{s-4},\end{aligned} \tag{18}$$

where,$Z_{s-4}=(\frac{1}{\eta}Z_{s-3}G_1-\frac{1}{\eta}Z_{s-2}F_1)$.

And so on, eventually obtaining

$$\begin{aligned}\alpha_{S-i,1}=&(-1)^i\alpha_{S0}W_i+(-1)^{i-1}\alpha_{S0}(F_1{G_2}^{-1})^{S-s-1}Z_{S-i},\\&S-s+1\le i\le S,\end{aligned} \tag{19}$$

where,

$$W_i=\frac{1}{\eta}W_{i-1}G_1-\frac{1}{\eta}W_{i-2}F_1, S-s+1\le i\le S,$$

$$Z_j=\begin{cases}\frac{1}{\eta}F_1, j=s-1,\\ \frac{1}{\eta}F_1G_1, j=s-2,\\ \frac{1}{\eta}Z_{j+1}G_1-\frac{1}{\eta}Z_{j+2}F_1, 0\le j\le s-3.\end{cases}$$

So far, all components of the steady-state probability vector α of H have been represented by the vector α_{S0} component.

According to equation $\alpha e=1$, we can obtain that

$$\begin{aligned}\alpha_{S0}=&[\sum_{i=1}^{S}(-1)^iW_ie_1\\&+\sum_{i=S-s+1}^{S}(-1)^{i-1}(F_1{G_2}^{-1})^{S-s-1}Z_{S-i}e_1\\&+\sum_{i=0}^{S-s-1}(-1)^i(F_1{G_2}^{-1})^ie_1]^{-1}.\end{aligned} \tag{20}$$

From the literature [25], the sufficiently necessary condition for the quasi-birth-and-birth process is

$$\alpha Ce<\alpha B_Ne. \tag{21}$$

Substituting the steady-state probability vector α and matrix into Eq. (21), we have

$$\begin{aligned}\lambda <&(-1)^{S}\alpha_{S0}[W_S-(F_1G_2{}^{-1})^{S-s-1}Z_0]\begin{pmatrix}N\xi_1\\N\xi_1\end{pmatrix}\\&+\alpha_{S0}[\sum_{i=1}^{S-1}(-1)^{i}W_i\\&+\sum_{i=S-s+1}^{S-1}(-1)^{i-1}(F_1G_2{}^{-1})^{S-s-1}Z_{S-i}\\&+\sum_{i=0}^{S-s-1}(-1)^{i}(F_1G_2{}^{-1})^{i}]\begin{pmatrix}\mu_\nu+(N-1)\xi_2\\\mu_b\end{pmatrix}.\end{aligned}\tag{22}$$

III.iii Matrix Geometry Solution

Define $X=(X_0,X_1,\cdots,X_n,\cdots)$ as the steady-state probability of the system, which satisfies the balance equation

$$\begin{cases}XQ=0,\\Xe=1,\end{cases}\tag{23}$$

which,

$$\begin{aligned}X_0=(&X_0(0,0,1),X_0(1,0,1),\cdots,X_0(s,0,1),\\&X_0(s+1,0,0),X_0(s+1,0,1),\cdots,\\&X_0(S-1,0,0),X_0(S-1,0,1),X_0(S,0,0)),\end{aligned}$$

$$\begin{aligned}X_n=(&X_n(0,0,1),X_n(0,1,1),\cdots,X_n(s,0,1),\\&X_n(s,1,1),X_n(s+1,0,0),X_n(s+1,1,0),\\&X_n(s+1,0,1),X_n(s+1,1,1),\cdots\\&X_n(S-1,0,0),X_n(S-1,1,0),X_n(S-1,0,1),\\&X_n(S-1,1,1),X_n(S,0,0),X_n(S,1,0)),\\&n=1,2,\cdots,\end{aligned}$$

e is a column vector with elements equal to 1 of the appropriate dimension.

From the literature [23], the system state process returns normally. If and only if the minimal non-negative solution *R* of the matrix quadratic equation $R^2B_N+RA_N+C=0$ has a positive solution with spectral radius $sp(R)<1$ and $(X_0,X_1,\cdots,X_N)B[R]=0$, where,

$$B[R]=\begin{pmatrix}A_0&C_0&&&\\B_1&A_1&C&&\\&\ddots&\ddots&\ddots&\\&&B_{N-1}&A_{N-1}&C\\&&&B_N&RB_N+A_N\end{pmatrix}.$$

The steady-state probability vector of the system has the following matrix geometric solution form

$$X_k=X_NR_{k-N},k\geq N,$$

and satisfies the following equation

$$\begin{cases}(X_0,X_1,\cdots,X_N)B[R]=0,\\X_k=X_NR_{k-N},k\geq N,\\(\sum\limits_{i=1}^{N-1}X_i+X_N(I-R)^{-1})e=1,\end{cases}$$

where *I* is an identity matrix of the order 4*S*-2*s*, *e* is the column vector whose elements of the appropriate dimension are equal to one. The key to solving the steady-state probability vector lies in the solution of the rate matrix *R*. We adopted the cyclic reduction algorithm to solve the rate matrix *R*, which is described in detail in the literature [26].

IV. System performance measures and cost functions

IV.i System performance measures

1) The mean number of customers is given by

$$E_q=\sum_{i=1}^{\infty}iX_ie.$$

2) The mean inventory level is given by

$$\begin{aligned}E_{inv}=&\sum_{i=0}^{\infty}\sum_{j=0}^{S-1}j\,[X_i(j,0,1)+X_i(j,1,1)]\\&+\sum_{i=0}^{\infty}\sum_{j=s+1}^{S}j\,[X_i(j,0,0)+X_i(j,1,0)].\end{aligned}$$

3) The mean number of customers lost due to impatience arising from inventory is given by

$$E_{loss}{}^1=\sum_{i=1}^{\infty}(i-1)[X_i(0,0,1)+X_i(0,1,1)]\xi_1.$$

4) The mean number of customers lost due to impatience arising from server's working vacation is given by

$$\begin{aligned}E_{loss}{}^2=&[\sum_{i=1}^{\infty}\sum_{j=0}^{S-1}(i-1)X_i(j,0,1)\\&+\sum_{i=1}^{\infty}\sum_{j=s+1}^{S}(i-1)X_i(j,0,0)]\xi_2.\end{aligned}$$

5) The mean customers loss rate is given by

$$E_{loss}=E_{loss}{}^1+E_{loss}{}^2.$$

6) The mean production start-up rate is given by

$$\begin{aligned}E_{open}=&\mu_\nu\sum_{i=1}^{\infty}X_i(s+1,0,0)\\&+\mu_b\sum_{i=1}^{\infty}X_i(s+1,1,0).\end{aligned}$$

7) The mean productivity is given by

$$E_{pro}=\eta\sum_{i=0}^{\infty}\sum_{j=0}^{S-1}(X_i(j,0,1)+X_i(j,1,1)).$$

8) The mean working vacation rate is given by

$$E_{vac}=\theta[\sum_{i=0}^{\infty}\sum_{j=0}^{S-1}X_i(j,0,1)+\sum_{i=0}^{\infty}\sum_{j=s+1}^{S}X_i(j,0,0)].$$

IV.ii Cost analysis

According to the performance measures, the cost function per time of the system can be established following:

$$C(s,S) = c_0 E_q + c_1 E_{inv} + c_2 E_{loss} + c_3 E_{pro} + c_4 E_{open}.$$

Where c_0 is a waiting cost per unit of time per customer, c_1 is the storage cost of unit product per unit of time, c_2 is the cost of loss per unit of customer per unit of time, c_3 is the production cost of unit product per unit of time, and c_4 is a fixed cost for each restart of the equipment.

Based on practical considerations, the customers loss rate cannot be too large, otherwise it will cause great losses to the system. Therefore, when optimizing the above cost function, a constraint is added to it so that the optimal policy and minimum cost can be derived when the mean customers loss rate is less than a certain range. Due to the characteristic of the cost function, a genetic algorithm is employed to obtain the optimal solution of the function optimization problem that has constraints. In this paper, the population size is set to 200, the crossover rate is 0.8, the variation rate is 0.01, and the termination condition is 100 iterations. After repeated experiments, 100 iterations can converge to the optimal solution stably.

The specific steps of the algorithm are following:

Step 1: Define the objective function and constraints. The objective function is the function to be minimized or maximized. Constraints are defined, which can be either equality constraints or inequality constraints. Step 2: Select the appropriate coding scheme. According to the characteristics of the problem choose the appropriate coding scheme to represent the candidate solution space. Step 3: Initialise the population. Initialise the population using an appropriate method to ensure that the individuals in the population meet the constraints. Step 4: Evaluate the fitness function. Calculate the fitness value for each individual based on the objective function. The fitness value can be used to evaluate how good or bad an individual is based on the objective function value and the constraints. Step 5: Selection operation. Use the selection operation to choose the better-adapted individual from the population as the parent. Step 6: Crossover Operation. Use crossover operation to generate offspring by combining the chromosome information of the parents. Step 7: Mutation Operation. Use mutation operation to make random changes in the offspring to increase the diversity of the population. Step 8: Update the population. Update the population based on the results of selection, crossover, and mutation operations. Step 9: Determine termination conditions. Repeat steps 5 to 8 until the maximum number of iterations is reached. Step 10: Output optimal solution. Output the optimal solution that satisfies the constraints.

Table 1: Parameters corresponding to Figures 1-4.

Figs	Parameters
Fig. 1	$(\lambda,\theta,\eta,\mu_\nu,\mu_b,\xi_2,s,S) = (5,5,7,5,10,1,3,10)$
Fig. 2	$(\lambda,\theta,\eta,\mu_\nu,\mu_b,\xi_1,s,S) = (5,5,7,5,10,1,3,10)$
Fig. 3	$(\lambda,\theta,\mu_\nu,\mu_b,\xi_1,\xi_2,s,S) = (5,5,5,10,1,1,3,10)$
Fig. 4	$(\lambda,\eta,\mu_\nu,\mu_b,\xi_1,\xi_2,s,S) = (5,7,5,10,1,1,3,10)$

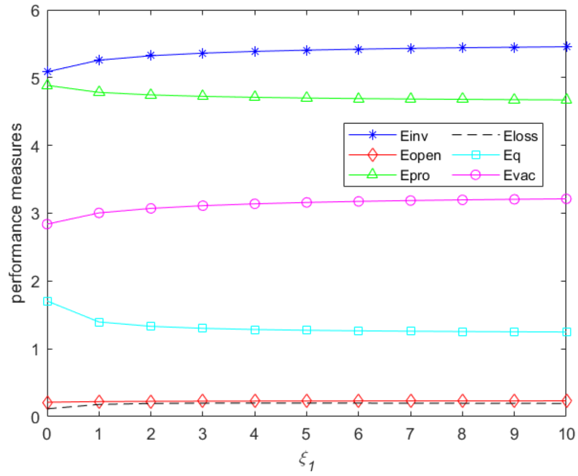

Fig. 1: The effect of changes in impatience rate ξ_1.

V. Numerical analysis

V.i Analysis of performance measures

In this section, the impact of system parameters change on various performance measures is analyzed through numerical examples. Figs. 1-4 Parameter settings details in Table 1. Tables 2-4 Parameters settings details in Table 5.

As shown in Fig. 1 and Fig. 2, ξ_1 and ξ_2 have the same effect on the system performance measures. When ξ_1 and ξ_2 increase, E_{inv}, E_{loss}, E_{vac} and E_{open} all increase, the remaining performance measures decrease. Because of when

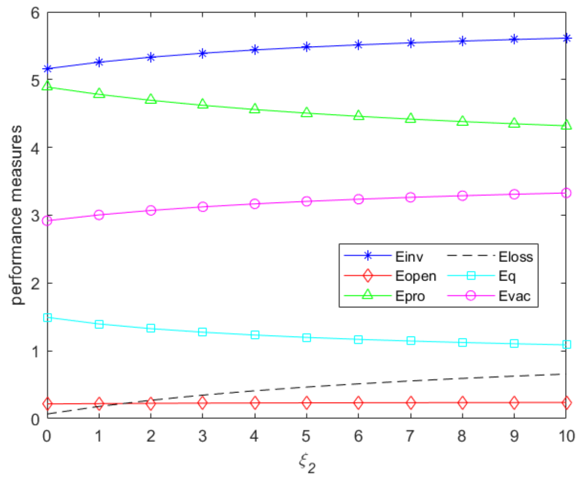

Fig. 2: The effect of changes in impatience rate ξ_2.

ξ_1 and ξ_2 increase, the mean customers' impatience time decreases, the probability of customers leaving the system becomes greater, resulting in fewer customers waiting in the queue, fewer customers needing fewer products, E_{inv} increases. By looking at Figs. 5-10, it is found that the effect of parameter ξ_2 on the system is higher than the effect of parameter ξ_1. In other words, the customers' impatience caused by working vacation has a higher impact than the customers' impatience caused by zero inventory.

As shown in Fig. 3, when η increases, E_{inv}, E_{pro}, E_{open} and E_{vac} all increase, and the remaining performance measures decrease.

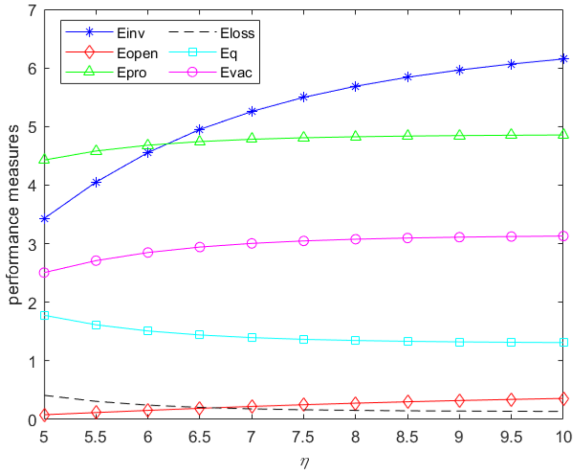

Fig. 3: The effect of changes in impatience rate η.

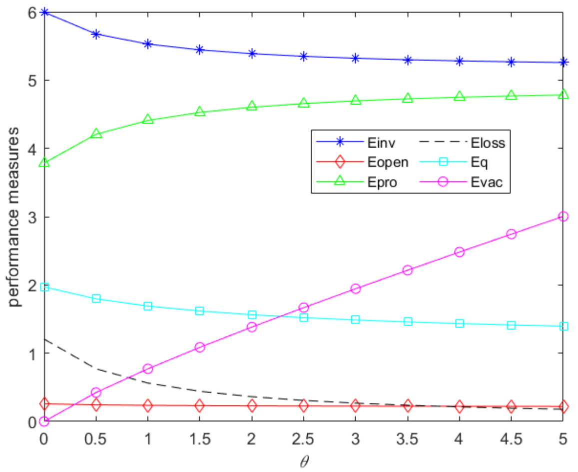

Fig. 4: The effect of changes in impatience rate θ.

As shown in Fig. 4, when parameter θ increases, E_{vac} andE_{pro} increase, the remaining performance measures decrease. That is because when θ increases, the server's vacation decreases, the impatient customers' number leaving the system decreases, resulting in a decrease in E_{inv}. The efficiency of service when server is during a regular busy period is much greater than when server is during a working vacation. Therefore, E_q decreases. The customers' number becomes larger, and the number of products needed to be stocked increases, so E_{pro} increases.

From Table 2, as the parameter μ_ν increases, E_q, E_{loss} and E_{open} decrease, E_{inv} first increases and then decreases, E_{pro} and E_{vac} increase. Parameter μ_ν has a greater effect on E_q, E_{loss} and E_{vac}.

As shown in Table 3, as the parameter μ_b increase, E_q decreases, E_{inv}, E_{loss} and E_{open} all decrease and then increase, and E_{vac} increases. Parameter μ_b has a greater effect on E_q and E_{vac} of server.

As shown in Table 4, as the parameter λ increases, E_q, E_{loss} and E_{pro} increase, E_{inv}, E_{open} and E_{vac} decrease. That is because the parameter λ increases, the waiting customers' number becomes larger, the products required by the customers becomes larger, and the production system produces more products, increasing E_q and a decrease in

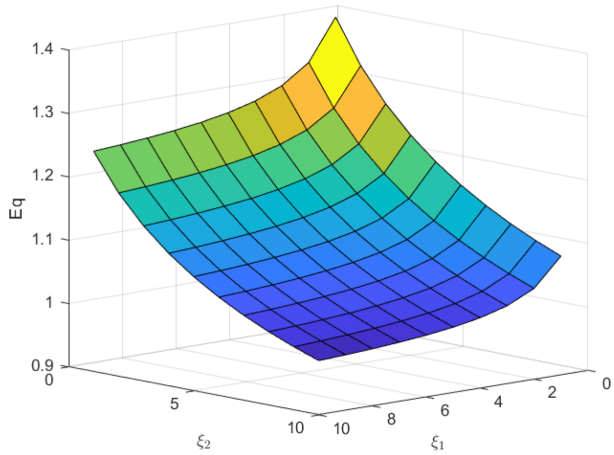

Fig. 5: The effect of ξ_1 and ξ_2 on mean number of customers E_q.

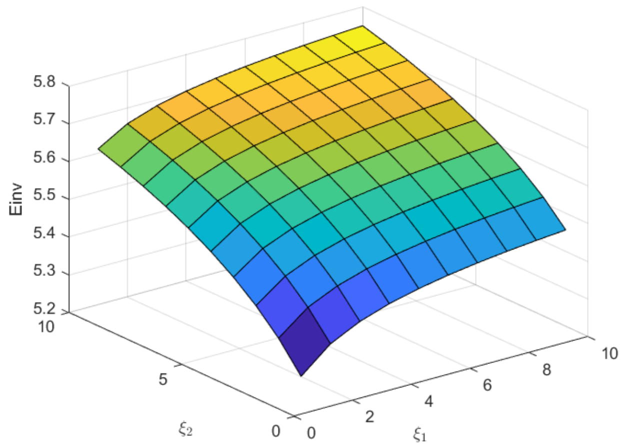

Fig. 6: The effect of ξ_1 and ξ_2 on mean inventory level E_{inv}.

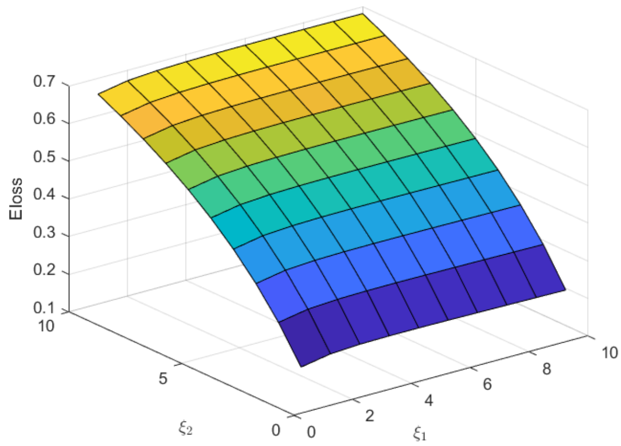

Fig. 7: The effect of ξ_1 and ξ_2 on ean customers loss rate E_{loss}.

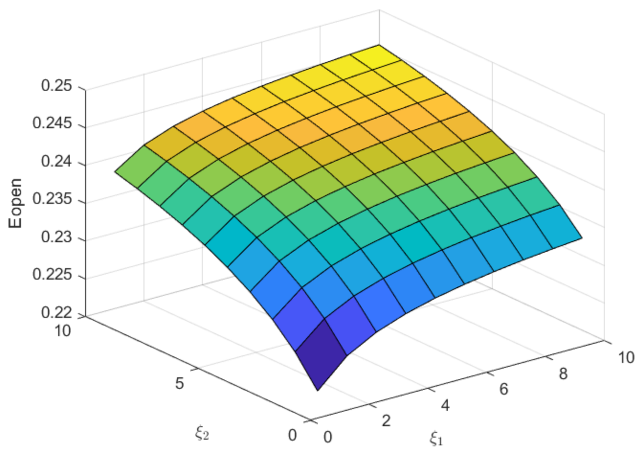

Fig. 8: The effect of ξ_1 and ξ_2 on ean production start-up rate E_{open}.

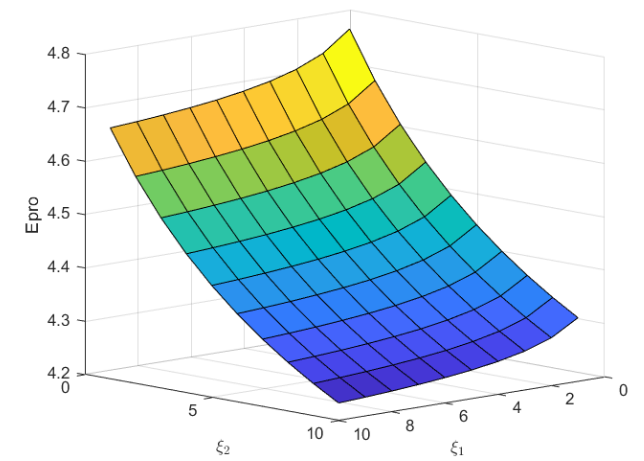

Fig. 9: The effect of ξ_1 and ξ_2 on mean productivity E_{pro}.

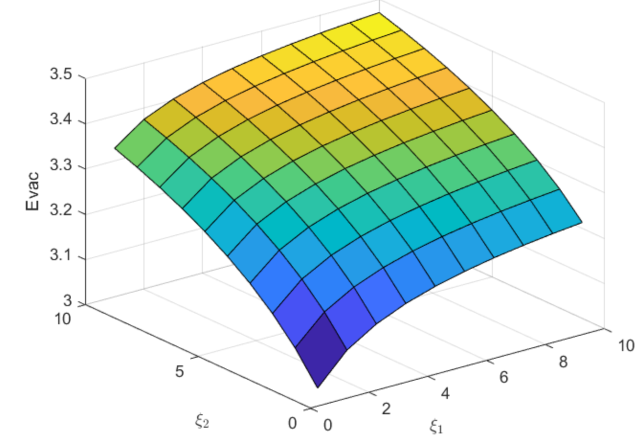

Fig. 10: The effect of ξ_1 and ξ_2 on mean working vacation rate E_{vac}.

Table 2: The effect of the parameter μ_ν on various measures.

μ_ν	E_q	E_{inv}	E_{loss}	E_{pro}	E_{open}	E_{vac}
0	1.8954	5.2620	0.2765	4.6818	0.2276	2.4669
0.5	1.8367	5.2624	0.2646	4.6943	0.2269	2.5251
1	1.7798	5.2625	0.2532	4.7063	0.2262	2.5827
1.5	1.7249	5.2625	0.2423	4.7177	0.2255	2.6396
2	1.6719	5.2623	0.2319	4.7286	0.2249	2.6956
2.5	1.6210	5.2618	0.2220	4.7389	0.2242	2.7504
3	1.5720	5.2612	0.2126	4.7487	0.2236	2.8041
3.5	1.5251	5.2605	0.2037	4.7579	0.2231	2.8564
4	1.4802	5.2596	0.1953	4.7666	0.2225	2.9072
4.5	1.4373	5.2586	0.1873	4.7748	0.2220	2.9566
5	1.3962	5.2574	0.1799	4.7825	0.2215	3.0045

Table 3: The effect of the parameter μ_b on various measures.

μ_b	E_q	E_{inv}	E_{loss}	E_{pro}	E_{open}	E_{vac}
6	4.7268	5.2834	0.1867	4.7944	0.2236	1.2325
6.5	3.4399	5.2665	0.1692	4.8017	0.2220	1.6081
7	2.7208	5.2584	0.1648	4.8017	0.2212	1.9199
7.5	2.2768	5.2552	0.1652	4.7995	0.2209	2.1797
8	1.9797	5.2542	0.1676	4.7963	0.2208	2.3982
8.5	1.7687	5.2543	0.1706	4.7928	0.2209	2.5841
9	1.6120	5.2550	0.1738	4.7892	0.2210	2.7440
9.5	1.4915	5.2561	0.1769	4.7857	0.2212	2.8828
10	1.3962	5.2574	0.1799	4.7825	0.2215	3.0045
10.5	1.3192	5.2590	0.1826	4.7795	0.2217	3.1119
11	1.2557	5.2606	0.1852	4.7767	0.2220	3.2075

E_{inv}. From Table 4, it can be seen that parameter λ has a relatively large impact.

V.ii *Optimal inventory and cost analysis*

It is assumed that the parameter $c_0 = 50$, $c_1 = 20$, $c_2 = 150$, $c_3 = 100$, $c_4 = 1000$ and E_{loss} are restricted to be less than 0.15, and the rest of the parameter settings are detailed in Table 5.

Combined Tables 6-12 we can obtain:

Table 4: The effect of the parameter λ on various measures.

λ	E_q	E_{inv}	E_{loss}	E_{pro}	E_{open}	E_{vac}
4	0.9232	5.9235	0.1086	3.8785	0.2497	3.5761
4.4	1.0879	5.6836	0.1322	4.2474	0.2420	3.3636
4.8	1.2836	5.4084	0.1619	4.6071	0.2294	3.1299
5.2	1.5207	5.0978	0.2006	4.9541	0.2125	2.8731
5.6	1.8126	4.7546	0.2528	5.2836	0.1921	2.5925
6	2.1756	4.3858	0.3238	5.5901	0.1694	2.2899
6.4	2.6286	4.0029	0.4200	5.8679	0.1455	1.9709
6.8	3.1914	3.6216	0.5484	6.1118	0.1219	1.6448
7.2	3.8823	3.2600	0.7158	6.3182	0.1000	1.3247
7.6	4.7145	2.9358	0.9285	6.4862	0.0809	1.0256
8	5.6927	2.6622	1.1919	6.6180	0.0653	0.7611

Table 5: Tables 6-12 show the corresponding parameters.

Table	Parameters
Table 2	$(\lambda, \theta, \eta, \mu_b, \xi_1, \xi_2, s, S)$=(5,5,7,10,1,1,3,10)
Table 3	$(\lambda, \theta, \eta, \mu_\nu, \xi_1, \xi_2, s, S)$ = (5,5,7,5,1,1,3,10)
Table 4	$(\theta, \eta, \mu_\nu, \mu_b, \xi_1, \xi_2, s, S)$ = (5,7,5,10,1,1,3,10)
Table 6	$(\lambda, \theta, \eta, \mu_\nu, \mu_b, \xi_2)$ = (5,5,7,5,10,1)
Table 7	$(\lambda, \theta, \eta, \mu_\nu, \mu_b, \xi_1)$ = (5,5,7,5,10,1)
Table 8	$(\theta, \eta, \mu_\nu, \mu_b, \xi_1, \xi_2)$ = (5,7,5,10,1,1)
Table 9	$(\lambda, \theta, \mu_\nu, \mu_b, \xi_1, \xi_2)$ = (5,5,5,10,1,1)
Table 10	$(\lambda, \eta, \mu_\nu, \mu_b, \xi_1, \xi_2)$ = (5,7,5,10,1,1)
Table 11	$(\lambda, \eta, \theta, \mu_b, \xi_1, \xi_2)$= (5,7,5,10,1,1)
Table 12	$(\lambda, \eta, \theta, \mu_\nu, \xi_1, \xi_2)$= (5,7,5,5,1,1)

Table 6: The effect of the ξ_2 on safety inventory, maximum inventory and minimum cost.

ξ_2	0.1	0.3	0.5	0.7	0.9	1.1
(s, S)	(1,14)	(1,14)	(1,14)	(1,16)	(2,19)	(5,22)
C (s, S)	825.7600	827.0018	828.2016	831.4052	850.2960	893.1953

Table 7: The effect of the ξ_1 on safety inventory, maximum inventory and minimum cost.

ξ_1	0.1	0.3	0.5	0.7	0.9	1.1
(s, S)	(1,14)	(1,16)	(3,17)	(3,19)	(3,20)	(4,17)
C(s, S)	835.5587	834.9533	854.5932	859.8781	863.8324	866.5782

Table 8: The effect of the λ on safety inventory, maximum inventory and minimum cost.

λ	4.5	4.7	4.9	5.1	5.3	5.5
(s, S)	(1,14)	(2,18)	(3,17)	(4,21)	(6,23)	(9,26)
C (s, S)	784.5623	819.9469	845.7170	886.1870	931.6207	992.8955

Table 9: The effect of the η on safety inventory, maximum inventory and minimum cost.

η	5.5	5.7	5.9	6.1	6.3	6.5
(s, S)	(15,29)	(12,25)	(9,24)	(8,20)	(6,22)	(5,20)
C (s, S)	913.0728	889.8227	875.01807	866.3208	866.2811	859.4551

Table 10: The effect of the θ on safety inventory, maximum inventory and minimum cost.

θ	4.5	5	5.5	6	6.5	7
(s, S)	(8,20)	(4,17)	(2,18)	(2,15)	(1,16)	(1,14)
C (s, S)	930.5628	866.6005	845.0227	837.3806	828.1802	824.8397

Table 11: The effect of the μ_ν on safety inventory, maximum inventory and minimum cost.

μ_ν	4.2	4.4	4.6	4.8	5	5.2
(s, S)	(6,20)	(5,19)	(4,21)	(4,18)	(4,17)	(3,19)
C (s, S)	902.4437	885.8823	880.0002	868.9471	866.6005	858.7131

Table 12: The effect of the μ_b on safety inventory, maximum inventory and minimum cost.

μ_b	7	8	9	10	11	12
(s, S)	(2,15)	(2,17)	(3,16)	(4,17)	(5,18)	(7,20)
C (s, S)	906.8008	871.0157	862.9165	866.6005	874.7581	900.7313

1) As the parameters ξ_2 and ξ_1 increase, the safety stock, the maximum inventory, and the minimum cost increase, and it can be seen from the degree of change in the inventory and the minimum cost that the parameter ξ_2 effects the system more than the effect of ξ_1 on the system.
2) When parameter λ increases, safety inventory level, maximum inventory level and minimum cost also increase. Because of when the number of arriving customers increases, the quantity of products required also rises, the inventory consumption becomes larger, and the mean cost of waiting customers, the customers' loss cost, the storing the products cost and the production of the product cost are all increased. Therefore, the system cost also increases.
3) When the parameter η increases, both the inventory level and minimum cost decrease. It means that with the production rate of the system increases, more products are produced per unit of time, a lower inventory level is sufficient to satisfy the demand of the system, and the storing products cost decreases, thus reducing the system cost.
4) As the parameter θ increases, both the inventory level and minimum cost of the system decrease. This indicates that with the mean working vacation decreases, the probability of customers leaving the system decreases, the storing products cost and the customers' loss cost decreases. So the system's required cost decreases as a result.
5) As μ_ν increases, the safety inventory level, maximum inventory level and minimum cost decrease. It indicates that with the working vacation service rate increases, E_q decreases, the mean waiting cost and the storing products cost decreases, so the system's required cost decreases as a result.
6) As μ_b increases, the safety inventory, and maximum inventory increase, and the minimum cost decreases and then increases, which shows that during the the regular busy period, μ_b is not the faster, the better. μ_b needs to choose the right value, which can make the system require the lowest cost.

VI. Conclusion

In this paper, an M/M/1 production queueing inventory system with impatient customers and server on multiple working vacations is investigated. And we mainly consider the effect on the system of the customers' impatient in two situations: When the inventory is zero and when the server is on working vacations. A four-dimensional Markov process was established. And the steady-state conditions, the steady-state probabilities and performance measures of the system were obtained by applying Neuts-Rao truncation method. Create a cost function and constrain the mean customers loss rate. Numerical experiments are conducted to analyze the effect of different parameter variations on performance measures, optimal inventory policy and minimum cost. The experimental results show that the impatience of customers during the working vacation period has a greater impact on the system; the service rate during the regular busy period is not as fast as possible, and the correct value chooses to minimize the cost spent on the system. Therefore, business decision-makers need to consider the efficiency of server during the regular busy periods in addition to customer impatience when making decisions. In the course of conducting numerical experiments, we found that optimization with genetic algorithms can take a long time. Consequently, it remains to investigate how to choose a more appropriate algorithm for optimisation. In addition, extending the distribution of service time and production time to phase-type distribution is also a direction we want to expand in the future.

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