Multiplicative Triple Fibonacci Sequence of Second Order under Three Specific Schemes and Third Order under Nine Specific Schemes

A. K. Awasthi, *Member, IAENG,* Vikas Ranga, Kamal Dutt

***Abstract*—The Fibonacci sequence (FS) can be found in various aspects of nature. This sequence has applications in multiple fields of mathematics and real-world scenarios. The FS is used to build various algebraic structures, including the Fibonacci group, Fibonacci graph, Fibonacci lattice, Fibonacci quaternion and Fibonacci octonion. This theory has gained significant attention recently and is now considered a major area of number theory. In recent years, there has been considerable interest in the growth of knowledge in the general area of Fibonacci numbers and related mathematical problems. Triple Fibonacci sequences (TFS) have gained popularity recently, although multiplicative triple equations of recurrence relations are less well-known. In 1202, Leonardo of Pisa, also known as Fibonacci (which means "son of Bonacci"), introduced the results of his investigation into expanding a rabbit population. The FS is recognized as a sequence with astonishing properties. In 1985, K.T. Attanasov introduced the Coupled Fibonacci Sequence (CFS), and further developments were made in 1987. However, compared to the additive form of TFS, the multiplicative form of TFS is less well-known. The multiplicative triple Fibonacci sequences (MTFS) of the second and third order represent a novel extension of the classical FS, introducing three specific schemes for the second order and nine specific schemes for the third order. This mathematical study explores the intricate relationships between numbers in a multiplicative context, revealing fascinating patterns and properties.**

***Index Terms*- FS, CFS, TFS, MTFS.**

I. INTRODUCTION

One well-known integer sequence is the Fibonacci sequence (FS). Mathematicians have long been fascinated by this series. The FS has applications in numerous fields, including architecture, engineering, computer science, physics, nature, art, and more. By altering the recurrence relation, the initial condition, or both, the FS can be generalized. This broader form is known as the generalized Fibonacci sequence. Several authors have explored second-order generalized Fibonacci sequences in the literature. The

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A. K. Awasthi is a Professor in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab 144411, India. (corresponding author phone: +91-9315517628; e-mail: dramitawasthi@gmail.com)

Vikas Ranga is a Ph.D. student in the Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara, Punjab 144411, India. ((corresponding author phone: +91- 9599514814; e-mail: vickyrangaphd@gmail.com)

Kamal Dutt is an Assistant Professor in the Department of Mathematics, Chaudhary Ranbir Singh University, Jind 126102, Haryana, India. (email: drkamalvats1994@gmail.com)

Fibonacci numbers appear in many remarkable scenarios andare abundant in nature, often represented in images of fruits, vegetables, and flowers. Mathematical scholars have been deeply interested in the study of Fibonacci numbers and related mathematics for centuries.

Triple Fibonacci sequences (TFS) represent a novel approach to generalizing the Coupled Fibonacci Sequence (CFS). The TFS is a significant advancement in the field of FS and extends the CFS, offering a wide range of intriguing properties and applications. The multiplicative triple Fibonacci sequences (MTFS), an extension of the classical FS, have garnered substantial interest in recent mathematical research, particularly in the context of second and third-order derivations under specific schemes. The FS, known for its ubiquity in nature and applications across diverse fields, serves as the foundation for exploring the multiplicative variations proposed in this study.

The TFS represents a fresh approach to the generalization of the CFS. It is a significant advancement in the field of FS and a generalization of the CFS, offering a wide range of fascinating properties and applications. The MTFS, an extension of the classical FS, has garnered substantial interest in recent mathematical research, particularly concerning second and third-order derivations under specific schemes. The FS, known for its ubiquity in nature and applications across diverse fields, composes the foundation for exploring the multiplicative variations proposed in this study.

There has been a great deal of research on the TFS. J. Z. Lee and J. S. Lee [1] were the first to propose the TFS. Koshy's book [2] is an excellent source for these applications. In 1985, Attanasov [3, 4] popularized the concept of the CFS and introduced a new TFS design. The TFS connects three integer sequences, where the elements of one sequence are part of the generalization of the others, and vice versa. Singh and Sikhwal [4, 7] computed the MCFS and additive TFS, both have significant properties.

Under two distinct schemes, Kiran Singh Sisodiya, Vandana Gupta, and Kiran Sisodiya [8] investigated several features of the fourth-order MCFS. Omprakash Sikhwal, Mamta Singh, and Shweta Jain [6] examined various aspects of the fifth-order CFS. In 2014, Krishna Kumar Sharma et al. [13] formulated the additive-linked Fibonacci sequences of r-th order and demonstrated their diverse features. Bijendra Singh and Omprakash Sikhwal [9] explored both the primitive aspects of second-order TFS and several features of additive TFS. The MTFS of the second order was examined from multiple perspectives by Mamta Singh, Shikha Bhatnagar, and Omprakash Sikhwal [10]. The properties of second-order MTFS were extensive by Satish Kumar, Hari Kishan, and Deepak Gupta [11]. Additionally, K.S. Sisodiya, V. Gupta, and V. H. Badshah [12] illustrated different characteristics of second-order TFS. B. Singh, Kiran Singh Sisodiya, and Kiran Sisodiya [14] further enhanced the second-order MTFS and provided convinced fundamental characteristics. Shoukralla [15] obtained a numerical solution to the first kind of Fredholm integral equation using the matrix form of the second-kind chebyshev polynomials.

The second-order MTFS introduces a novel dimension to the classical sequence by incorporating three distinct initial values and employing three specific schemes for its evolution. This extension beyond the traditional Fibonacci paradigm unveils a richer tapestry of numerical relationships and behaviors, prompting a deeper investigation into the underlying mathematical structure. Building upon this exploration, the study delves into the third-order MTFS, expanding its complexity by introducing nine specific schemes. This extension amplifies the intricacies of the sequence, offering a more nuanced understanding of its behavior and potential applications. The literature surrounding FS and its derivatives has witnessed a surge in interest due to their relevance in various scientific and computational domains. Previous studies have often focused on additive properties and relationships within the Fibonacci framework. However, the current research contributes significantly by extending the scope to multiplicative operations under specific schemes, thereby paving the way for novel insights into the mathematical landscape. This literature review sets the stage for a comprehensive analysis of MTFS, emphasizing its potential impact on both theoretical mathematics and practical applications.

Overall, the MTFS of the second and third order, with three and nine specific schemes appropriately, presents a unique and intricate exploration of mathematical sequences, contributing to a broader understanding of Fibonacci-related structures and their potential applications. In the second order, the sequence is generated by considering three initial values and using a set of rules that dictate the multiplication of the last three terms to obtain the subsequent term. Exploring different schemes adds complexity and diversity to the sequence, uncovering unique numerical behaviors. Moving into the third order, the investigation expands to nine distinct schemes, each contributing to the richness and complexity of the sequence. The interplay of these schemes yields an MTFS sequence with intricate dynamics, offering mathematicians and researchers a wealth of material for analysis and exploration.

This introduction encapsulates a pioneering study in the realm of mathematical sequences, showcasing the remarkable versatility and adaptability of the Fibonacci framework when subjected to multiplicative operations. The exploration of specific schemes introduces a nuanced understanding of the sequence's evolution, offering a solid platform for further research and diverse applications in various mathematical and computational domains.

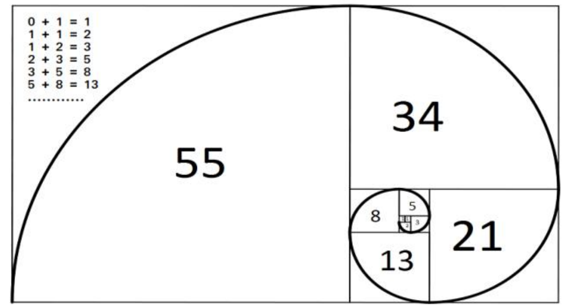

Fig. 1: Fibonacci Numbers Spiral

In Fig. 1, The Fibonacci spiral in the figure is constructed by arranging squares whose side lengths correspond to Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.). Each square's dimensions represent the sequence's increasing values. By connecting the corners of these squares with quarter-circle arcs, the figure forms a spiral. This spiral visually demonstrates the Fibonacci sequence's exponential growth pattern and its approximation of the golden ratio. Such spirals are commonly found in nature, such as in the arrangement of sunflower seeds, shells, and galaxies, highlighting the connection between mathematics and natural phenomena.

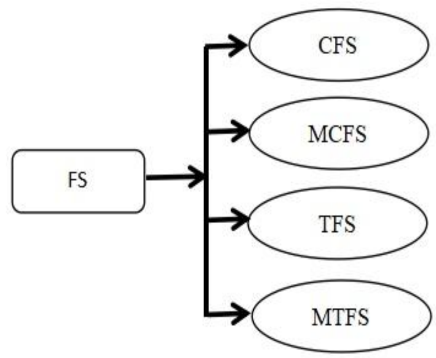

Fig. 2: Types of Fibonacci Sequence

Fig. 2 illustrates different variations of the Fibonacci sequence. CFS are modified version where each term is generated based on a coupling between previous terms. MCFS variations where the relationship between the terms involves multiplication and coupling of previous terms.TFS is an extension of the Fibonacci sequence where the next term is calculated based on the previous three term instead of two. MTFS is an extension where the terms are calculated based on a multiplicative relationship among three previous terms.

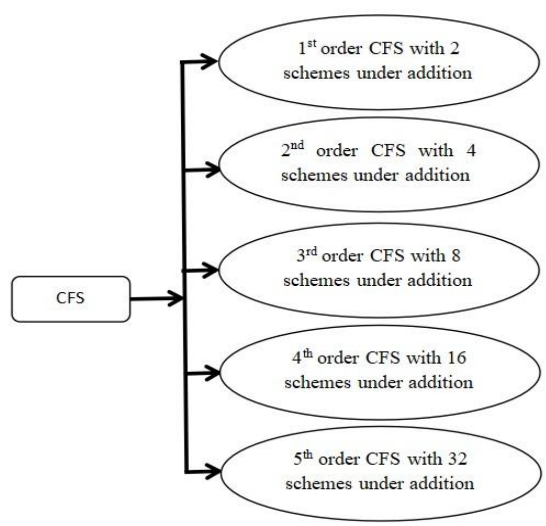

Fig. 3: Hierarchical Structure of CFS Under Addition

Fig. 3 illustrates hierarchical structure of the CFS under addition, with different orders and schemes: 1st order CFS represents the basic CFS with two schemes, where the terms are derided by adding two coupled sequences. 2nd order CFS are more complex sequence with four schemes, extending the coupling process to a second level. 3rd order CFS involves eight schemes, further expanding the coupling and addition process. 4th order CFS are more advanced version with sixteen schemes, counting the pattern of CFS under addition. 5th order CFS are most complex, involving thirty-two schemes, representing the highest order of coupling in this structure.

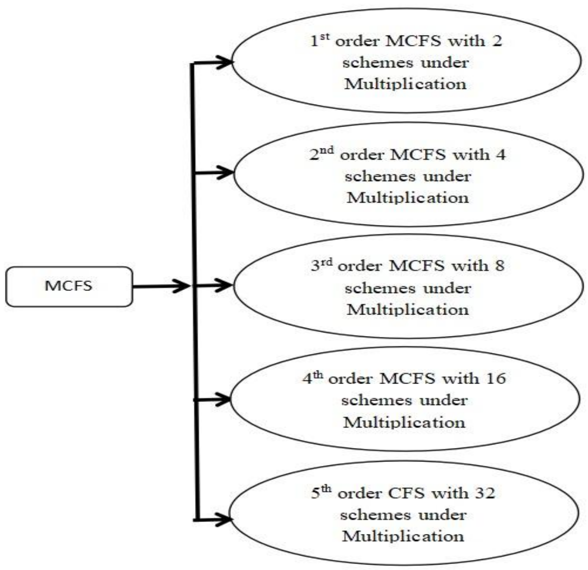

Fig. 4: Structure of MCFS Under Multiplication

Fig. 4 outlines the structure of the MCFS under multiplication , showcasing different orders and schemes. 1st order MCFS is most basic form of MCFS with two schemes, where terms are generated using a multiplication process between coupled sequences. 2nd order MCFS is more advanced version with four schemes, extending the multiplication based coupling to a second level. 3rd order MCFS increases in complexity with eight schemes, involving further multiplication of coupled sequence. 4th order MCFS is higher level sequence with sixteen schemes, expanding the multiplicative coupling process even further. 5th order MCFS is most complex sequence, involving thirty-two schemes, representing the highest level of multiplicative coupling in the FS structure.

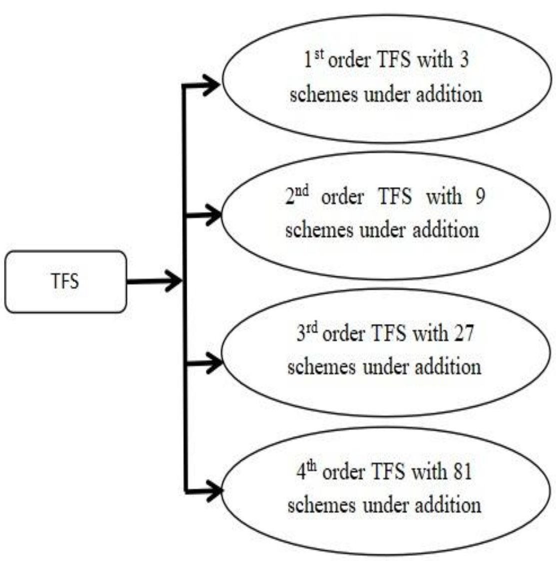

Fig. 5: Structure of TFS

Fig. 5 represents the structure of the TFS under addition, featuring different orders and schemes. 1st order TFS is the basic form of the TFS, where each term is derived from the sum of the previous three terms, with three schemes for generating the sequence. 2nd order TFS is more complex extension, incorporating with nine schemes, where the coupling and addition process are applied at the second level. 3rd order TFS involves twenty-seven schemes, expanding the addition process to further include previous terms at an even higher level. 4th order TFS is most complex version in this series, with eighty-one schemes, involving a highly intricate addition process across multiple levels.

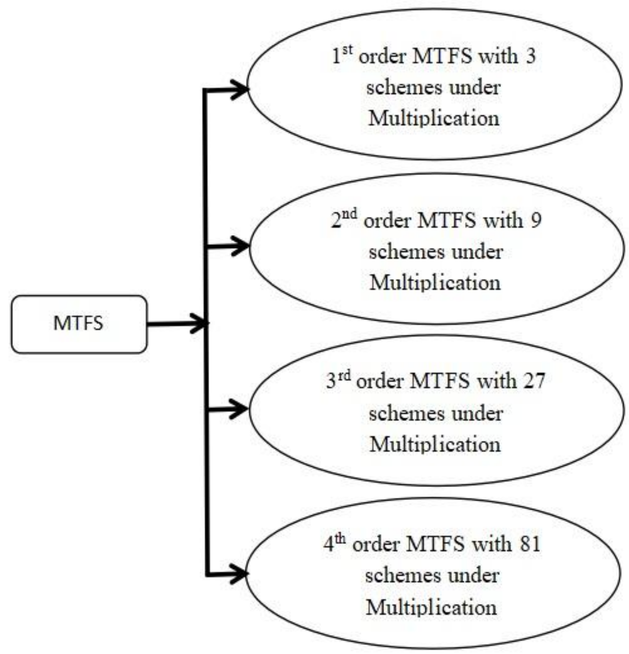

Fig. 6: Structure of MTFS

Fig. 6 illustrates the structure of the MTFS, showcasing increasing complexity through various orders and schemes. 1st order MTFS generates terms by multiplying the previous three terms, utilizing three schemes for sequence generation. 2nd order MTFS is more complex version that applies the multiplicative relationship at a second level, incorporating nine schemes to enhance the sequence formation. 3rd order MTFS further expands the multiplicative structure, using twenty-seven schemes for generating terms through the multiplication of three previous terms in more intricate patterns. 4th order MTFS is most advanced level in this series, with eighty-one schemes, where the multiplicative relationships become increasingly elaborate across multiple levels.

II. SECOND ORDER MTFS

Let $\{X_i\}_{i=0}^{\infty}$, $\{Y_i\}_{i=0}^{\infty}$ and $\{Z_i\}_{i=0}^{\infty}$ be three infinite sequences with initial values $a, b, c, d,$ e and f which are referred to as the 3-F Sequence or TFS.

If $X_0 = a$, $Y_0 = b$, $Z_0 = c$, $X_1 = d$, $Y_1 = e$ and $Z_1 = f$

Then the there are nine different Multiplicative Triple Fibonacci Sequence schemes, each defined by initial values a, b and c .These sequences evolve through distinct multiplicative relationships, generating unique patterns and behaviors. Additionally, we will introduce parameters d , e, and f to further enhance the complexity and richness of these sequences. J. Z.Lee and J. S.Lee [1] defined following nine different schemes of multiplicative triple Fibonacci sequences are as follows:

Table I:All the schemes of MTFS of Second order

Scheme	X_{n+2}	Y_{n+2}	Z_{n+2}
1	$Y_{n+1}.Z_n$	$Z_{n+1}.X_n$	$X_{n+1}.Y_n$
2	$Z_{n+1}.Y_n$	$X_{n+1}.Z_n$	$Y_{n+1}.X_n$
3	$X_{n+1}.Y_n$	$Y_{n+1}.Z_n$	$Z_{n+1}.X_n$
4	$Y_{n+1}.X_n$	$Z_{n+1}.Y_n$	$X_{n+1}.Z_n$
5	$X_{n+1}.Z_n$	$Y_{n+1}.X_n$	$Z_{n+1}.Y_n$
6	$Z_{n+1}.X_n$	$X_{n+1}.Y_n$	$Y_{n+1}.Z_n$
7	$X_{n+1}.X_n$	$Y_{n+1}.Y_n$	$Z_{n+1}.Z_n$
8	$Y_{n+1}.Y_n$	$Z_{n+1}.Z_n$	$X_{n+1}.X_n$
9	$Z_{n+1}.Z_n$	$X_{n+1}.X_n$	$Y_{n+1}.Y_n$

Properties of seventh, eighth and ninth scheme. Below are the first few terms of the seventh schemes:

Table II: Some initial values of seventh scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	ad	be	cf
3	ad^2	be^2	cf^2
4	a^2d^3	b^2e^3	c^2f^3
5	a^3d^5	b^3e^5	c^3f^5

The eight scheme's initial terms are listed below:

Table III: Some initial terms of eighth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	be	cf	ad
3	be^2	cf^2	ad^2
4	b^2e^3	c^2f^3	a^2d^3
5	b^3e^5	c^3f^5	a^3d^5

Following are the first few terms of the 9th schemes:

Table IV:Some initial terms of ninth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	cf	ad	be
3	cf^2	ad^2	be^2
4	c^2f^3	a^2d^3	b^2e^3
5	c^3f^5	a^3d^5	b^3e^5

O. P. Sikhwal, M. Singh, and S. Bhatnagar [10] examined a wide range of second-order results.

III. MAIN RESULTS OF 2ND ORDER MTFS

We will present some other results on the MTFS of Second order under three specific schemes and Third Order under nine schemes in this paper.

Now, under Schemes 7th, 8th and 9th, we introduce some results of the MTFS of Second Order:

Theorem 1: For each whole number n:

(a) $X_{n+1} = X_0^{F_n} X_1^{F_{n+1}}$

(b) $Y_{n+1} = Y_0^{F_n} Y_1^{F_{n+1}}$

(c) $Z_{n+1} = Z_0^{F_n} Z_1^{F_{n+1}}$

Proof: These results are confirmed by the induction hypothesis.

(a) If $n = 0$, then

$$X_1 = X_0^{F_0} X_1^{F_1}$$

$$= X_1$$

For $n = 1$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $n \geq 1$. Then

$$X_{n+2} = X_{n+1} X_n$$

$$= X_0^{F_n} X_1^{F_{n+1}} X_0^{F_{n-1}} X_1^{F_n}$$

$$= X_0^{F_n + F_{n-1}} X_1^{F_{n+1} + F_n}$$

$$= X_0^{F_{n+1}} X_1^{F_{n+2}}$$

The conclusion is valid for all integers $n \geq 0$. Similar evidence is available for the remaining parts (b) and (c).

Example based on Theorem 1

Consider a Fibonacci sequence F_n, where each term is the sum of the two preceding terms. This sequence typically starts with 0 and 1.

0,1,1,2,3,5,8,13, 21.........

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on..

Let $\{X_n\}_{n=0}^{\infty}, \{Y_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ be three sequences where each term is the product of the two preceding ones, such that

$$\{X_n\}_{n=0}^{\infty} = 1,3,3,9,27,243,\ldots\ldots\ldots\ldots$$

Where,

$X_0 = 1, X_1 = 3, X_2 = 3, X_3 = 9, X_4 = 27, X_5 = 243$ and so on...

$$\{Y_n\}_{n=0}^{\infty} = 2,3,6,18,108,1944\ldots\ldots..$$

Where,

$Y_0 = 2$, $Y_1 = 3$, $Y_2 = 6$, $Y_3 = 18$, $Y_4 = 108$, $Y_5 = 1944$ and so on…

$$\{Z_n\}_{n=0}^{\infty} = 1,4,4,16,64,1024,\ldots\ldots..$$

Where,

$Z_0 = 1$, $Z_1 = 4$, $Z_2 = 4$, $Z_3 = 16$, $Z_4 = 64$, $Z_5 = 1024$ and so on...

Now we are going to apply the result of part (a) of theorem 1

$$X_{n+1} = X_0^{F_n} X_1^{F_{n+1}}$$

Put $n = 4$, $\quad X_{4+1} = X_0^{F_4} X_1^{F_{4+1}}$

$$\Rightarrow \quad X_5 = X_0^{F_4} X_1^{F_5}$$

$$= (1)^3 (3)^5$$

$$= 243$$

Now we are going to apply the result of part (b) of theorem 1 $\quad Y_{n+1} = Y_0^{F_n} Y_1^{F_{n+1}}$

Put $n = 4$, $\quad Y_{4+1} = Y_0^{F_4} Y_1^{F_{4+1}}$

$$\Rightarrow \quad Y_5 = Y_0^{F_4} Y_1^{F_5}$$

$$= (2)^3 (3)^5$$

$$= 1944$$

Now we are going to apply the result of part (c) of theorem 1 $\quad Z_{n+1} = Z_0^{F_n} Z_1^{F_{n+1}}$

Put $n = 4$, $\quad Z_{4+1} = Z_0^{F_4} Z_1^{F_{4+1}}$

$$\Rightarrow \quad Z_5 = Z_0^{F_4} Z_1^{F_5}$$

$$= (1)^3 (4)^5$$

$$= 1024$$

Hence the result is confirmed.

Theorem 2: For each natural number n;

$$\left(X_n Y_n Z_n\right) = (X_0 Y_0 Z_0)^{F_{n-1}} (X_1 Y_1 Z_1)^{F_n}$$

Proof: We will confirm this result with the help of induction hypothesis

If $n = 1$, then

$$\left(X_1 Y_1 Z_1\right) = (X_0 Y_0 Z_0)^{F_0} (X_1 Y_1 Z_1)^{F_1}$$

$$= (X_1 Y_1 Z_1)$$

For n=1, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $n \geq 2$.

Then

$$\left(X_{n+1} Y_{n+1} Z_{n+1}\right) = (X_n X_{n-1})(Y_n Y_{n-1})(Z_n Z_{n-1})$$

$$= \left(X_n Y_n Z_n\right)\left(X_{n-1} Y_{n-1} Z_{n-1}\right)$$

$$= (X_0 Y_0 Z_0)^{F_{n-1}} (X_1 Y_1 Z_1)^{F_n} (X_0 Y_0 Z_0)^{F_{n-2}} (X_1 Y_1 Z_1)^{F_{n-1}}$$

$$=(X_0 Y_0 Z_0)^{F_{n-1}+F_{n-2}} (X_1 Y_1 Z_1)^{F_n+F_{n-1}}$$

$$=(X_0 Y_0 Z_0)^{F_n} (X_1 Y_1 Z_1)^{F_{n+1}}$$

The conclusion is valid for all integers $n \geq 1$.

Example based on Theorem 2

Consider a Fibonacci sequence F_n, where each term is obtained by adding the two preceding terms. This sequence usually begins with 0 and 1.

0,1,1,2,3,5,8,13, 21...........

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on.

Let $\{X_n\}_{n=0}^{\infty}$, $\{Y_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ be three sequences where each term is the product of the two preceding ones, such that

$$\{X_n\}_{n=0}^{\infty} = 2,4,8,32,256\ldots\ldots\ldots..$$

Where,

$X_0 = 2$, $X_1 = 4$, $X_2 = 8$, $X_3 = 32$, $X_4 = 256$ and so on...

$$\{Y_n\}_{n=0}^{\infty} = 1,1,1,1,1,1\ldots\ldots..$$

Where,

$Y_0 = 1$, $Y_1 = 1$, $Y_2 = 1$, $Y_3 = 1$, $Y_4 = 1$ and so on...

$$\{Z_n\}_{n=0}^{\infty} = 2,3,6,18,108,1944,\ldots\ldots..$$

Where,

$Z_0 = 2$, $Z_1 = 3$, $Z_2 = 6$, $Z_3 = 18$, $Z_4 = 108$ and so on...

Now we are going to apply the result of theorem 2

$$\left(X_n Y_n Z_n\right) = (X_0 Y_0 Z_0)^{F_{n-1}} (X_1 Y_1 Z_1)^{F_n}$$

Put $n = 4$

$$(X_4 Y_4 Z_4) = (X_0 Y_0 Z_0)^{F_{4-1}} (X_1 Y_1 Z_1)^{F_4}$$

$$\Rightarrow (256 \times 1 \times 108) = (2 \times 1 \times 2)^2 (4 \times 1 \times 3)^3$$

$$\Rightarrow \quad 27648 = (4)^2 (12)^3$$

$$= 16 \times 1728$$

$$= 27648$$

Hence the result is verified.

Theorem 3: For each whole number n;

(a) $X_n X_{n+1} X_{n+2} = {X_0}^{2F_{n+1}} {X_1}^{2F_{n+2}}$
(b) $Y_n Y_{n+1} Y_{n+2} = {Y_0}^{2F_{n+1}} {Y_1}^{2F_{n+2}}$
(c) $Z_n Z_{n+1} Z_{n+2} = {Z_0}^{2F_{n+1}} {Z_1}^{2F_{n+2}}$

Proof: These results are confirmed by the induction hypothesis.

If $n = 0$, then $\quad X_0 X_1 X_2 = {X_0}^{2F_1} {X_1}^{2F_2}$

$$= {X_0}^2 {X_1}^2$$

$$= X_0 X_0 X_1 X_1$$

$$= X_0 X_1 X_0 X_1$$

$$= X_0 X_1 X_2 \quad \text{(By Scheme 7)}$$

For $n = 0$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer n ≥ 1.

Then

$$X_{n+1} X_{n+2} X_{n+3} = (X_{n+1} X_{n+2} X_{n+1} X_{n+2}) \quad \text{(By Scheme 7)}$$

$$= (X_{n+1} X_{n+1})(X_{n+2} X_{n+2})$$

$$= (X_{n-1} X_n X_{n+1})(X_n X_{n+1} X_{n+2})$$

(By given Hypothesis)

$$= ({X_0}^{2F_n} {X_1}^{2F_{n+1}})({X_0}^{2F_{n+1}} {X_1}^{2F_{n+2}})$$

$$= {X_0}^{2F_n + 2F_{n+1}} {X_1}^{2F_{n+1} + 2F_{n+2}}$$

$$= {X_0}^{2F_{n+2}} {X_1}^{2F_{n+3}}$$

The conclusion is valid for all integers $n \geq 0$.

For the remaining sections (b) and (c), comparable evidence is provided.

Example based on Theorem 3

Consider a Fibonacci sequence F_n, where each term is obtained by adding the two preceding terms. This sequence usually begins with 0 and 1.

0,1,1,2,3,5,8,13,21,...........

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on...

Let $\{X_n\}_{n=0}^{\infty}, \{Y_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ be three sequences where each term is the product of the two preceding ones, such that

$$\{X_n\}_{n=0}^{\infty} = 4,5,20,100,2000,200000\ldots\ldots\ldots\ldots$$

Where,

$X_0 = 4, X_1 = 5, X_2 = 20, X_3 = 100, X_4 = 2000$ and so on...

$$\{Y_n\}_{n=0}^{\infty} = 1,7,7,49,343,16807\ldots\ldots\ldots$$

Where,

$Y_0 = 1, Y_1 = 7, Y_2 = 7, Y_3 = 49, Y_4 = 343$ and so on...

$$\{Z_n\}_{n=0}^{\infty} = 2,4,8,32,256,8192\ldots\ldots\ldots$$

Where,

$Z_0 = 2, Z_1 = 4, Z_2 = 8, Z_3 = 32, Z_4 = 256$ and so on...

Now we are going to apply the result part (a) of theorem 3

$$X_n X_{n+1} X_{n+2} = {X_0}^{2F_{n+1}} {X_1}^{2F_{n+2}}$$

For $n = 2$, $X_2 X_3 X_4 = {X_0}^{2F_3} {X_1}^{2F_4}$

$$\Rightarrow \quad X_2 X_3 X_4 = {X_0}^{2F_3} {X_1}^{2F_4}$$

$$\Rightarrow \quad 20 \times 100 \times 2000 = 4^4 5^6$$

$$\Rightarrow \quad 20000 = 20000$$

Now we are going to apply the result part (b) of theorem 3

$$Y_n Y_{n+1} Y_{n+2} = {Y_0}^{2F_{n+1}} {Y_1}^{2F_{n+2}}$$

For $n = 2$, $Y_2 Y_3 Y_4 = {Y_0}^{2F_3} {Y_1}^{2F_4}$

$$\Rightarrow \quad Y_2 Y_3 Y_4 = {Y_0}^{2F_3} {Y_1}^{2F_4}$$

$$\Rightarrow \quad 7 \times 49 \times 343 = 1^4 7^6$$

$$\Rightarrow \quad 117649 = 117649$$

Now we are going to apply the result part (c) of theorem 3

$$Z_nZ_{n+1}Z_{n+2} = Z_0^{\,2F_{n+1}} Z_1^{\,2F_{n+2}}$$

For $n = 2$, $\;Z_2Z_3Z_4 = Z_0^{\,2F_3} Z_1^{\,2F_4}$

$\Rightarrow \qquad Z_2Z_3Z_4 = Z_0^{\,2F_3} Z_1^{\,2F_4}$

$\Rightarrow \qquad 8 \times 32 \times 256 = 2^4 4^6$

$\Rightarrow \qquad 65536 = 65536$

Hence the result is applicable.

Theorem 4: For each whole number n and every natural no. $k \geq 2$;

(a) $X_{n+k+1}Y_{n+k-1} = X_n^{F_k} X_{n+1}^{F_{k+1}} Y_n^{F_{k-2}} Y_{n+1}^{F_{k-1}}$
(b) $Y_{n+k+1}Z_{n+k-1} = Y_n^{F_k} Y_{n+1}^{F_{k+1}} Z_n^{F_{k-2}} Z_{n+1}^{F_{k-1}}$
(c) $Z_{n+k+1}X_{n+k-1} = Z_n^{F_k} Z_{n+1}^{F_{k+1}} X_n^{F_{k-2}} X_{n+1}^{F_{k-1}}$

Proof: These results are confirmed by the induction hypothesis.

If $k = 2$ then $X_{n+3}Y_{n+1} = X_{n+2}X_{n+1}Y_{n+1}$

$$= X_{n+1}X_nX_{n+1}Y_{n+1}$$

$$= X_n^1 X_{n+1}^2 Y_n^0 Y_{n+1}^1$$

$$= X_n^{F_2} X_{n+1}^{F_3} Y_n^{F_0} Y_{n+1}^{F_1}$$

For $k = 2$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $k \geq 3$.

Then

$$X_{n+k+2}Y_{n+k} = X_{n+k+1}X_{n+k}Y_{n+k-1}Y_{n+k-2}$$

$$= (X_{n+k+1}Y_{n+k-1})(X_{n+k}Y_{n+k-2})$$

$$= X_n^{F_k} X_{n+1}^{F_{k+1}} Y_n^{F_{k-2}} Y_{n+1}^{F_{k-1}} X_n^{F_{k-1}} X_{n+1}^{F_k} Y_n^{F_{k-3}} Y_{n+1}^{F_{k-2}}$$

$$= X_n^{F_k+F_{k-1}} X_{n+1}^{F_{k+1}+F_k} Y_n^{F_{k-2}+F_{k-3}} Y_{n+1}^{F_{k-1}+F_{k-2}}$$

$$= X_n^{F_{k+1}} X_{n+1}^{F_{k+2}} Y_n^{F_{k-1}} Y_{n+1}^{F_k}$$

The conclusion is valid for all integers $n \geq 0, k \geq 2$.

Similar evidence is available for the remaining parts (b) and (c).

Example based on Theorem 4

Let F_n be a Fibonacci sequence whose terms are the sum of the two preceding ones. This sequence commonly starts from 0 and 1.

0,1,1,2,3,5,8,13,21,...........

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$ and so on...

Let $\{X_n\}_{n=0}^{\infty}, \{Y_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ be three sequences whose terms is the multiplication of the two preceding ones such that

$$\{X_n\}_{n=0}^{\infty} = 1,3,3,9,27,243,\ldots\ldots\ldots\ldots$$

Where,

$X_0 = 1, X_1 = 3, X_2 = 3, X_3 = 9, X_4 = 27, X_5 = 243, X_6 = 6561$ and so on...

$$\{Y_n\}_{n=0}^{\infty} = 2,3,6,18,108,1944\ldots\ldots\ldots$$

Where,

$Y_0 = 2, Y_1 = 3, Y_2 = 6, Y_3 = 18, Y_4 = 108, Y_5 = 1944$ and so on...

$$\{Z_n\}_{n=0}^{\infty} = 1,4,4,16,64,1024,\ldots\ldots\ldots$$

Where,

$Z_0 = 1, Z_1 = 4, Z_2 = 4, Z_3 = 16, Z_4 = 64, Z_5 = 1024$ and so on...

Now we are going to apply the result of part (a) theorem 4

$$X_{n+k+1}Y_{n+k-1} = X_n^{F_k} X_{n+1}^{F_{k+1}} Y_n^{F_{k-2}} Y_{n+1}^{F_{k-1}}$$

For $n = 2$ and $k = 3$

$$X_6Y_4 = X_2^{F_3} X_3^{F_4} Y_2^{F_1} Y_3^{F_2}$$

$\Rightarrow \qquad 108 \times 6561 = 3^2 9^3 6^1 18^1$

$\Rightarrow \qquad 708588 = 708588$

Similarly, we can apply the result in parts (b) and (c).

Theorem 5: For every integer $n \geq 0, k \geq 2$;

(a) $X_{n+k+1}Z_{n+k-1} = X_n^{F_k} X_{n+1}^{F_{k+1}} Z_n^{F_{k-2}} Z_{n+1}^{F_{k-1}}$
(b) $Y_{n+k+1}X_{n+k-1} = Y_n^{F_k} Y_{n+1}^{F_{k+1}} X_n^{F_{k-2}} X_{n+1}^{F_{k-1}}$
(c) $Z_{n+k+1}Y_{n+k-1} = Z_n^{F_k} Z_{n+1}^{F_{k+1}} Y_n^{F_{k-2}} Y_{n+1}^{F_{k-1}}$

Proof: A similar proof can be given as in theorem 4.

Theorem 6: For every integer $n \geq 0$;

(a) $X_0X_{n+4} = X_0^{F_{n+3}-1} X_1^{F_{n+4}}$
(b) $Y_0Y_{n+4} = Y_0^{F_{n+3}-1} Y_1^{F_{n+4}}$
(c) $Z_0Z_{n+4} = Z_0^{F_{n+3}-1} Z_1^{F_{n+4}}$

Proof: We can prove the theorem by the method of mathematical induction.

We can also prove theorem 1 to theorem 6 with the help of schemes 8th and 9th.

IV. 3rd ORDER MTFS

Let $\{X_i\}_{i=0}^{\infty}$, $\{Y_i\}_{i=0}^{\infty}$ and $\{Z_i\}_{i=0}^{\infty}$ be three infinite sequences with initial values a, b, c, d, e, f, g, $\hbar$ and i, which are referred to as the 3-F Sequence or TFS.

If $X_0 = a, Y_0 = b, Z_0 = c, X_1 = d, Y_1 = e, Z_1 = f, X_2 = g, Y_2 = \hbar, Z_2 = i,$

Then the following are twenty-seven different MTFS schemes:

Table V: All the schemes of MTFS of third order

Scheme	X_{n+3}	Y_{n+3}	Z_{n+3}
1	$Y_{n+2}.Z_{n+1}.X_n$	$Z_{n+2}.X_{n+1}.Y_n$	$X_{n+2}.Y_{n+1}.Z_n$
2	$X_{n+2}.X_{n+1}.X_n$	$Y_{n+2}.Y_{n+1}.Y_n$	$Z_{n+2}.Z_{n+1}.Z_n$
3	$X_{n+2}.Z_{n+1}.Y_n$	$Y_{n+2}.X_{n+1}.Z_n$	$Z_{n+2}.Y_{n+1}.X_n$
4	$Z_{n+2}.Y_{n+1}.X_n$	$X_{n+2}.Z_{n+1}.Y_n$	$Y_{n+2}.X_{n+1}.Z_n$
5	$X_{n+2}.Y_{n+1}.Z_n$	$Y_{n+2}.Z_{n+1}.X_n$	$Z_{n+2}.X_{n+1}.Y_n$
6	$X_{n+2}.X_{n+1}.Y_n$	$Y_{n+2}.Y_{n+1}.Z_n$	$Z_{n+2}.Z_{n+1}.X_n$
7	$X_{n+2}.Y_{n+1}.X_n$	$Y_{n+2}.Z_{n+1}.Y_n$	$Z_{n+2}.X_{n+1}.Z_n$
8	$Y_{n+2}.X_{n+1}.X_n$	$Z_{n+2}.Y_{n+1}.Y_n$	$X_{n+2}.Z_{n+1}.Z_n$
9	$X_{n+2}.X_{n+1}.Z_n$	$Y_{n+2}.Y_{n+1}.X_n$	$Z_{n+2}.Z_{n+1}.Y_n$
10	$X_{n+2}.Z_{n+1}.X_n$	$Y_{n+2}.X_{n+1}.Y_n$	$Z_{n+2}.Y_{n+1}.Z_n$
11	$Z_{n+2}.X_{n+1}.X_n$	$X_{n+2}.Y_{n+1}.Y_n$	$Y_{n+2}.Z_{n+1}.Z_n$
12	$Y_{n+2}.Y_{n+1}.Z_n$	$Z_{n+2}.Z_{n+1}.X_n$	$X_{n+2}.X_{n+1}.Y_n$
13	$Y_{n+2}.Z_{n+1}.Y_n$	$Z_{n+2}.X_{n+1}.Z_n$	$X_{n+2}.Y_{n+1}.X_n$
14	$Z_{n+2}.Y_{n+1}.Y_n$	$X_{n+2}.Z_{n+1}.Z_n$	$Y_{n+2}.X_{n+1}.X_n$
15	$Y_{n+2}.Z_{n+1}.Z_n$	$Z_{n+2}.X_{n+1}.X_n$	$X_{n+2}.Y_{n+1}.Y_n$
16	$Z_{n+2}.Y_{n+1}.Z_n$	$X_{n+2}.Z_{n+1}.X_n$	$Y_{n+2}.X_{n+1}.Y_n$
17	$Z_{n+2}.Z_{n+1}.Y_n$	$X_{n+2}.X_{n+1}.Z_n$	$Y_{n+2}.Y_{n+1}.X_n$
18	$Z_{n+2}.X_{n+1}.Y_n$	$X_{n+2}.Y_{n+1}.Z_n$	$Y_{n+2}.Z_{n+1}.X_n$
19	$Y_{n+2}.X_{n+1}.Y_n$	$Z_{n+2}.Y_{n+1}.Z_n$	$X_{n+2}.Z_{n+1}.X_n$
20	$X_{n+2}.Y_{n+1}.Y_n$	$Y_{n+2}.Z_{n+1}.Z_n$	$Z_{n+2}.X_{n+1}.X_n$
21	$Y_{n+2}.Y_{n+1}.X_n$	$Z_{n+2}.Z_{n+1}.Y_n$	$X_{n+2}.X_{n+1}.Z_n$
22	$X_{n+2}.Z_{n+1}.Z_n$	$Y_{n+2}.X_{n+1}.X_n$	$Z_{n+2}.Y_{n+1}.Y_n$
23	$Z_{n+2}.X_{n+1}.Z_n$	$X_{n+2}.Y_{n+3}.X_n$	$Y_{n+2}.Z_{n+1}.Y_n$
24	$Z_{n+2}.Z_{n+1}.X_n$	$X_{n+2}.X_{n+1}.Y_n$	$Y_{n+2}.Y_{n+1}.Z_n$
25	$Y_{n+2}.X_{n+1}.Z_n$	$Z_{n+2}.Y_{n+1}.X_n$	$X_{n+2}.Z_{n+1}.Y_n$
26	$Y_{n+2}.Y_{n+1}.Y_n$	$Z_{n+2}.Z_{n+1}.Z_n$	$X_{n+2}.X_{n+1}.X_n$
27	$Z_{n+2}.Z_{n+1}.Z_n$	$X_{n+2}.X_{n+1}.X_n$	$Y_{n+2}.Y_{n+1}.Y_n$

Below are the first few terms of the first scheme:

Table VI: Some initial terms of first scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	$\hbar$	i
3	$\hbar f a$	$i d b$	g e c
4	$i^2 d^2 b$	$g^2 e^2 c$	$\hbar^2 f^2 a$
5	$g^4 e^3 c^2$	$\hbar^4 f^3 a^2$	$i^4 d^3 b^2$

Following are the first few terms of the second scheme:

Table VII:Some initial terms of second scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	$\hbar$	i
3	adg	$be\hbar$	cfi
4	ad^2g^2	$be^2\hbar^2$	cf^2i^2
5	$a^2d^3g^4$	$b^2e^3\hbar^4$	$c^2f^3i^4$

The initial terms of the third scheme are listed below:

Table VIII:Some initial terms of third scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	$\hbar$	i
3	gfb	$\hbar dc$	iea
4	$gfbic$	$\hbar dcgf$	$ieahd$

Below are the first few terms of the 4th scheme:

Table IX: Some initial terms of fourth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	$\hbar$	i
3	$a$ei	bgf	$cd\hbar$
4	$cd^2\hbar^2$	ae^2i^2	bf^2g^2

Below are the first few terms of the 5th scheme:

Table X: Some initial terms of fifth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	$\hbar$	i
3	ge$c$	$\hbar f a$	idb
4	$\hbar^2 f^2 a$	$i^2 d^2 b$	g^2e^2c

Below are the first few terms of the 18th scheme:

Table XI: Some initial terms of eighteenth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	h	i
3	idb	gec	hfa
4	g^2e^2c	h^2f^2a	i^2d^2b

The initial terms of the twenty-fifth scheme are listed below:

Table XII: Some initial terms of twenty-fifth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	h	i
3	hdc	iea	gfb
4	$hdcgf$	$ieahd$	$gfbic$
5	g^2f^2hdcbi	h^2d^2ieacg	g^2c^2fbieh

The following are the first few terms of the 26th scheme:

Table XIII: Some initial terms of twenty-sixth scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	h	i
3	beh	cfi	adg
4	be^2h^2	cf^2i^2	ad^2g^2
5	$b^2e^3h^4$	$c^2f^3i^4$	$a^2d^3g^4$

The following are the first few terms of the 27th scheme:

Table XIV: Some initial terms of twenty-seventh scheme of MTFS

n	X_n	Y_n	Z_n
0	a	b	c
1	d	e	f
2	g	h	i
3	cfi	adg	beh
4	cf^2i^2	ad^2g^2	be^2h^2
5	$c^2f^3i^4$	$a^2d^3g^4$	$b^2e^3h^4$

V. MAIN RESULTS OF 3RD ORDER MTFS

Now we present some results of MTFS of third order under 1st, 2nd, 3rd, 4th, 5th, 18th, 25th, 26th and 27th:

Theorem 7: For each natural no. $n \geq 2$:

$$\frac{\prod_{k=0}^{n} X_{k+6} Y_{k+6} Z_{k+6}}{\prod_{k=0}^{n} X_{k+4} Y_{k+4} Z_{k+4}} = \frac{(X_{n+5} Y_{n+5} Z_{n+5})(X_{n+6} Y_{n+6} Z_{n+6})}{(X_4 Y_4 Z_4)(X_5 Y_5 Z_5)}$$

Proof: We demonstrate these findings through induction hypothesis:

If $n = 2$, then

$$\frac{\prod_{k=0}^{2} X_{k+6} Y_{k+6} Z_{k+6}}{\prod_{k=0}^{2} X_{k+4} Y_{k+4} Z_{k+4}} = \frac{(X_6 Y_6 Z_6)(X_7 Y_7 Z_7)(X_8 Y_8 Z_8)}{(X_4 Y_4 Z_4)(X_5 Y_5 Z_5)(X_6 Y_6 Z_6)}$$

$$= \frac{(X_7 Y_7 Z_7)(X_8 Y_8 Z_8)}{(X_4 Y_4 Z_4)(X_5 Y_5 Z_5)}$$

For $n = 2$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $n \geq 2$.

Then

$$\frac{\prod_{k=0}^{n+1} X_{k+6} Y_{k+6} Z_{k+6}}{\prod_{k=0}^{n+1} X_{k+4} Y_{k+4} Z_{k+4}} = \frac{(X_{n+7} Y_{n+7} Z_{n+7}) \prod_{k=0}^{n} X_{k+6} Y_{k+6} Z_{k+6}}{(X_{n+5} Y_{n+5} Z_{n+5}) \prod_{k=0}^{n} X_{k+4} Y_{k+4} Z_{k+4}}$$

$$= \frac{(X_{n+7} Y_{n+7} Z_{n+7})(X_{n+5} Y_{n+5} Z_{n+5})(X_{n+6} Y_{n+6} Z_{n+6})}{(X_{n+5} Y_{n+5} Z_{n+5})(X_4 Y_4 Z_4)(X_5 Y_5 Z_5)}$$

$$= \frac{(X_{n+6} Y_{n+6} Z_{n+6})(X_{n+7} Y_{n+7} Z_{n+7})}{(X_4 Y_4 Z_4)(X_5 Y_5 Z_5)}$$

The conclusion is valid for all integers $n \geq 0$.

Theorem 8: For each whole no. n:

$$\frac{(X_n Y_n Z_n)(X_{n+1} Y_{n+1} Z_{n+1})}{(X_{n+3} Y_{n+3} Z_{n+3})} = \frac{1}{(X_{n+2} Y_{n+2} Z_{n+2})}$$

Proof: By induction hypothesis, we have

If $n = 0$, then

$$\frac{(X_0 Y_0 Z_0)(X_1 Y_1 Z_1)}{(X_3 Y_3 Z_3)} = \frac{(X_0 Y_0 Z_0)(X_1 Y_1 Z_1)}{(Y_2 Z_1 X_0)(Z_2 X_1 Y_0)(X_2 Y_1 Z_0)}$$

$$= \frac{(X_0 Y_0 Z_0)(X_1 Y_1 Z_1)}{(X_0 Y_0 Z_0)(X_1 Y_1 Z_1)(X_2 Y_2 Z_2)}$$

$$= \frac{1}{(X_2 Y_2 Z_2)}$$

For $n = 0$, the conclusion is correct.

We'll proceed by assuming that the outcome is accurate for some integer $n \geq 1$. Then

$$\frac{(X_{n+1}Y_{n+1}Z_{n+1})(X_{n+2}Y_{n+2}Z_{n+2})}{(X_{n+4}Y_{n+4}Z_{n+4})}$$

$$= \frac{(X_{n+1}Y_{n+1}Z_{n+1})(X_{n+2}Y_{n+2}Z_{n+2})}{(X_{n+3}X_{n+2}X_{n+1})(Y_{n+3}Y_{n+2}Y_{n+1})(Z_{n+3}Z_{n+2}Z_{n+1})}$$

$$= \frac{(X_{n+1}Y_{n+1}Z_{n+1})(X_{n+2}Y_{n+2}Z_{n+2})}{(X_{n+1}Y_{n+1}Z_{n+1})(X_{n+2}Y_{n+2}Z_{n+2})(X_{n+3}Y_{n+3}Z_{n+3})}$$

$$= \frac{1}{(X_{n+3}Y_{n+3}Z_{n+3})}$$

The conclusion is valid for all integers $n \geq 0$.

VI. CONCLUSION

The study of the MTFS of the second order under three specific schemes and the third order under nine specific schemes has illuminated a fascinating realm of mathematical intricacies and potential applications. This investigation into these extended FS has deepened our understanding of their structural properties and unveiled diverse numerical relationships beyond the classical Fibonacci framework. The second-order exploration, featuring three distinct initial values and evolving through three specific schemes, has demonstrated the sequence's adaptability and complexity in response to multiplicative operations. The numerical patterns and relationships discovered within this context contribute valuable insights to the existing body of FS literature. Extending the inquiry to the third order, with nine specific schemes guiding the multiplicative evolution, has further enriched the narrative. The intricate dynamics introduced by these schemes underscore the sequence's versatility and offer a wealth of patterns for exploration. This research goes beyond the traditional additive perspectives of FS, opening new avenues for mathematical inquiry and potential applications in cryptography, number theory, and computational algorithms. Reflecting on the MTFS under specific schemes, this study not only expands the theoretical understanding of mathematical sequences but also holds promise for practical applications in various scientific and computational domains. The findings underscore the importance of exploring unconventional operations within well-established mathematical frameworks, paving the way for future research and innovative applications in mathematics and related disciplines.

The explorations of the MTFS of the second order under three specific schemes and the third order under nine specific schemes have revealed a rich tapestry of mathematical intricacies and potential applications. The study not only extended the classical TFS but also introduced multiplicative factors that add a layer of complexity and depth to the sequences' behavior. Through a systematic analysis of the recurrence relations and initial conditions, we observed the emergence of distinct patterns under each specific scheme.

The second-order MTFS exhibited unique properties influenced by carefully designed schemes, demonstrating the sensitivity of the sequence to the choice of initial conditions. Expanding our exploration to the third-order case, introducing nine specific schemes further diversified the mathematical landscape. The interaction of three factors within each scheme resulted in a wide range of dynamic behaviors, highlighting the versatility and complexity of these multiplicative sequences. The results of this research enhance the theoretical understanding of multiplicative sequences and create opportunities for practical applications. The periodicities and convergence behaviors observed provide valuable insights into potential applications in mathematical modeling, cryptography, and coding theory.

Moreover, the specific schemes identified in this study offer a roadmap for future investigations into other order multiplicative sequences and their applications. In conclusion, the MTFS of the second and third order exemplifies the elegance and versatility of mathematical sequences. The interweaving of factors and the influence of carefully crafted schemes deepen our appreciation for the inherent beauty and complexity present in these sequences, motivating further research and exploration in the broader field of mathematical sequences and their applications.

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