A Vertically Averaged Groundwater Quality Measurement With Monitored Boundary Data

Prattana Maneechay, Nopparat Pochai and Suriyun Khatbanjong*

Abstract— **Landfills are a major source of environmental groundwater pollution. Many medium- and high-income developing countries find landfill operations to be the most viable option for waste disposal. Groundwater contaminant measurements are essential to control the quality of drinking water. If a landfill project is to remain in operation in an area under consideration, the expected influence on groundwater quality must be estimated using a mathematical model. This study proposes a long-term groundwater quality assessment using a heterogeneous soil model with a vertically averaged two-dimensional advection–diffusion equation. The standard forward time-centered space finite difference technique is used to estimate the concentration of groundwater pollutants in an area around a landfill. The usual procedure produces an acceptable approximation result.**

Index Terms— **Groundwater quality measurement, monitored boundary data, vertically averaged, two-dimensional advection–diffusion equation.**

I. Introduction

Groundwater pollution has numerous sources, including landfills, mine spoils, municipal waste, industrial effluents, and cemeteries. Many studies on groundwater contamination at local and global scales have been conducted over the past few decades by hydrologists, civil engineers, geo-environmentalists, groundwater scientists, and others. Analytical solutions have been found for the longitudinal dispersion problem in porous media, taking into account the effects of variable dispersion coefficients and non-uniform flows [1].

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Each person in Thailand and Indonesia generates about 0.65 kilograms of trash per day on average. Indonesia produced 1.75 billion tons of garbage annually, compared with 1.80 billion tons by Indonesia. The composition of solid waste disposed of at landfill sites, the inability to separate wet and dry solid waste, and inadequate landfill site management are the main causes of groundwater pollution [2]. A landfill is a site where waste materials are buried. It is also the earliest method of solid waste treatment, and historically the most popular means of organized garbage disposal, remaining so in many nations. Landfills can be internal waste disposal sites where a waste producer disposes of their own garbage at the point of production, or locations used by several producers. Much land is used for waste management purposes, such as temporary storage, consolidation, transfer, sorting, treatment, or recycling [2].

Developing countries, such as India, are grappling with a major solid waste problem as their economies grow and urbanization accelerates. According to a 2006 assessment by Japan's Ministry of the Environment, the quantity of garbage generated globally in 2000 was around 12.7 billion tons, which is expected to increase to approximately 19 billion tons by 2025 and to nearly 27 billion tons by 2050. In India, municipal solid waste (MSW) generation was approximately 0.46 kg per person per day in 1995, which is expected to increase to 0.70 kg per person per day by 2025[3]. The majority of developing countries are currently dealing with the problem of increased MSW due to urbanization and industrialization.

Real-time monitoring and analysis of groundwater can be used to assess the geographical size, migration pattern, and possible risk level of particulate pollution. Particulate pollution, and its migration via groundwater, may be measured and tracked with high accuracy, down to 0.1 mm particles [4]. Monitoring systems for groundwater quality aid in assessing the likelihood and severity of contamination. Glass fragments, metals, papers, rags, plastics, ashes, and combustible materials are all included in the composition of MSW [5]. Other materials found in solid waste include leftover chemicals, paints, scrap metal, hazardous waste from decomposing animals, leftovers from industry, agriculture, and horticulture, and waste concrete and building materials resulting from demolition. Fuzzy logic models have been used to validate groundwater quality indices [6].

Any liquid that seeps through solid waste and extracts solutes, suspended solids, or any other harmful element from the material it has passed through is known as leachate. In developing nations, improperly designed landfill sites and a lack of adequate leachate collection and control systems lead to major environmental problems. Because a possible pollution source for leachate is located nearby, groundwater in landfill-adjacent areas is more vulnerable to contamination; numerous studies have examined the detrimental effects of landfill leachate on both the surface and groundwater in these areas [7–9]. Because leachate migrates readily from its origin through groundwater, point sources-like landfills-can discharge high concentrations of pollutants into the system [10]. Pollution and water disposal are inextricably linked. Numerous environmental risks, including water pollution, land pollution, air pollution, and health risks, are brought on by the open dumping of waste. The health impacts of leachate from landfill sites contaminating groundwater are a major concern for a large number of researchers and professionals worldwide.

The use of groundwater contaminated by leachate is a commonly reported threat to human health [11]. A suitable mathematical model for a realistic groundwater quality assessment can be formed by taking 1-D ADE. A numerical groundwater quality assessment model using a new fourth-order scheme with the Saulyev scheme has been presented. Landfill is a cause of environmental groundwater pollution. In many middle-income developing nations, landfill operations are the most practical. The measurement of groundwater pollutants is necessary to regulate the quality of drinking water. A mathematical model is presented. If a landfill project is intended to remain in the area under consideration, the anticipated impact on groundwater quality must be taken into account. A long-term groundwater quality assessment in a heterogeneous soil model has been suggested, with two numerical models presented. The groundwater pollution concentration in the vicinity of a landfill is estimated using the conventional forward time-centered space finite difference method. The solution is also approximated using the new Saulyev scheme in conjunction with the fourth-order finite difference technique. The estimated and ideal exact solutions are compared. Both numerical techniques give good approximate solutions. The proposed new fourth-order scheme with the Saulyev scheme provides better approximate solutions than the traditional method [12].

Long-term groundwater quality assessments surrounding landfill sites necessitate the use of a long-term numerical model that incorporates a modified fourth-order finite difference method with a Saulyev scheme. When reporting the environmental impact assessment of landfill site projects, a groundwater quality prediction is used. Two different fourth-order finite difference methods with Saulyev schemes have been proposed: the standard method and the modified method [13]. In a given scenario, the approximate and exact solutions are contrasted. Precise approximate solutions are obtained using the proposed modified fourth-order finite difference technique. Numerous forms of soil physics can be addressed using this numerical approach.

Groundwater flow and solute transport in homogeneous and heterogeneous porous media are described by mathematical models in the literature. Gardener and Yule [14] used traditional implicit and explicit finite difference methods, as well as alternating direction methods, to simulate groundwater. In their case, the finite difference methods were accurate. The forward time-centered space (FTCS) finite difference method was the fastest, followed by ADEM, ADIM, and BTCS, in that order. A water-driven head was added to the groundwater model to provide the groundwater level. The groundwater flow was estimated using the recognized limited distinction technique. The model's intricate geometry was taken into account by changing the grid sizes, aquifer parameters, and sink and source terms.

A system in which the aquifer is initially free of any contamination, was modeled using 2-D ADE [17]. Additionally, solute transport was modeled in porous media in geochemistry, geomorphology, and carbon cycling using percolation theory. Analytical solutions were derived for aqueous and solid-phase colloid concentrations in a porous medium, in which colloids are subject to advective transport and reversible retention, depending on time and/or depth. Previous researchers had taken into consideration the dependence of contamination on time and/or space [15–16].

For surface water and groundwater flow, a fully coupled depth-integrated model has been taken into consideration and solved; however, it was not the same as the problem studied in this work [18]. A comparison of the finite volume and finite difference methods for solving advection–diffusion–reaction equations numerically ware presented. For water pollution in reservoirs with one or two entrance gates and one exit gate, numerical solutions for 2-D ADRE models, for low and high rates of diffusion for three representative source terms, were also obtained using finite volume and finite difference methods. With the exception of areas close to the entrance and exit gates, it was discovered that the results from both methods generally agreed well [19]. The Lax–Wendroff method and the conventional upwind method are both two-level explicit methods that were used in a numerical groundwater quality assessment model to precisely estimate a better solution to the problem than the upwind method. In the future, the proposed simulation could be used to warn of risks posed by groundwater pollution near landfills [20].

In this research, a groundwater pollutant concentration dispersion flow problem through heterogeneous soil is considered. A groundwater pollutant dispersion model is introduced. A vertically averaged groundwater quality measurement with monitored boundary data is used to solve the problem, employing a two-dimensional advection–diffusion equation.

II. Governing Equation

A. Model of Groundwater Pollution Dispersion Flow Through Inhomogeneous Soil

The governing equation is a two-dimensional advection–diffusion partial differential equation:

$$\frac{\partial C}{\partial t}+V\cdot\nabla C=\nabla\cdot(\overline{K}\otimes C) \tag{1}$$

when V is the (average) fluid velocity field (mass per unit volume), K is the eddy – diffusivity or dispersion tensor, and $C(x,z,t)$ is the concentration of a pollutant at point (x,z,t). This equation may be written in 2-D form:

$$\frac{\partial C}{\partial t}=D_x f_1\frac{\partial^2 C}{\partial x^2}+D_z f_2\frac{\partial^2 C}{\partial z^2}-ug_1\frac{\partial C}{\partial x}-vg_2\frac{\partial C}{\partial z},$$

$$\frac{\partial C(x,z,t)}{\partial t}=D_x f_1(x,t)\frac{\partial^2 C(x,z,t)}{\partial x^2}+D_z f_2(z,t)\frac{\partial^2 C(x,z,t)}{\partial z^2}$$

$$-ug_1(x,t)\frac{\partial C(x,z,t)}{\partial x}-vg_2(z,t)\frac{\partial C(x,z,t)}{\partial z}. \quad (2)$$

Where u is a constant flow speed in the $X-$ direction, v is a constant flow speed in the $Z-$ direction, D_x is a constant dispersion coefficient in the horizontal direction, D_z is a constant dispersion coefficient in the vertical direction which $0 \le t \le T$

$$x_i = i\Delta x,\ i = 0,1,2,\dots,N\ ;$$
$$z_i = j\Delta z,\ j = 0,1,2,\dots,M\ ;$$
$$t_n = n\Delta t,\ n = 0,1,2,\dots,R\ ;$$

By using approximations $C_{i,j}^n$ to $C(i\Delta x, j\Delta z, n\Delta t)$ grid spacing are $\Delta z = \frac{W}{M}, \Delta x = \frac{L}{N}$ and $\Delta t = \frac{T}{R}$ where W is the wait, L is the length, and T is time.

B. *Initial and boundary conditions*

The initial state of the soil, free of groundwater contamination, implies the following initial condition for a hot start landfill source:

$$C(x,z,0) = f(x,z),\ f(x,z) = xz,\ t = 0, \quad (3)$$

which considered domain is depended on horizontal (x-axis) and vertical(z-axis) respectively by using sign $\forall(x,z)\in[0,1]+[0,1]$ that it is $0 \le f(x,z) \le 2$. Such this $f(x,z)$ is a given initially measured groundwater pollutant function. Due to a continuous input groundwater pollutant concentration is introduced at the origin, whereas the concentration gradient at the ended point is defined by the average rate of chance of groundwater pollutant concentration around them, the following boundary conditions are obtained,

$$C(x,z,t) = C_0, \quad t > 0, \quad (4)$$

$$\frac{\partial^2 C(x,z,t)}{\partial x^2} = C_s,\ \ x = L, \quad t \ge 0. \quad (5)$$

Where C_0 is a given averaged groundwater pollutant concentration at the considered landfill and C_s is rate of change of the pollutant concentration around the far field monitoring station.

III. Numerical techniques

A. *The Traditional Forward Time-Centered Space Method (FTCS)*

We have central difference scheme and forward difference scheme. The parameters in the horizontal term are

$$\frac{\partial^2 C(z,t)}{\partial x^2} = \frac{C(x_i+\Delta x, z_j, t_n) - 2C(x_i,z_j,t_n) + C(x_i-\Delta x, z_j, t_n)}{(\Delta x)^2} + O(\Delta x)^2$$
$$= \frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2} \quad (6)$$

$$\frac{\partial C(x,z,t)}{\partial x} = \frac{C(x_i+\Delta x, z_j, t_n) - C(x_i-\Delta x, z_j, t_n)}{2\Delta x} + O(\Delta x)^2$$
$$= \frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x} \quad (7)$$

$$\frac{\partial C(x,z,t)}{\partial t} = \frac{C(x_i, z_j, t_n+\Delta t) - C(x_i, z_j, t_n)}{\Delta t} + O(\Delta t)$$
$$= \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t} \quad (8)$$

and parameter z in vertical term, there are

$$\frac{\partial^2 C(x,z,t)}{\partial z^2} = \frac{C(x_i, z_j+\Delta z, t_n) - 2C(x_i,z_j,t_n) + C(x_i, z_j-\Delta z,, t_n)}{(\Delta z)^2} + O(\Delta z)^2$$
$$= \frac{C_{i,j+1}^n - 2C_{i,j}^n - C_{i,j-1}^n}{(\Delta z)^2} \quad (9)$$

$$\frac{\partial C(x,z,t)}{\partial z} = \frac{C(x_i, z_j+\Delta z, t_n) - C(x_i, z_j-\Delta z, t_n)}{2\Delta z} + O(\Delta z)^2$$
$$= \frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta z} \quad (10)$$

$$\frac{\partial C(x,z,t)}{\partial t} = \frac{C(x_i, z_j, t_n+\Delta t) - C(x_i, z_j, t_n)}{\Delta t} + O(\Delta t)$$
$$= \frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t} \quad (11)$$

The finite difference scheme approximates the solution; we will use the FTCS scheme as previously described by substituting Eqs.(3–8) into Eq(2).

$$\frac{\partial C(x,z,t)}{\partial t} = \frac{D_x f_1(x,t)\partial^2 C(x,z,t)}{\partial x^2} + \frac{D_z f_2(z,t)\partial^2 C(x,z,t)}{\partial z^2}$$
$$-\frac{ug_1(x,t)\partial C(x,z,t)}{\partial x} - \frac{vg_2(z,t)\partial C(x,z,t)}{\partial z} \quad (12)$$

$$\frac{C_{i,j}^{n+1} - C_{i,j}^n}{\Delta t} = D_x f_1(x,t)\left(\frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2}\right)$$
$$+D_z f_2(z,t)\left(\frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta z)^2}\right) - ug_1(x,t)\left(\frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x}\right)$$
$$-vg_2(z,t)\left(\frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta z}\right) \quad (13)$$

$$C_{i,j}^{n+1} - C_{i,j}^n = D_x f_1(x,t)\Delta t\left(\frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2}\right)$$
$$+D_z f_2(z,t)\Delta t\left(\frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta z)^2}\right)$$
$$-ug_1(x,t)\Delta t\left(\frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x}\right) - vg_2(z,t)\Delta t\left(\frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta z}\right). \quad (14)$$

$$C_{i,j}^{n+1} = D_x f_1(x,t)\Delta t\left(\frac{C_{i+1,j}^n - 2C_{i,j}^n + C_{i-1,j}^n}{(\Delta x)^2}\right)$$
$$+D_z f_2(z,t)\Delta t\left(\frac{C_{i,j+1}^n - 2C_{i,j}^n + C_{i,j-1}^n}{(\Delta z)^2}\right)$$
$$-ug_1(x,t)\Delta t\left(\frac{C_{i+1,j}^n - C_{i-1,j}^n}{2\Delta x}\right)$$
$$-vg_2(z,t)\Delta t\left(\frac{C_{i,j+1}^n - C_{i,j-1}^n}{2\Delta z}\right) + C_{i,j}^n \quad (15)$$

$$C_{i,j}^{n+1} = \frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i+1,j}^n - \frac{2D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i,j}^n$$
$$+\frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i-1,j}^n + \frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j+1}^n$$
$$-\frac{2D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j}^n + \frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j-1}^n$$
$$-\frac{ug_1(x,t)\Delta t}{2\Delta x}C_{i+1,j}^n + \frac{ug_1(x,t)\Delta t}{2\Delta x}C_{i-1,j}^n$$
$$-\frac{vg_2(z,t)\Delta t}{2\Delta z}C_{i,j+1}^n + \frac{vg_2(z,t)\Delta t}{2\Delta z}C_{i,j-1}^n + C_{i,j}^n \quad (16)$$

$$C_{i,j}^{n+1} = \frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i+1,j}^n + \frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j+1}^n$$
$$-\frac{ug_1(x,t)\Delta t}{2\Delta x}C_{i+1,j}^n - \frac{vg_2(z,t)\Delta t}{2\Delta z}C_{i,j+1}^n$$
$$-\frac{2D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i,j}^n - \frac{2D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j}^n + C_{i,j}^n$$
$$+\frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i-1,j}^n + \frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j-1}^n$$
$$+\frac{ug_1(x,t)\Delta t}{2\Delta x}C_{i-1,j}^n + \frac{vg_2(z,t)\Delta t}{2\Delta z}C_{i,j-1}^n \quad (17)$$

$$C_{i,j}^{n+1} = \frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i-1,j}^n + \frac{ug_1(x,t)\Delta t}{2\Delta x}C_{i-1,j}^n$$
$$+\frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j-1}^n + \frac{vg_2(z,t)\Delta t}{2\Delta z}C_{i,j-1}^n$$
$$-\frac{2D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i,j}^n - \frac{2D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j}^n + C_{i,j}^n$$
$$+\frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2}C_{i+1,j}^n - \frac{ug_1(x,t)\Delta t}{2\Delta x}C_{i+1,j}^n$$
$$+\frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2}C_{i,j+1}^n - \frac{vg_2(z,t)\Delta t}{2\Delta z}C_{i,j+1}^n \quad (18)$$

The explicit finite difference method (EFDM)

$$C_{i,j}^{n+1} = \left(\frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2} + \frac{ug_1(x,t)\Delta t}{2\Delta x}\right)C_{i-1,j}^n$$
$$+\left(\frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2} + \frac{vg_2(z,t)\Delta t}{2\Delta z}\right)C_{i,j-1}^n$$
$$+\left(-\frac{2D_x f_1(x,t)\Delta t}{(\Delta x)^2} - \frac{2D_z f_2(z,t)\Delta t}{(\Delta z)^2} + 1\right)C_{i,j}^n$$
$$+\left(\frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2} - \frac{ug_1(x,t)\Delta t}{2\Delta x}\right)C_{i+1,j}^n$$
$$+\left(\frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2} - \frac{vg_2(z,t)\Delta t}{2\Delta z}\right)C_{i,j+1}^n \quad (19)$$

Where:

$$E_i = \frac{D_x f_1(x,t)\Delta t}{(\Delta x)^2},\ F_j = \frac{D_z f_2(z,t)\Delta t}{(\Delta z)^2},\ G_i = \frac{ug_1(x,t)\Delta t}{2\Delta x},$$
$$H_i = \frac{vg_1(z,t)\Delta t}{2\Delta z}$$

Thus form the EFDM become to an equation

$$C_{i,j}^{n+1} = (E_i + G_i)C_{i-1,j}^n + (F_j + H_j)C_{i,j-1}^n$$
$$+(1 - 2E_i - 2F_j)C_{i,j}^n - (E_i - G_i)C_{i+1,j}^n$$
$$+(F_j - H_j)C_{i,j+1}^n \quad (20)$$

To obtain the approximate solution of the Eq. (1) with the boundary and initial conditions using the explicit finite difference method (EFDM).Its considered domain is depended on horizontal (x-axis) and vertical(z-axis) respectively by using sign $\forall(x,z) \in [0,1]+[0,1]$ that it is $0 \le f(x,z) \le 2$. Such this $f(x,z)$ is a given initially measured groundwater pollutant function.

IV. Numerical experiments

Suppose that the measurement of groundwater pollutant concentration under a landfill and their vicinity is considered. The considered area is aligned between horizontal distance and vertical dept, 1.0 km^2 total area. There is a landfill which discharging leachate as pollutant source into the underground and when the pollutant parameters at the considered landfill are D_x =0.71 km^2/year, D_z= 0.71 km^2/year, a=1 km, u_x=0.60 km^2/year, v_z =0.20 km^2/year and a=1km^{-2}. In the numerical experiment, the space and time are discretized by km^2, and year. The groundwater concentration is approximated by using the explicit finite difference method (EFDM).

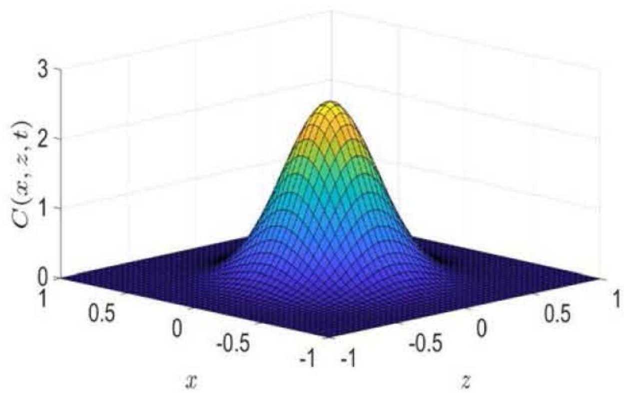

Fig.1. Contaminant concentration variation with depth and distance, for a fixed time period of T = 1 year.

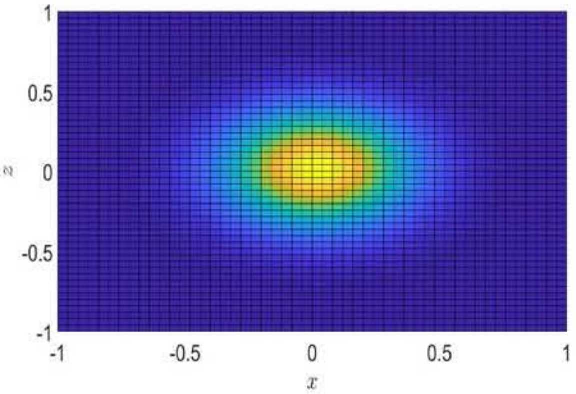

Fig.2. Contour plot showing the variation of contamination level with depth and distance.

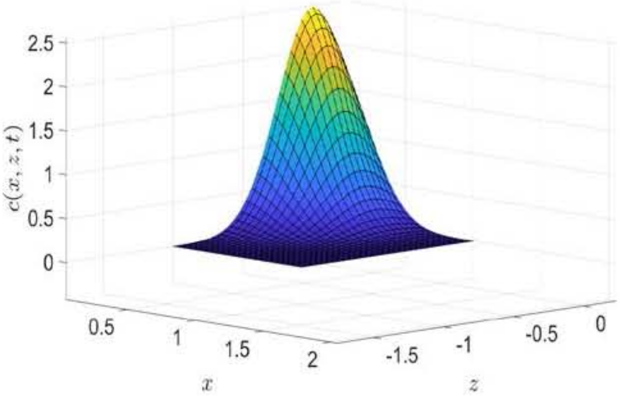

Fig.3. Concentration of contaminants with depth and distance using a deformable mesh.

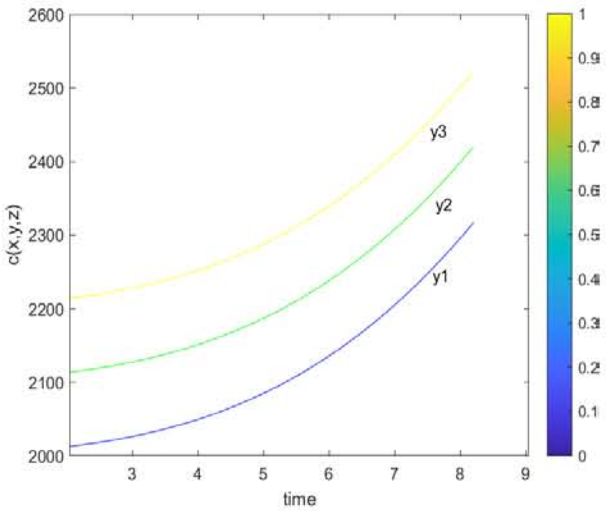

Fig.4. Concentration of contaminants with depth and distance along a fixed line parallel to the Y-axis.

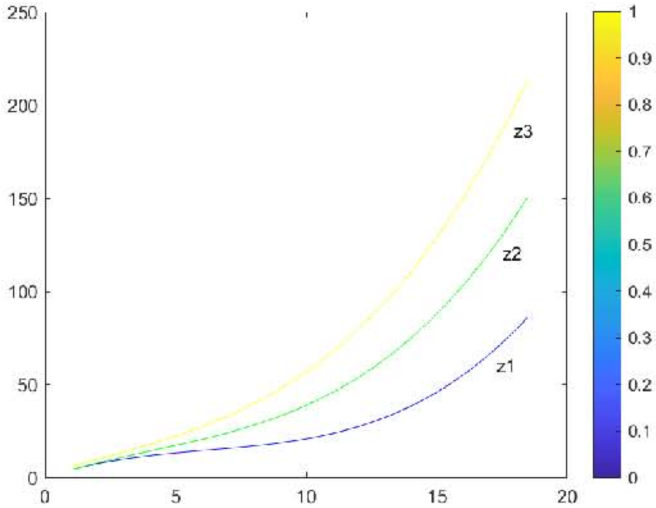

Fig.5. Contaminant concentration with depth and distance along a fixed line parallel to the Z-axis.

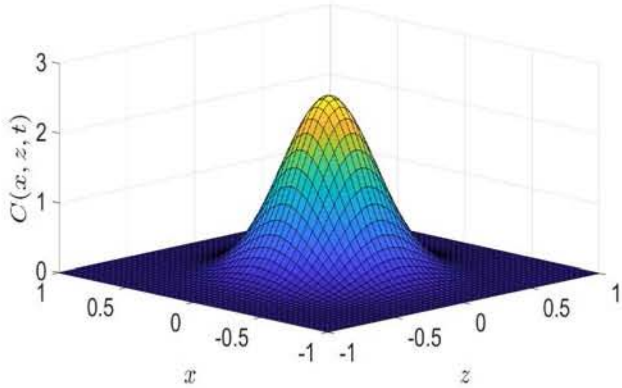

Fig.6. Contaminant concentration variation with depth and distance, for a fixed time period of $T = 5$ years.

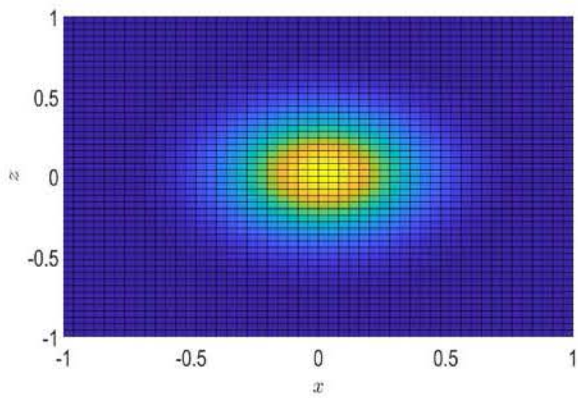

Fig.7. Contour plot of the variation of contamination level with depth and distance.

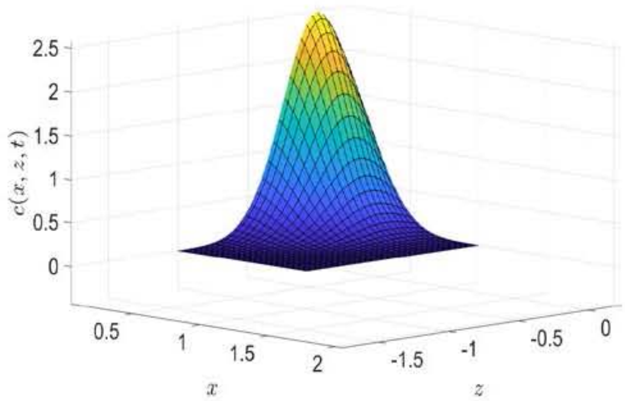

Fig.8. Variation of the concentration of contaminants with depth and distance using a deformable mesh.

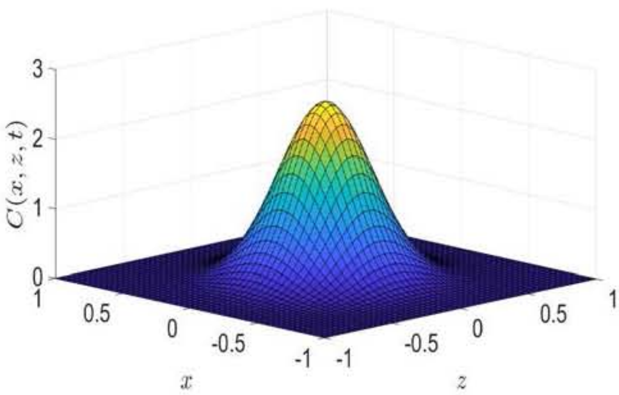

Fig.11. Variation of contaminant concentration with depth and distance, for a fixed time period of $T = 10$ years.

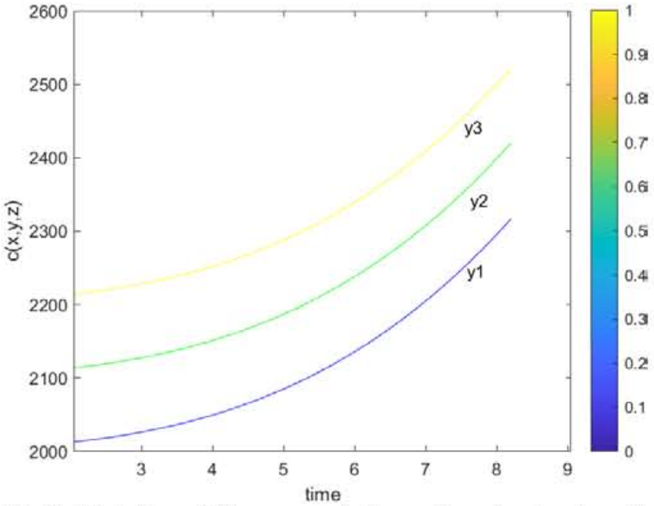

Fig.9. Variation of the concentration of contaminants with depth and distance along a fixed line parallel to the Y-axis.

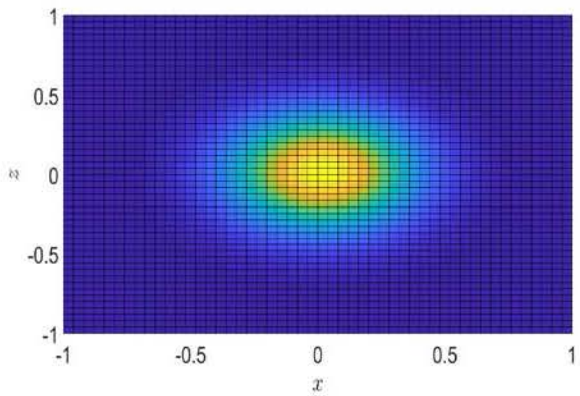

Fig.12. Contour plot showing the variation of contamination level with depth and distance.

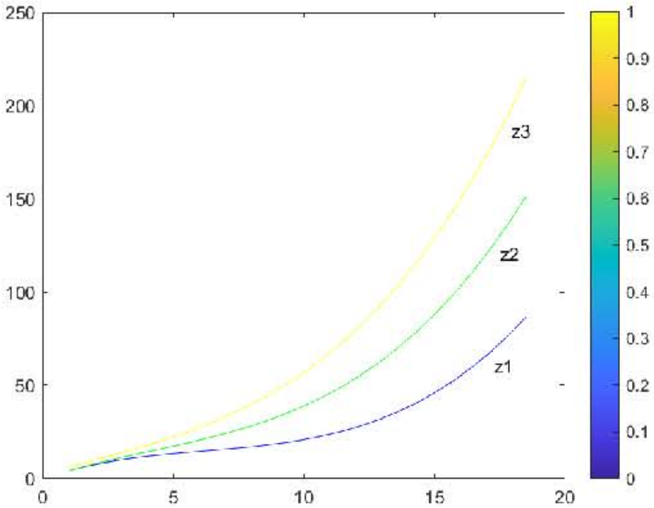

Fig.10. Variation of the contaminant concentration with depth and distance along a line parallel to the Z-axis.

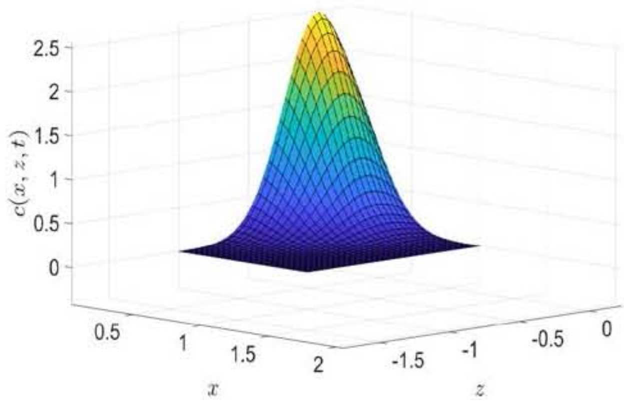

Fig.13. Variation of the concentration of contaminants with depth and distance using a deformable mesh.

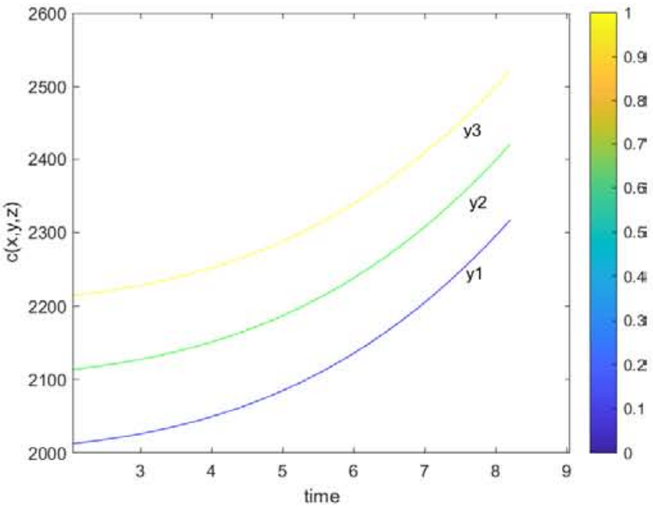

Fig.14. Variation of the concentration of contaminants with depth and distance along a fixed line parallel to the Y-axis.

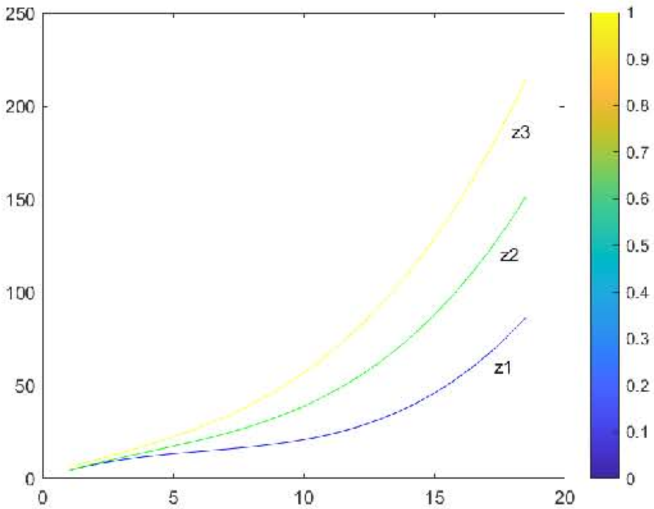

Fig.15. Variation of contaminant concentration with depth and distance along a fixed line parallel to the Z-axis.

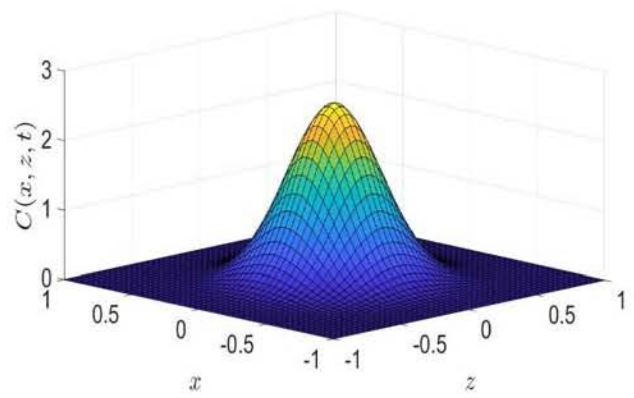

Fig.16. Variation of contaminant concentration with depth and distance for a fixed time period of $T = 15$ years.

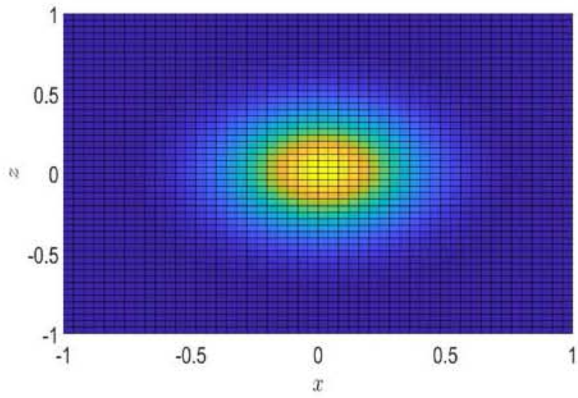

Fig.17. Contour plot showing the variation of contamination level with depth and distance.

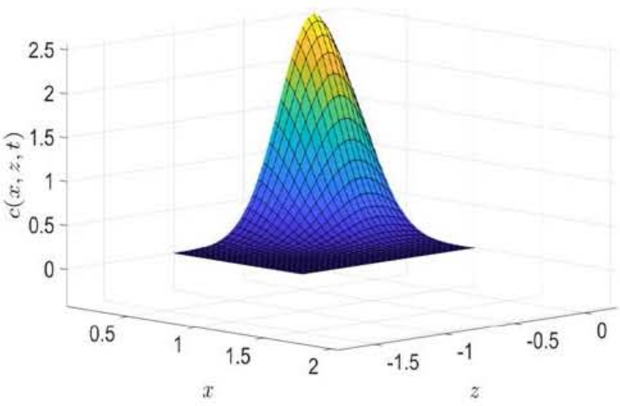

Fig.18. Variation of the concentration of contaminants with depth and distance using a deformable mesh.

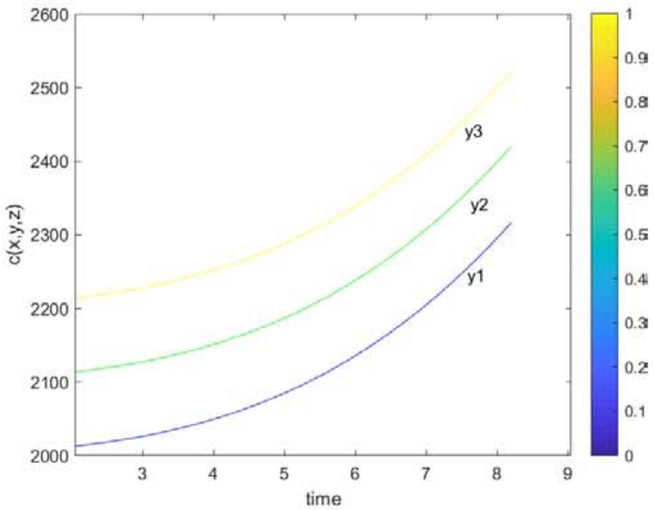

Fig.19. Variation of the concentration of contaminants with depth and distance along a fixed line parallel to the Y-axis.

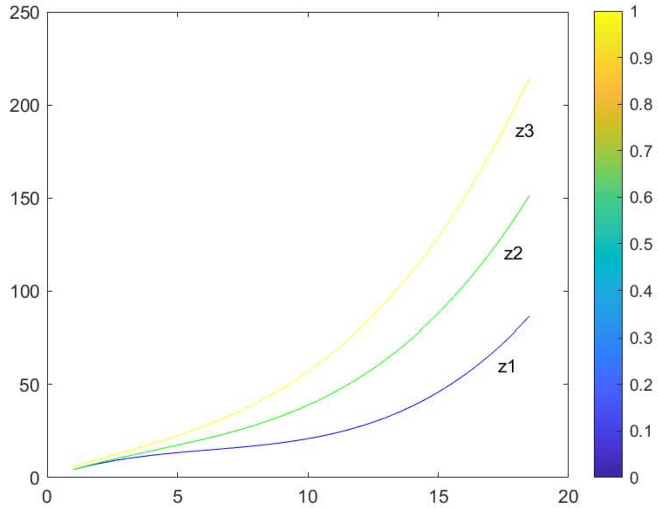

Fig.20. Variation of the contaminant concentration with depth and distance along a fixed line parallel to the Z-axis.

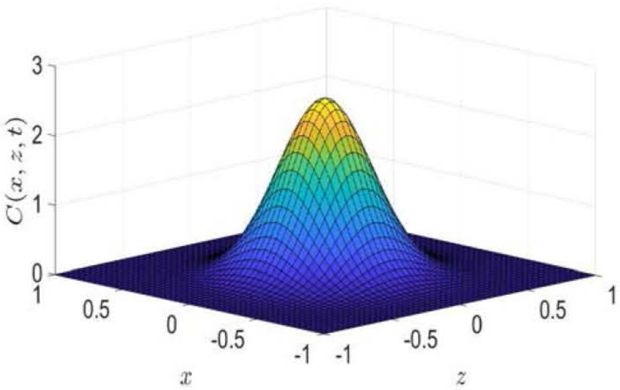

Fig.21. Variation of contaminant concentration with depth and distance for a fixed time period of T = 21 years.

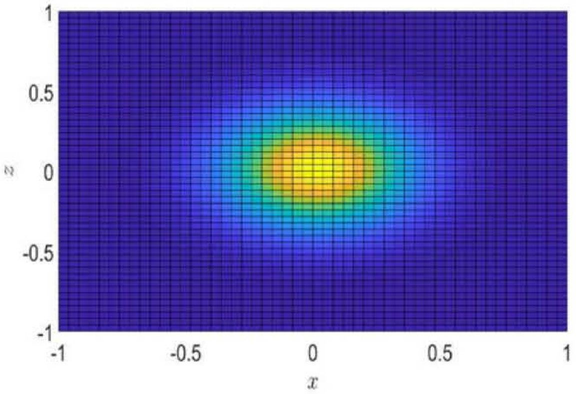

Fig.22. Contour plot showing the variation of contamination level with depth and distance

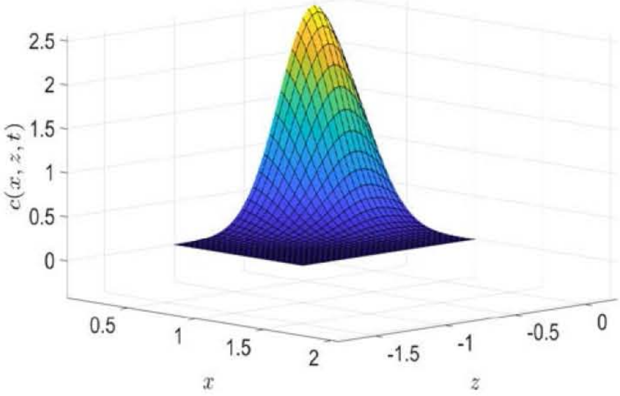

Fig.23. Variation of the concentration of contaminants with depth and distance using a deformable mesh.

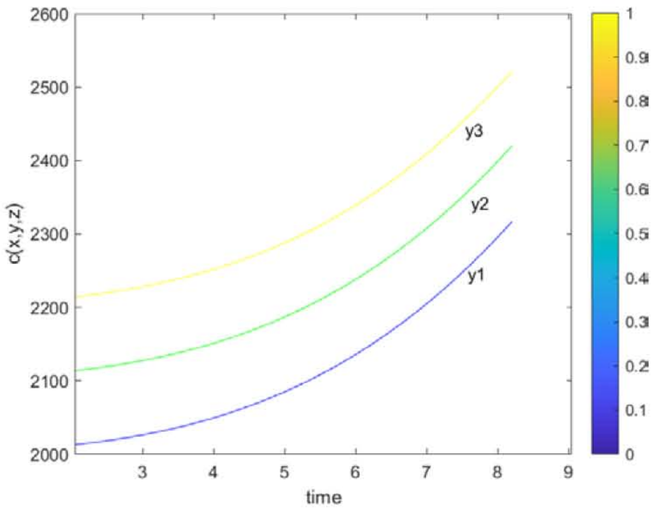

Fig.24. Variation of the concentration of contaminants with depth and distance along a fixed line parallel to the Y-axis.

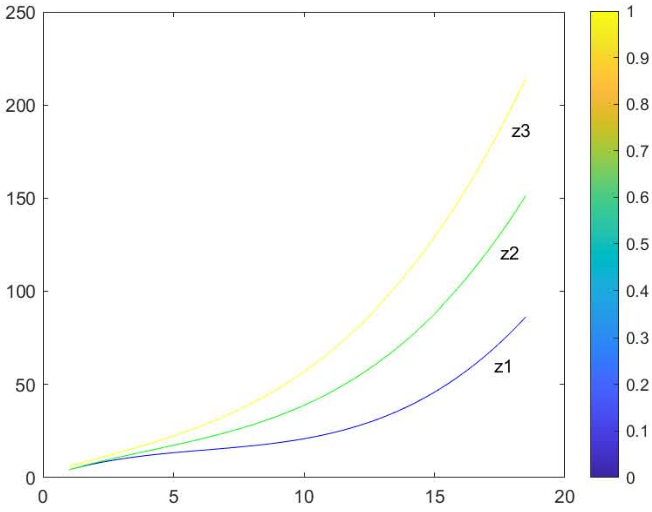

Fig.25. Variation of the contaminant concentration with depth and distance along a fixed line parallel to the Z-axis.

TABLE I GROUNDWATER POLLUTANT CONCENTRATION AFTER 1 YEAR APPROXIMATED BY USING THE EFDM.

j \ i	0	1	2	3	4
0	0.0000	0.8895	0.8750	0.8895	1.0000
1	0.8895	0.7644	03468	0.7254	0.8125
2	0.8750	0.3588	0.0000	0.3228	0.7501
3	0.8895	0.8351	0.4464	0.7961	0.8125
4	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE II GROUNDWATER POLLUTANT CONCENTRATION AFTER 5 YEARS APPROXIMATED BY USING THE EFDM.

j \ i	0	1	2	3	4
0	0.0000	0.8195	0.8010	0.8195	0.9909
1	0.8195	0.7999	0.4897	0.7999	0.80560
2	0.8010	0.4001	0.0000	0.4001	0.7003
3	0.8195	0.7835	0.4464	0.7961	0.80560
4	0.9909	0.9909	0.9909	0.9909	0.9909

TABLE III GROUNDWATER POLLUTANT CONCENTRATION AFTER 10 YEARS APPROXIMATED BY USING THE EFDM.

j \ i	0	1	2	3	4
0	0.0000	0.8006	0.8750	0.8005	0.9905
1	0.8006	0.6244	0.5690	0.6254	0.8003
2	0.8750	0.5988	0.00000081	0.5988	0.6989
3	0.8005	0.6998	0.5702	0.6096	0.7587
4	0.9905	0.9905	0.9905	0.9905	0.9905

TABLE IV GROUNDWATER POLLUTANT CONCENTRATION AFTER 15 YEARS APPROXIMATED BY USING THE EFDM.

j \ i	0	1	2	3	4
0	0.0000	0.8001	0.8050	0.8001	0.9899
1	0.8001	0.6899	0.6008	0.6899	0.7798
2	0.8050	0.7002	0.0000098	0.7106	0.6804
3	0.8001	0.6905	0.6104	0.6910	0.6951
4	0.9899	0.9899	0.9899	0.9899	0.9899

TABLE V GROUNDWATER POLLUTANT CONCENTRATION AFTER 20 YEARS APPROXIMATED BY USING THE EFDM.

j \ i	0	1	2	3	4
0	0.0000	0.7879	0.7989	0.7899	0.8999
1	0.7879	0.7009	0.7098	0.7174	0.7201
2	0.7989	0.7588	0.00008891	0.7598	0.7006
3	0.7899	0.7065	0.7069	0.7071	0.7028
4	0.8999	0.8999	0.8999	0.8999	0.8999

TABLE VI ASSESSMENT AND COMPARISON RESULTS OF GROUNDWATER POLLUTION CONCENTRATION DURING 1 YEAR, 5 YEARS, 10 YEARS, 15 YEARS, AND 20 YEARS BY FIXED Y-AXIS.

i=4 / j	1 YEAR	5 YEARS	10 YEARS	15 YEARS	20 YEARS
0	1.0000	0.9909	0.9905	0.9899	0.8999
1	1.0000	0.9909	0.9905	0.9899	0.8999
2	1.0000	0.9909	0.9905	0.9899	0.8999
3	1.0000	0.9909	0.9905	0.9899	0.8999
4	1.0000	0.9909	0.9905	0.9899	0.8999

TABLE VII ASSESSMENT AND COMPARISON RESULTS OF GROUNDWATER POLLUTION CONCENTRATION DURING 1 YEAR, 5 YEARS, 10 YEARS, 15 YEARS, AND 20 YEARS BY FIXED Z-AXIS.

j=4 / i	1 YEAR	5 YEARS	10 YEARS	15 YEARS	20 YEARS
0	1.0000	0.9909	0.9905	0.9899	0.8999
1	0.8125	0.8056	0.8003	0.7798	0.7201
2	0.7501	0.7003	0.6989	0.6804	0.7006
3	0.8125	0.8056	0.7587	0.6950	0.7028
4	1.0000	0.9909	0.9905	0.9899	0.8999

V. Discussion

An EFDM is considered for a two-dimensional advection–diffusion equation by employing an FTCS scheme to give a good agreement with the approximated groundwater pollutant concentration in an ideal case, as shown in Figs. 1, 6, 11, 16, and 21. In this case, the groundwater pollutant measurement is simulated for a long period of time, around 1–20 years, as shown in Tables I–VII and Figs. 3, 8, 13, 18, and 23. The proposed numerical techniques provide an accurate approximate solution.

VI. Conclusion

In 2-D heterogeneous soil, a vertically averaged groundwater quality measurement with monitored boundary data was tested. The groundwater quality model was updated and used over an extended period of time. The concentrations of groundwater pollutants at the monitoring stations were taken to be the initial and boundary conditions of the model, and the polluting concentration positions were estimated through numerical techniques. An EFDM, namely, the conventional forward time-centered space finite difference technique, was used to approximate the model solution in the two-dimensional advection–diffusion equation. It is possible to forecast future groundwater contamination events by using the proposed simulation technique. These numerical methods produce an approximate solution that is accurate and do not lead to an excessive amount of numerical diffusion.

Furthermore, this research could be beneficial in mathematics education. It could be used to teach Grade 12 calculus-related concepts in a project-based learning approach, helping students to develop their skills so that they can apply mathematics to realistic problems.

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