Analytical Study of a Semilinear Problem With Dirichlet Boundary Conditions

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***Abstract*—This paper presents a qualitative analysis of the critical set of semilinear equations with Dirichlet boundary conditions in multiply-connected two-dimensional domains with corners, employing the method of moving planes to examine nodal lines associated with the solution. Additionally, comprehensive numerical investigations are conducted to validate the theoretical findings.**

***Index Terms*—Critical points, moving plane method, elliptic equations, semilinear problems, Dirichlet conditions, Poisson's equation.**

I. Introduction

Let us consider the boundary value problem given by the equation:

$$\begin{array}{rclll} \Delta u & = & f(u) & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega. \end{array} \tag{1}$$

The solution to this problem is generally not well known as it significantly depends on the domain Ω. Only in very few cases, an explicit solution for u is known. Therefore, a qualitative analysis of the solution is sought through the study of the critical set, defined as:

$$\mathbb{K} = \{x \in \overline{\Omega} : \nabla u(x) = 0\}.$$

The study of $\mathbb{K}$ has been extensively documented in cases where it is assumed that Ω is a convex planar region with a smooth boundary. Influential research in this area includes the work of Makar-Limanov [15] and Cabré and Chanillo [5]. Further references that explore the problem under the convexity condition include [8] and [14].

However, when the convexity assumption is removed from the domain Ω, studying the solution to (1) becomes more complex. Recent research in this context includes the work of Finn [8], Arango and Gómez [2], [3], as well as contributions from Grossi and Molle [12], Gladiali and Grossi [10] and Grecco [11]. Furthermore, in [4], the critical set of (1) is investigated in the three-dimensional case.

Analyzing the qualitative aspects of problem (1) becomes more intricate when the boundary contains vertices, i.e., when Ω is a domain with a non-smooth boundary. This research specifically aims to contribute to the study of cases in which Ω is an annular region whose outer boundary contains vertices. In particular, this study focuses on scenarios where Ω represents a "nut-like" domain, distinguished by its outer boundary taking the shape of a regular convex polygon, with its inner edge being a circle centered within the polygon.

Furthermore, we will analyze the critical set for solutions to the boundary value problem (1) when f is a constant equal to w, a positive constant. To provide valuable insights into this examination, we will present multiple simulations illustrating the anticipated characteristics of the critical set within non-concentric annular domains.

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II. General Assumptions

The objective of this study is to expand upon the results presented by Arango and Gómez to encompass domains featuring cornered geometries. To achieve this, we will employ a methodology reminiscent of the one described in [1], while enhancing the underlying assumptions and providing numerical calculations to bolster our argument.

In the absence of specific declarations to the contrary, it shall be understood that the function $f : \mathbb{R} \to \mathbb{R}$ is a real-analytic, non-decreasing function with $f(0) > 0$. Additionally, Ω denotes an open region with a ring-like shape and its boundary denoted as $\partial\Omega$ comprises two Jordan curves, both possessing a convex interior. We propose that the external border is a Lipschitz continuous curve composed of analytic pieces and a limited number of vertices. In contrast, the internal boundary is described as a smooth curve.

A point p on the boundary will be termed a corner or vertex point if there exist functions g_k of class C^1 defined on the interval $I = [0, 1]$ and with values in $\partial\Omega$, $k = 1, 2$, such that $g_1(I) \cap g_2(I) = \{p\}$,

$$g_1(1) = g_2(0) = p \quad \text{and} \quad \lim_{t\to 1^-} g_1'(t) \neq \lim_{t\to 0^+} g_2'(t).$$

We will further elaborate on the results presented in [3, Theorem 3.1] by proving that the set $\mathbb{K}$ consists of a finite number of discrete points and at most, a single Jordan curve. Additionally, we will establish that domains falling under the category of nut-like do not encompass critical curves. Consequently, $\mathbb{K}$ will be finite in such domains.

III. Theoretical Framework-moving Planes Technique

To establish a theoretical framework that supports the numerical evidence presented later, which is the main contribution of this manuscript, we will start by ensuring the smoothness, existence, and uniqueness of the solution to the equation represented by (1). In this regard, we will follow the standard propositions and lemmas previously described in Section II of [1], [2], [4].

Proposition 3.1: A solution u to problem (1) exists and is unique. This solution u is characterized by the following features: negativity, real analyticity within the domain Ω and continuity over the closure of Ω, denoted as $\overline{\Omega}$.

The verification of Proposition 3.1 can be found dispersed across the specialized texts dedicated to elliptic equations. Notably, pertinent results regarding existence and uniqueness can be located in [9, Theorems 8.15 and 12.5]. The analytic nature of the solution can be cross-referenced in [17]. For a more comprehensive exploration of the aspects related to existence and uniqueness, we advise interested readers to consult [19, Theorem 1.16]. Let's initiate with foundational findings concerning $\mathbb{K}$.

Lemma 3.1: Let $\Omega \subset \mathbb{R}^2$ be a bounded planar domain and let $f \in C^1(\mathbb{R})$ be a non-decreasing function with $0 < f(0)$. A solution u of equation (1) satisfies $\Delta u > 0$. Additionally, the Hessian matrix H_u does not vanish anywhere within Ω. The demonstration closely follows the approach delineated in Lemma 2 and Corollary 1 of [2]. Furthermore, we draw upon [3], which presents a slightly broader rendition of Lemma 3.1. The insights from Lemma 3.1 define the critical points as "semi-Morse", locations where the Hessian matrix H_u remains non-singular. For a more comprehensive understanding, additional insights can be found in [2] and [3].

Proposition 3.2: [1, Proposition 2] A Jordan critical curve within the domain Ω signifies the absence of simple connectedness in Ω. Moreover, the existence of multiple Jordan critical curves within Ω is ruled out. In the scenario where a Jordan critical curve does exist, it will form a solitary, non-intersecting loop encircling the internal edge of $\partial\Omega$.

When Ω exhibits a smooth boundary, a direct application of Lemma 3.1 combined with Hopf's boundary point lemma (as stated in [18, Theorem 2.8.3]) leads to the conclusion that u doesn't exhibit any critical points at the boundary. Nevertheless, when there are corner points present at the boundary, the situation undergoes a significant alteration. See examples 2, 3 and 4.

In the subsequent section, our aim is to prove the non-existence of critical points that accumulate at the boundary. When the boundary displays a smooth contour, this assertion directly stems from the implications of Hopf's lemma. However, in situations involving corner points, we are obliged to employ the moving plane method. For a more detailed exposition of this technique, we direct interested readers to the seminal work by Serrin [20].

Lemma 3.2: A critical point on the boundary of the solution u for equation (1) is necessarily a corner. Moreover, boundary points are not limit points of $\mathbb{K}$. Therefore, a compact set S contained within Ω encompasses all interior critical points of the solution u for (1).

Proof: For $p \in \partial\Omega$ excluding corners, Lemma 3.1 facilitates the direct application of Hopf's lemma, leading to the ensuing assertion:

$$\frac{\partial u}{\partial \nu}(p) \neq 0,$$

where $\partial u/\partial \nu$ denotes the outer normal derivative. Consequently, this implies $\nabla u(p) \neq 0$ and as a result, critical points cannot accumulate at point p.

Let's proceed by contradiction. Assume $p \in \partial\Omega$ is a corner, and consider $(p_n)_n$ as a sequence in $\mathbb{K}$ converging to p. As Ω constitutes an annular region with a convex interior along its outer boundary, for every x sufficiently close to p, there exists a line L such that $\Omega(L) \subset \Omega$ and $\Omega^*(L) \subset \Omega$. Here, $\Omega(L)$ stands for the region inside Ω, bordered by L and $\partial\Omega$, encompassing point p, whereas $\Omega^*(L)$ refers to the mirrored counterpart (reflection) of $\Omega(L)$ in relation to L. Consequently, if n is sufficiently large, we can select L such that $p_n \in L$. Inside $\Omega^*(L)$, we introduce the function:

$$\tau(x) = u(x) - u(x^*),$$

here, x^* indicates the point x mirrored in relation to the line L. Upon a simple computation, the following is obtained: $\Delta\tau(x) = f'(g(x))\tau(x)$, In this context, $g(x) \in C^\infty(\Omega)$ that relies on both $u(x)$ and $u(x^*)$. A straightforward calculation confirms that τ satisfies:

$$\begin{aligned} \Delta\,\tau - f'(g(x))\tau &= 0 \quad \text{in} \quad \Omega^*(L), \\ \tau &< 0 \quad \text{on} \quad \partial\Omega^*(L) \setminus L, \\ \tau &= 0 \quad \text{on} \quad L \cap \partial\Omega^*(L). \end{aligned}$$

As f' is non-negative, applying the maximum principle we can conclude that τ attains its maximum value on the boundary $\partial\Omega^*(L)$. In other words, τ reaches its maximum along L. Furthermore, as a consequence of Hopf's boundary point lemma:

$$\frac{\partial \tau}{\partial \eta} > 0 \quad \text{on} \quad L \cap \partial\Omega^*(L).$$

Since $p_n \in L$, this leads to a contradiction. ■

IV. Primary Findings

With the necessary groundwork established, we can now outline the primary results of this research.

Theorem 4.1: If u is the solution to the boundary problem (1), then its critical set is formed by a finite collection of individual points and at most, a single Jordan curve.

Proof: According to Lemma 3.2, there exists a compact set S within Ω that encompasses all internal critical points of u. Now, consider δ_0 positive such that $f(-\delta_0) > 0$ and $\sup_S u < -\delta_0$. In the context of $0 < \delta < \delta_0$, we define γ as the level curve $-\delta$ of u, that is $u(x) = -\delta$. It's crucial to note that γ is composed of precisely two smooth curves, outlining the edge of an open annular domain denoted as Ω_δ. Next, let's define:

$$\tau(x) = u(x) + \delta, \quad x \in \Omega_\delta.$$

According to Proposition 3.2, there can be at most a single critical curve.

Now, if we consider $h(z) = f(z - \delta)$, we note that $h(0) = f(-\delta) > 0$. Additionally, since $H_u = H_\tau$ in Ω_δ, we can apply Lemma 3.1 to conclude that H_τ does not vanish. Consequently, the function τ satisfies:

$$\begin{aligned} \Delta\,\tau &= h(\tau) \quad \text{in} \quad \Omega_\delta, \\ \tau &= 0 \quad \text{on} \quad \partial\Omega_\delta. \end{aligned}$$

Referring to Theorem [3, Theorem 3.1], we can deduce that τ has a finite count of isolated critical points and at most, a single Jordan critical curve. Additionally, it's noteworthy that both u and τ have identical critical points within Ω_δ. Outside Ω_δ, any critical points of u, if present, are restricted to the corners. ■

In case the external edge of a nut-like domain Ω assumes the shape of a regular convex n-sided figure (regular convex polygon), the rays $e^{i\,k\pi/n}$ establish axes of symmetry denoted as L_k within the domain Ω, where k ranges from 1 to n. Moreover, these symmetries are reflected in the solution u for Problem (1). This implies that, for any x within Ω and for every k, $u(x)$ is equal to $u(x^{(k)})$, with $x^{(k)}$ signifying the reflection of x over the axis L_k.

Theorem 4.2: If Ω is a nut-like domain, then $\mathbb{K}$ does not contain any Jordan curve.

Proof: We will proceed with the proof under the assumption that the outer boundary is a square, which allows us to describe Ω as follows:

$$\Big\{x = (x_1, x_2) : \ \|x\|^2 > r^2, \, -\ell < x_1 < \ell, \, -\ell < x_2 < \ell\Big\},$$

in this context, r and ℓ are constants such that $0 < r < \ell$. Let R represent the $\pi/4$ rotation and define $\Omega^* = R(\Omega)$. We also introduce Q as the intersection of Ω and Ω^*, as depicted in Figure 1. Now, we can express $u^* = u \circ R^{-1}$ and it's worth noting that u^* satisfies (1) within Ω^*. Examining Q, we can divide it into 8 congruent regions, as illustrated in Figure 1. One of these regions, which contains the x_2 axis and lies above the x_1 axis, is denoted as E. Additionally, it's important to observe that ∂E can be decomposed into the following components: T_1, T_2, α and β. In this scenario, T_1 and T_2 represent sections along the directions of $e^{i\,3\pi/8}$ and $e^{i\,5\pi/8}$, respectively. The parameter α signifies an arc $r\,e^{i\,\theta}$, where $\theta \in (3\pi/8, 5\pi/8)$. The parameter β represents a section along the line $x_2 = \ell$ line, confined between the directions defined by $e^{i\,3\pi/8}$ and $e^{i\,5\pi/8}$.

We exclude the edges of β in a way that guarantees $R^{-1}(\beta) \subset \Omega$. Given that $u^*(\beta) = u \circ R^{-1}(\beta)$, Proposition 3.1 indicates the negativity of $u^*(x)$ for $x \in \beta$. Additionally, each symmetry axis within the domain Ω also acts as a symmetry axis for u. Next, let's define

$$\tau(x) = u(x) - u^*(x),$$

and notice that:

- $\tau(x) = 0$ on $T_1 \cup T_2$.
- $\tau(x) = 0$ on α.
- $\tau(x) > 0$ on β.

Hence, the conclusion is:

$$\tau \geq 0 \quad \text{ on } \partial E.$$

Subsequently, a simple computation shows that τ satisfies:

$$\Delta\tau = f'\left(g\left(x\right)\right)\,\tau \quad \text{ in } E,$$

here $g(x)$ is a function that depends on $\tau(x)$ (i.e., on both $u(x)$ and $u^*(x)$). It's important to note that $f'\left(g\left(x\right)\right)$ remains non-negative throughout the region E. Utilizing a conventional approach based on the maximum principle (illustrated for instance in [18, Theorem 2.1.1]), we can infer that:

$$\tau > 0 \quad \text{ in } E.$$

To conclude the proof, let's consider a contradiction where a critical curve Γ exists. According to Proposition 3.2, Γ encircles the inner boundary of Ω, implying the presence of a point $p \in \Omega \cap \Gamma \cap T_1$. Notably, at this point $\nabla\tau(p) = 0$. However, employing Hopf's lemma results in an inconsistency.

■

To produce graphical representations and perform the majority of numerical analysis for the solution of the semilinear equation (1), we utilized the finite element method through the PDE Toolbox routine within the MATLAB software

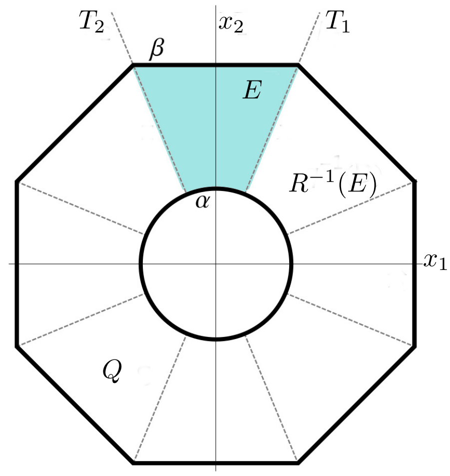

Fig. 1. Q and E regions in the proof of Theorem 4.2.

[13]. It is noteworthy that this software is provided by the university for the benefit of the academic community.

Example 1: In the Equation (1), consider the function $f(z) = 2 + (z-1)^3$, where $z \in (-1, 1)$. In this case, we define Ω as an annular region with an outer boundary represented by a regular pentagon whose vertices lie on the unit circle and an inner boundary defined by a circle with a radius of 0.3. Figure 2 illustrates the graph of the solution u to (1) within this domain, displaying 5 minimum and 5 saddle points, totaling 10 critical points.

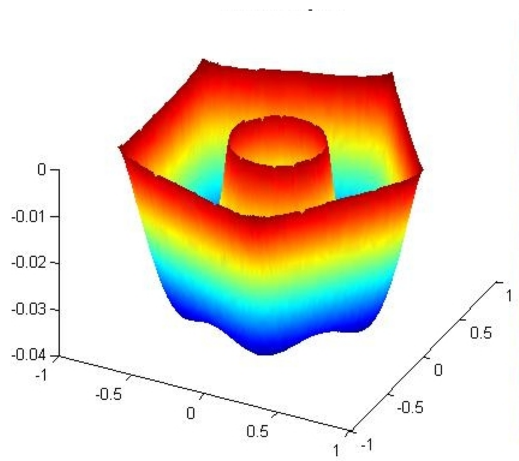

Fig. 2. Graph of the solution u in Example 1.

Example 2: Let u represent the solution to equation (1) within the domain

$$\Omega = \Big\{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{3}|x_1| < \tfrac{5}{\sqrt{3}} - x_2, \, x_2 + \tfrac{5}{2\sqrt{3}} > 0\Big\} \\ \setminus \big\{(x, y) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1/4\big\},$$

here $f(z) = \ln(z+2)$, $z \in (-1, 1)$. Using the MatLab program, we plotted the solution u. As evident in Figure 3, there are 3 minimum points and 3 saddle points, totaling 6 critical points.

Examples 1 and 2 support the theoretical result of Theorem 4.2 by demonstrating that the solution to the boundary problem (1) does not exhibit a critical curve when Ω is a nut-type domain.

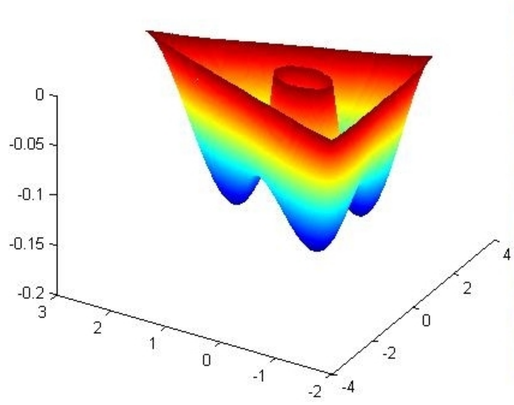

Fig. 3. Graph of the solution u of Example 2.

V. Nodal Sets and Numerical Calculations

In this section we will focus on a specific case of Equation (1), where $f \equiv w$ is a positive constant. In other words, we consider the following problem:

$$\begin{aligned} \Delta u &= w \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

To characterize the set $\mathbb{K}$, we will utilize an analytical technique based on the study of nodal sets of directional derivatives. For $\theta \in S^1$, let us denote

$$u_\theta(x) = \nabla u(x) \cdot \theta, \quad N_\theta = \{x \in \overline{\Omega} : u_\theta(x) = 0\}.$$

Observe that for each $\theta \in S^1$, we have $\mathbb{K} \subset N_\theta$. Moreover, if θ and ϕ are two non-collinear directions in S^1, then $\mathbb{K} = N_\theta \cap N_\phi$. In fact, if $x \in N_\theta \cap N_\phi$, then $\nabla u(x)$ is perpendicular to both θ and ϕ, hence $\nabla u(x) = 0$ and thus $x \in \mathbb{K}$. Keeping in mind that u satisfies (2), it is clear that the mapping

$$x \to RH_u(x)\theta, \quad R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

belongs to $C(\Omega)$, where R is the $\pi/2$-rotation matrix. If $p \in N_\theta$ is a regular point of u_θ, then N_θ can be locally parameterized by the solution of the ODE:

$$x'(t) = RH_u(x(t))\theta, \quad x(0) = p.$$

Indeed, $\nabla u_\theta = H_u\theta$ is orthogonal to the level curve N_θ and by taking its rotation, we obtain a tangent vector to N_θ. The aforementioned ordinary differential equation satisfies the conditions of the Picard's existence and uniqueness theorem for initial value problems of ordinary differential equations, see [16, Theorem 3.1].

Example 3: Let us consider the rectangular region of sides a and b

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < \frac{a}{2}, |x_2| < \frac{b}{2}\right\},$$

substituting $w \equiv 1$ into (2). An explicit formula in terms of elementary functions for the solution of problem (2) in this region remains elusive. However, it's established that the solution can be represented by a Fourier series:

$$u(x) = \sum_{m,n=0}^{\infty} u_{mn} \cos\left(\frac{(2n+1)\pi x_1}{a}\right) \cos\left(\frac{(2m+1)\pi x_2}{b}\right),$$

where $x = (x_1, x_2)$ and

$$u_{mn} = \frac{16a^2b^2(-1)^{m+n+1}}{\pi^4(2m+1)(2n+1)\big((2m+1)^2a^2 + (2n+1)^2b^2\big)}.$$

Since an explicit solution is unavailable, it's necessary to obtain information about the critical points through the nodal curves. It can be observed that in this instance, the nodal curves intersect at the corners. See Figure 4.

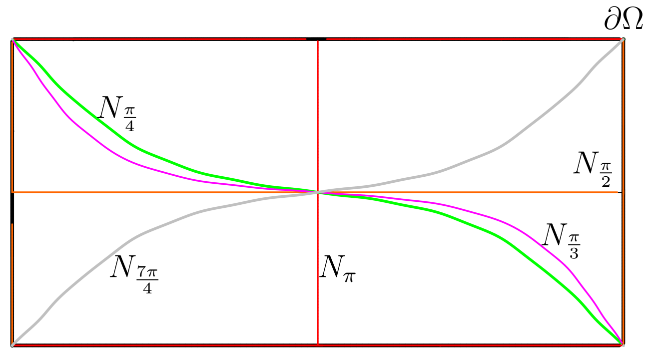

Fig. 4. Nodal sets and corresponding curves of the functions u_θ associated with the solution of Example 3 for various values of θ, with $a = 4$ and $b = 2$.

The subsequent example has been previously employed in [1, Example 1]; nevertheless, its inclusion at this juncture is pivotal for two reasons. First, we possess an explicit expression for the solution to (1). Second, it is meant to enable a comparison with another example in a nut-like domain, as demonstrated in Example 2.

Example 4: Let Ω represent the interior region of an equilateral triangle centered at the origin with a side length of $2\sqrt{3}$:

$$\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2 + 1, |x_1| < \frac{2 - x_2}{\sqrt{3}}\right\}.$$

Substituting $w \equiv 1$ into (2), the solution u is expressed as:

$$u(x_1, x_2) = \frac{(x_2+1)\Big(3x_1^2 - (x_2-2)^2\Big)}{12}.$$

Upon direct computation, it becomes evident that the gradient of u, denoted as ∇u, becomes zero at the points $(0, 2)$, $(\sqrt{3}, -1)$ and $(-\sqrt{3}, -1)$. These points happen to coincide with the corners.

Remark 5.1: In figure 4, the color scheme for the nodal curves of the functions u_θ associated with the solution of Example 3 is as follows: the black line denotes the boundary $\partial\Omega$, the red line represents the nodal set N_π, the orange line corresponds to $N_{\frac{\pi}{2}}$, the pink line indicates $N_{\frac{\pi}{3}}$, the gray line represents $N_{\frac{7\pi}{4}}$, and the green line denotes $N_{\frac{\pi}{4}}$.

In figure 5, the color scheme for the nodal curves of the functions u_θ associated with Example 4 is as follows: the black line denotes the boundary $\partial\Omega$, the red line represents the nodal set N_π, the orange line corresponds to $N_{\frac{\pi}{2}}$, the gray line indicates $N_{\frac{7\pi}{6}}$, the pink line represents $N_{\frac{\pi}{3}}$, the brown line denotes $N_{\frac{15\pi}{9}}$, and the blue line represents the line defined by $x_2 = \frac{-1}{\sqrt{3}}x_1$. It is important to note that the nodal lines intersect at the vertices and at the centroid of the triangle, where the function reaches its minimum.

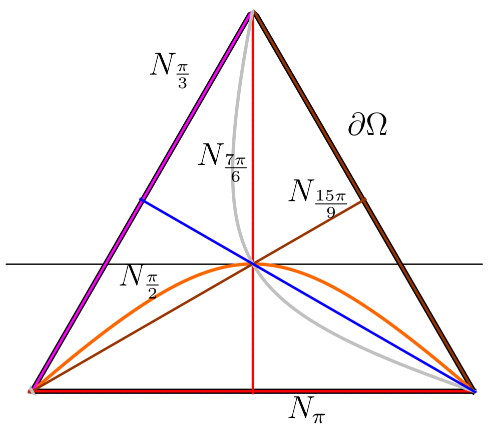

Fig. 5. Nodal sets and curves of the functions u_θ associated with Example 4 for various values of θ, with intersections at the vertices and centroid of the triangle.

Let's delve into the local structure of N_θ near a critical point of u_θ. If u satisfies (2), note that u_θ is harmonic, i.e., $\Delta u_\theta = 0$. A harmonic function can be approximated by harmonic polynomials. Particularly, from [6, Lemma 2.1], it follows that the nodal lines of a harmonic function can be approximated by those of a harmonic polynomial of degree m, where its nodal set consists of m straight lines intersecting at the origin and forming an equiangular system of $2m$ rays.

Lemma 5.1: If u satisfies (2) with $w > 0$, then u is a semi-Morse function. Therefore, given a point $q \in \mathbb{K}$, there exists at most one direction θ such that q is a critical point for u_θ .

Proof: $\Delta u > 0$, then u is a semi-Morse function and $\nabla u_\theta = H_u(q)\theta$. ■

Lemma 5.2: Suppose that u satisfies (2) with $w > 0$. If for some θ, N_θ contains a Jordan curve $\Gamma \subset \Omega$, then the subregion of Ω inside Γ is not simply connected.

Proof: Let Ω_Γ denote the subregion of Ω contained within Γ. In the case where Ω_Γ is simply connected, we would expect Γ to coincide with the boundary $\partial\Omega_\Gamma$. However, since u_θ is a harmonic function in Ω_Γ and satisfies $u_\theta = 0$ on Γ, it would imply that u_θ identically vanishes throughout Ω_Γ. Nevertheless, due to the analyticity of u, this would mean that u_θ must vanish over the entire domain Ω, thereby contradicting Hopf's boundary point lemma. ■

As per Theorem 4.1, it's established that $\mathbb{K}$ consists of a limited set of individual points and a finite count of Jordan curves. Additionally, the presence of an analytic Jordan curve within the critical set indicates that the domain Ω cannot be simple connected.

A. Numerical Examples

This section presents a collection of numerical examples that illustrate the investigation of the critical set of equations (2), showcasing significant original and unpublished contributions.

Example 5: Considering that Ω is the circle of radius R, we can easily characterize the set $\mathbb{K}$ by explicitly writing the function u as follows:

$$u(x_1, x_2) = \frac{w}{4}(x_1^2 + x_2^2 - R^2).$$

In this case, there is a unique critical point, and we have $\mathbb{K} = (0, 0)$.

Example 6: For $0 < r_1 < 1$, let us consider the case of a concentric ring, $\Omega = \{x \in \mathbb{R}^2 : r_1 < \|x\| < 1\}$. We can switch to polar coordinates using the transformation:

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= w \quad \text{in} \quad \Omega, \\ u(r_1, \theta) &= 0, \quad \text{on} \quad \partial\Omega, \\ u(1, \theta) &= 0, \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3}$$

and we look for a radially symmetric solution, $u(r, \theta) = u(r)$. For this situation, we need to solve the following ordinary differential equation (ODE):

$$\begin{aligned} \frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} &= w \quad \text{in} \quad \Omega, \\ u(r_1) = u(1) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

which leads to the solution:

$$u(r, \theta) = u(r) = \frac{w}{4 \ln r_1}\Big((r^2 - 1)\ln r_1 + (1 - r_1^2)\ln r\Big).$$

In this case the critical set is a curve, that is $\mathbb{K} = \left\{x \in \mathbb{R}^2 : \|x\|^2 = \dfrac{r_1^2 - 1}{2\ln(r_1)}\right\}$.

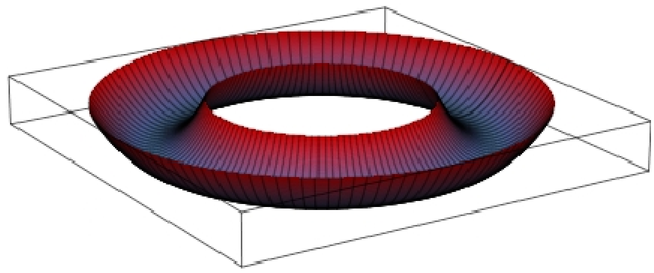

Fig. 6. Graph of the solution of equation (3) with $r_1 = 1/2$ and $w = 1$ in Example 6, indicating the minimum value on the critical curve.

Remark 5.2: Examples 5 and 6 highlight the significance of a seemingly small variation in the domain Ω, which results in a remarkable impact on the set $\mathbb{K}$. Instead of being a single point, the set undergoes a substantial transformation and becomes a Jordan curve.

Moreover, consider $0 < r_1 < 1$, u_{r_1} as the solution from Example 6, and u as the solution from Example 5 with $R = 1$. It can be observed that:

$$u_{r_1} \to u \quad \text{pointwise}, \quad \text{as} \quad r_1 \to 0.$$

Similarly, this convergence behavior is observed for the corresponding set $\mathbb{K}$.

Example 7: Let's consider an ϵ-perturbation in the harmonic part of the solution presented in Example 6.

$$u(x_1, x_2) = \frac{w}{8}\Big(2(r^2 - 1) + \frac{1 - r_1^2}{\ln r_1}\ln(r^2 + \epsilon^2 - 2x_1\epsilon)\Big),$$

where $r^2 = x_1^2 + x_2^2$, let's examine the level curve $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : u(x_1, x_2) = 0\}$. We define Ω as the connected region bounded by Γ, indicating that $\Gamma = \partial\Omega$. Notably, it is important to observe that Ω is dependent on the parameters r_1 and ϵ.

The three nodal lines, denoted as $N_{\frac{\pi}{4}}$, N_π and $N_{\frac{\pi}{2}}$, intersect only at two interior points within Ω. As stated earlier, the condition $\mathbb{K} = N_\theta \cap N_\phi$ remains valid when θ and ϕ are non-collinear, indicating that the critical set consists solely of these points. Refer to Figure 7 for a representation.

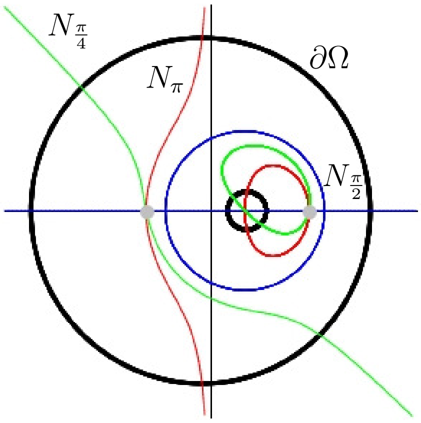

Fig. 7. Nodal sets and curves of the functions u_θ associated with the solution in Example 7 for various values of θ. Parameters: $w = 1$, $\epsilon = \frac{1}{5}$, and $r_1 = \frac{1}{10}$.

Remark 5.3: Although an explicit expression for the solution of (2) is unknown when $\Omega = B(0;1) \setminus \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - \epsilon)^2 + x_2^2 = r_1^2\}$, Example 7 provides insight into the behavior of the critical set for such a solution.

Example 8: If

$$u(x_1, x_2) = w\frac{x_2^2}{2} + 3x_1^4 - x_1^3 x_2 - 18x_1^2 x_2^2 + x_1 x_2^3 + 3x_2^4 - \frac{1}{125},$$

let's define Ω as the connected interior region of $\partial\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : u(x_1, x_2) = 0\}$. Through a simple calculation, it can be observed that u satisfies (2). It is

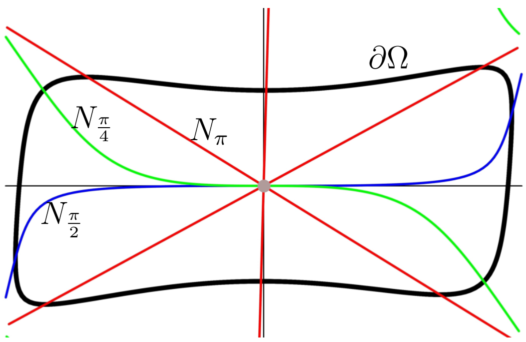

Fig. 8. Nodal sets and curves of the functions u_θ associated with the solution in Example 8 for various values of θ. Here, $w = 2$.

noteworthy that by calculating the nodal lines (Figure 8), we can conclude the following:

$$\mathbb{K} = \{(x_1, x_2) \in \Omega : \nabla u(x_1, x_2) = 0\} = \{(0, 0)\},$$

while the Hessian matrix $H_u(x_1, x_2)$ is given by:

$$\begin{pmatrix} 36x_1^2 - 6x_1x_2 - 36x_2^2 & 3x_2^2 - 72x_1x_2 - 3x_1^2 \\ 3x_2^2 - 72x_1x_2 - 3x_1^2 & w - 36x_1^2 + 6x_1x_2 + 36x_2^2 \end{pmatrix},$$

indicating that $H_u(0, 0)$ is a singular matrix.

Remark 5.4: It is important to observe that the function u in Example 8 takes the form:

$$u(x_1, x_2) = w\frac{x_2^2}{2} + Q(x_1, x_2),$$

where Q is a homogeneous harmonic polynomial of degree 4. The polynomial Q is obtained by appropriately combining the real and imaginary parts of $(x_1 + ix_2)^4$. Note that in Example 6, each critical point is degenerate; however, in that case, it corresponds to a curve rather than an isolated point.

Example 9: Now, we will consider two perturbations ϵ_1, ϵ_2 in the harmonic part of the solution in Example 6.

$$\begin{aligned} u(x) &= \frac{w}{8}\Big[2(r^2 - 1) + \frac{1 - r_1^2}{\ln r_1}\Big(\ln(r^2 + \epsilon_1^2 - 2x_1\epsilon_1) + \\ &\qquad \ln(r^2 + \epsilon_2^2 - 2x_1\epsilon_2)\Big)\Big], \end{aligned}$$

where $x = (x_1, x_2)$ and $r^2 = x_1^2 + x_2^2$. As before, let us consider the level curve $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : u(x_1, x_2) = 0\}$. We define Ω as the connected region with Γ as its boundary, specifically $\Gamma = \partial\Omega$. Once again, it is worth noting that Ω depends on the parameters r_1, ϵ_1 and ϵ_2. By calculating the nodal lines, we observe that $\mathbb{K}$ consists of 5 points. See Figure 9 for reference.

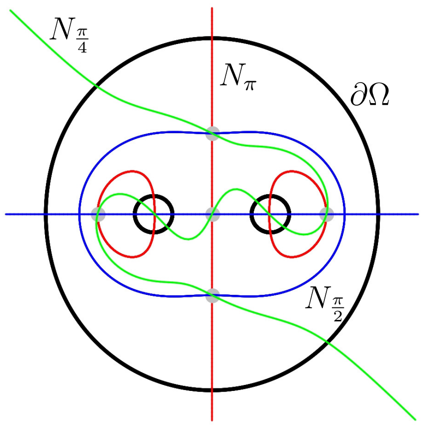

Fig. 9. Nodal sets and curves of the functions u_θ associated with the solution in Example 9 for various values of θ. Parameters: $w = 1$, $\epsilon_1 = \epsilon_2 = \frac{1}{3}$, and $r_1 = \frac{1}{20}$.

Remark 5.5: In figures 7, 8, 9, and 10, the following color scheme is used to represent the curves and points: the black line denotes the boundary $\partial\Omega$, the red line corresponds to the nodal set N_π, the blue line represents the nodal set $N_{\frac{\pi}{2}}$, and the green line indicates $N_{\frac{\pi}{4}}$. Gray points mark the critical points.

Example 10: Finally, let us consider four perturbations in the harmonic part of the solution in Example 6.

$$\begin{aligned} u(x) = \frac{w}{8}\Big[2(r^2 - 1) + \\ &\frac{1 - r_1^2}{\ln r_1}\Big(\ln(r^2 + \epsilon_1^2 + 2x_2\epsilon_1) \\ &\quad + \ln(r^2 + \epsilon_1^2 - 2x_2\epsilon_1) \\ &\quad + \ln(r^2 + \epsilon_2^2 - 2x_1\epsilon_2) \\ &\quad + \ln(r^2 + \epsilon_2^2 + 2x_1\epsilon_2)\Big)\Big], \end{aligned}$$

where $x = (x_1, x_2)$ and $r^2 = x_1^2 + x_2^2$, $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : u(x_1, x_2) = 0\}$. We define Ω as the connected region with Γ as its boundary, such that $\Gamma = \partial\Omega$. By calculating the nodal lines, we observe that $\mathbb{K}$ is comprised of 13 points, see Figure 10.

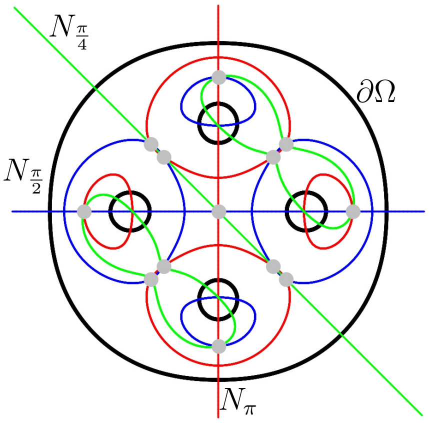

Fig. 10. Nodal sets and curves of the functions u_θ associated with the solution in Example 10 for various values of θ. Parameters: $w = 1$, $\epsilon_1 = \epsilon_2 = \frac{1}{2}$, and $r_1 = \frac{1}{50}$.

VI. Conclusion

As shown in Theorem 4.2, the set $\mathbb{K}$ is finite in the case of Ω being a nut-like domain. While it's relatively straightforward to confirm that any symmetry axis in an annular domain must possess at least two critical points, the demonstration that a nut-like domain bounded by an n-sided figure has precisely $2n$ critical points remains an unresolved inquiry. We present the following conjecture: *The solution to* (1) *has precisely* $2n$ *critical points within nut-like domains. Furthermore, we posit that, in the two-dimensional scenario, only concentric annuli display a critical curve.*

Notably, the only documented instances of problem (1) having a critical curve arise exclusively when Ω is structured as a concentric annulus. Based on numerical findings and various specific instances (explored in [2]), we conjecture that, given the assumptions about Ω and f in this study, *the concentric ring stands as the only domain where the solution to* (1) *exhibits a critical curve*. In [4], the equation (1) is studied in a three-dimensional domain, specifically a solid of revolution. In [4, Theorem IV.1], it is demonstrated that the solution has exactly one critical curve in this three-dimensional setting.

The document contributes to the understanding of the critical set in the context of the equation (1) and provides a clear and detailed description of its structure. These results are of interest in both pure and applied mathematics, as elliptic equations have numerous applications in various areas such as differential geometry, elasticity theory and fluid mechanics.

The reader is alerted to the fact that in this study, the problem (2) is not directly solved in a disk with two or four holes, but rather a perturbation is introduced to the solution of problem (2) in the concentric ring, thereby creating a domain with a geometry that is "similar" to a circle with two or four holes. Furthermore, through various additional calculations (not presented here), numerical evidence was found that when an even (odd) number of perturbations was introduced to the solution, an odd (even) number of critical points was obtained. This leads to the conjecture that the number of critical points of the solution to (2) in multiply connected domains depends on the number of "holes" in the domain: *an even (odd) number of holes produces an odd (even) number of critical points.*

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