Single Server Queueing-inventory System with Impatient Customers and Hybrid Vacations

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Abstract—**This paper investigates a single-server queuing inventory system with impatient customers and multiple vacation strategies, where the first vacation is a working vacation. After a customer completes the service, he or she will bring a product away with probability r or not to bring a product away with probability $1 - r$. Based on the M/M/1 queuing model, we construct a three-dimensional Markov process to represent the number of customers, inventory level, and server state in the system. The steady-state conditions of the system are obtained by the Neuts-Rao truncation method. Using the matrix geometric solution method, the steady-state performance indicators are derived. In addition, the cost function is established to do optimization analysis of the system. Finally, numerical experiments are conducted to analyze the effects of variations in system parameters on the performance indicators, and to determine the optimal inventory strategy and minimum cost under specific system parameters.**

Index Terms—**inventory system, impatient customers, optional consumption, quasi-birth and death process**

I. INTRODUCTION

THE queuing phenomenon is very common in daily life, such as when buying things in the mall, handling business in the bank, etc. The inventory level in a service system will reduce as customers consume products. When the inventory reaches a certain level, the manager should send a replenishing request in time to meet customer demand. If the customer demand is not satisfied, the customer will leave the system dissatisfied, thus causing a loss to the system. Therefore, proper inventory control is of great significance to the system. Sigman and Simchi-Levi [1] first introduced the inventory policy to the queuing system. Based on the M/G/1 queuing model, they investigated the queuing inventory system with limited inventory and obtained this system's performance indicators using a matrix geometric solution. Berman et al. [2] investigated an inventory management system with service facilities. Under the assumption that demand and service rates are fixed constants, it is possible to determine both the optimal order quantity and the associated system cost. Schwarz et al. [3] formally defined the system combining queuing and inventory as a queuing-inventory system. In this system, customer arrivals follow a Poisson distribution. Service times and replenishment times follow an exponential distribution. They analyzed several different replenishment strategies for a queuing-inventory system, based on the M/M/1 model with lost sales when inventory is empty. Finally, the steady-state probability distribution of the system can be obtained. Saffari et al. [4] continued to study the M/M/1 queuing inventory system more deeply. They incorporated the concept of lost sales and extended the distribution of replenishment times to a general distribution. Performance metrics and cost functions were established to determine the optimal order quantity.

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Daniel and Ramanarayanan [5] first integrated vacation theory into the (s, S) inventory system. When inventory reaches zero, the server immediately enters a vacation state, and customers are immediately lost. They used renewal theory and convolution theorem to determine the steady-state probability of the system. Krishnamoorthy and Viswanath [6] investigated a production inventory system with server vacations, where production time and customer arrivals have Markovian. Finally, they derived the system's stability, state distribution, and various performance metrics. Yue and Qin [7] analyzed a production-inventory system with server vacations and production equipment downtime. Additionally, the steady-state joint distribution of inventory level and queue length can be determined by using the quasi-birth and death process. Then they established a cost function and conducted numerical analysis. Zhang [8] studied a vacation queuing-inventory system with an (s, S) inventory strategy and multiple vacation strategies. Moreover, matrix analysis along with the theory of Markov processes was utilized to ascertain the steady-state probability distribution of the system. The relationship between vacation rate and performance indicators was analyzed. At last, under service level constraints, the optimal inventory strategy and cost were determined using a genetic algorithm. Ye and Yue [9] studied a queuing inventory system with synchronized multiple vacations at partial servers. They derived the system's steady-state probability and performance indicators using a quasi-birth-and-death process and matrix-geometric solution. Finally, they analyzed the effect of each parameter on the cost function through numerical experiments. Xu et al. [10] introduced a hybrid vacation mechanism that combines multiple vacations with single working vacations. They studied the queuing inventory system model with lost sales and (s, S) strategy. Using Markov theory, they derived the steady-state probability vector of the system and related performance indicators. Besides, they also established the optimal cost function and analyzed the impact of various parameters on this function.

In real life situations, customers tend to become impatient while waiting in line and may exit the queue, causing disruptions to the system. Haight [11] was the first to study the M/M/1 queuing model with stops in 1957. In this model, when the number of people in the system exceeds k, customers no longer enter the system. Moreover, Haight [12] in 1959 continued to study the M/M/1 queuing model with customers exiting midway. If a customer's waiting time exceeds a certain threshold, they choose to leave the system. Kumar [13] studied that customers who enter the system with probability p and probability $q(= 1 - p)$ do not enter the system. When the system population does not exceed k, differential equations were used to derive the steady-state probabilities. In 2015, Ammar [14] studied the M/M/1 model with impatient customers and system vacations. The server enters a vacation state after completing service, and during this period, only impatient customers are observed. If a customer arrives while the server is on vacation and their waiting time exceeds the maximum threshold, they will exit the system. In addition, the system state was analyzed and numerically illustrated to determine the impact of each parameter on performance indicators. Melikov et al. [15] investigated queuing inventory systems with impatient customers, considering both finite and infinite waiting spaces. They used exact and approximate algorithms to analyze the models and conduct numerical experiments. Shan and Yue [16] studied the M/M/1/N queuing inventory system model with impatient customers and multiple working vacations. They considered two cases: the server takes a vacation when inventory is empty, and the server takes a vacation when there are no customers. The steady-state probability distribution and related performance indicators were obtained using matrix-iterative methods. Fu et al. [17] examined two fault types involving a standby service station and initiation time within the framework of an M/M/1 queuing model. They derived the steady-state equilibrium conditions and probability vectors for the system through matrix geometric methods, calculated the steady-state queue length, and ultimately performed numerical analysis using Matlab to interpret their findings. Yang et al. [18] investigated the M/M/1 repairable queuing system characterized by two types of server failures and passive clients, employing quasi-birth-and-death (QBD) processes along with matrix geometric techniques for their analysis. They presented the steady-state conditions, derived the probability vectors at equilibrium, and calculated various steady-state queuing metrics as well as reliability measures.

In the literature on queuing-inventory systems, it is often assumed that customers must take a product when they leave the system. However, Krishnamoorthy et al. [19] studied a queuing-inventory system with service times, considering both (s, Q) and (s, S) inventory strategies. They examined customers either take one product or not after receiving the service with a certain probability. By calculating the marginal product of the joint distribution of customer and inventory numbers, the optimal inventory strategy and the optimal cost are determined. Manikandan and Nair [20] studied retry M/M/1/1 queueing inventory system with impatient customers, where a customer may takes one product with a certain probability after service. The study also considers the lost sales when the inventory is empty. Based on the (s, Q) inventory policy, they calculated the system's steady-state probabilities, provided performance indicators and a cost function. Dinkai et al. [21] studied the inventory system model with geometric batch demand. In this model, when the inventory is empty, the server begins multiple vacations. The number of products required by the customer to receive the service obeys the geometric distribution. What's more, they used the quasi-birth-and-death process and matrix-geometric solutions to obtain the system's steady-state distribution. Additionally, a genetic algorithm was used to perform a sensitivity analysis of the system's parameters.

This paper considers both impatient customers and customers who consume a product with probability r. It adopts the hybrid vacation model from [10] to make the model more realistic. Based on the (s, S) inventory strategy, the paper studies an M/M/1 queuing-inventory model with impatient customers and hybrid vacations. When the inventory is empty, the server initiates the first vacation as a working vacation in the multiple vacation strategy. A three-dimensional Markov process is established, incorporating the number of customers, inventory level, and server status. Finally, the Neuts-Rao truncation method is used to determine the steady-state probability vector, and we provide several performance indicators. Furthermore, numerical experiments using a genetic algorithm are conducted to find the optimal production strategy and cost.

Section 2 of this paper provides a detailed description of the system model. Section 3 conducts a steady-state analysis of the system, deriving the steady-state balance conditions and presenting the steady-state distribution using matrix-geometric solutions. Section 4 gives some relevant system performance indicators. Section 5 examines the impact of varying system parameters on system performance indicators, constructs a mean cost function, and analyzes parameter sensitivities through numerical experiments.Section 6 gives the conclusion of this paper.

II. Model Description

1) Service Rules: There is a single server present in the system. The service time of each customer follows the exponential distribution, and the server only serves one customer. The system adopts the First-Come-First-Served ($FCFS$) discipline.

2) Customer Arrival: Customer arrival follows the Poisson process with rate $\lambda\,(\lambda > 0)$. The customer accepts the service when the inventory in the system is not zero. After the service is completed, the customer will either take away one product with the probability $r(0 < r < 1)$ or not to take away one product with the probability $1 - r$.

3) Vacation Policy: The server takes a multiple vacation policy where the first vacation is a working vacation. When the system inventory level is zero, the server initiates the first vacation, which is a working vacation. During the working vacation period, the server stops serving if the inventory is zero. If the products arrive, the server serves a customer at a lower service rate μ_w, similar to the regular busy period. The customer still takes a product with probability r. When the working vacation ends and the inventory level is not zero, the server goes directly into the regular busy period. When the working vacation ends and the inventory level is still zero, the server initiates a full vacation. During the full vacation period, the server stops any service. If the inventory is non-zero after the full vacation, the server enters the regular busy period, during which the service rate is

$\mu_b\,(\mu_b > \mu_w)$. Otherwise, if the inventory remains empty, the server initiates another full vacation. If the inventory level is non-zero at the end of the working vacation, the server goes directly into the regular busy period. Working vacation time and full vacation time follow exponential distributions with parameters $\theta_w\,(\theta_w > 0)$ and $\theta_v\,(\theta_v > 0)$, respectively. As shown in Figure 1:

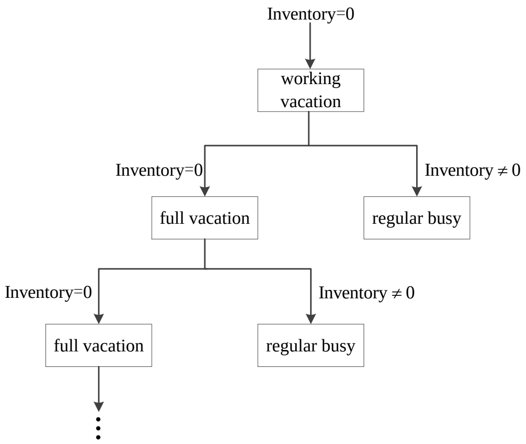

Fig. 1. The server vacation process.

4) Customer State: When the server on a working vacation, the waiting customers in the system may become impatient due to the low service rate of the server. The impatient waiting time T follows an exponential distribution with parameter ξ. When the server is in the regular busy period, the waiting customers will not be impatient.

5) The system is loss-based: When a customer enters the system, if it is during a working vacation or a standard busy period, they will wait until the service is completed before leaving. Conversely, if the system is on the full vacation, customers will not enter the system.

6) Replenishment Strategy: The system takes the (s, S) replenishment strategy. A replenishment demand will be sent when the system inventory level drops to s. The inventory level will get to $S(s < S)$ one time after a replenishment time which obeys an exponential distribution with parameter $\eta(\eta > 0)$.

7) The customer arrival process, server vacation time, system replenishment time, customer impatience time, and server service time during working vacations and regular busy periods are all independent of each other.

III. Steady-State Analysis

A. Steady-state Distribution

Let $N(t)$ denotes the number of customers in the system at time t, and $I(t)$ denotes the inventory level of the system at time t and $J(t)$ denotes the state of the server in the system at time t. The definition of $J(t)$ is as follows:

$$J(t) = \begin{cases} 0, \text{system is in full vacation period} \\ \quad \text{at time t,} \\ 1, \text{system is in working vacation period} \\ \quad \text{at time t,} \\ 2, \text{system is in working vacation period} \\ \quad \text{at time t.} \end{cases}$$

Then $\phi(t) = \{(N(t), I(t), J(t)), t \geq 0\}$ is a Markov process with the state space: $\Omega = \{(n, 0, j), j = 0, 1\} \cup \{(n, i, j), j = 1, 2\} \cup \{(n, S, j), j = 0, 1, 2\}$.

The state transition diagram of the system is shown in Figure 2.

The state process $\phi(t)$ is a level-dependent quasi-birth-and-death $(LDQBD)$ process, and the infinitesimal generator of the process can be written as follows:

$$Q = \begin{pmatrix} A_0 & C & & & & & \\ & B_1 & A_1 & C & & & \\ & & B_2 & A_2 & C & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & B_k & A_k & C \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} a_{00} & & & & & & & h_0 \\ & a_{01} & & & & & & h_1 \\ & & \ddots & & & & & \vdots \\ & & & a_{01} & & & & h_1 \\ & & & & a_{02} & & & \\ & & & & & \ddots & & \\ & & & & & & a_{02} & \\ & & & & & & & a_{03} \end{pmatrix},$$

where

$$a_{00} = \begin{pmatrix} -\eta & 0 \\ \theta_w & -(\eta + \lambda + \theta_w) \end{pmatrix},$$

$$a_{01} = \begin{pmatrix} -(\eta + \lambda + \theta_w) & \theta_w \\ 0 & -(\eta + \lambda) \end{pmatrix},$$

$$a_{02} = \begin{pmatrix} -(\lambda + \theta_w) & \theta_w \\ 0 & -\lambda \end{pmatrix},$$

$$a_{03} = \begin{pmatrix} -\theta_v & 0 & \theta_v \\ 0 & -(\lambda + \theta_w) & \theta_w \\ 0 & 0 & -\lambda \end{pmatrix},$$

$$h_0 = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \end{pmatrix}, h_1 = \begin{pmatrix} 0 & \eta & 0 \\ 0 & 0 & \eta \end{pmatrix},$$

$$A_i\,(i \geq 1) = \begin{pmatrix} a_{i0} & & & & & & & h_0 \\ & a_{i1} & & & & & & h_1 \\ & & \ddots & & & & & \vdots \\ & & & a_{i1} & & & & h_1 \\ & & & & a_{i2} & & & \\ & & & & & \ddots & & \\ & & & & & & a_{i2} & \\ & & & & & & & a_{i3} \end{pmatrix},$$

where

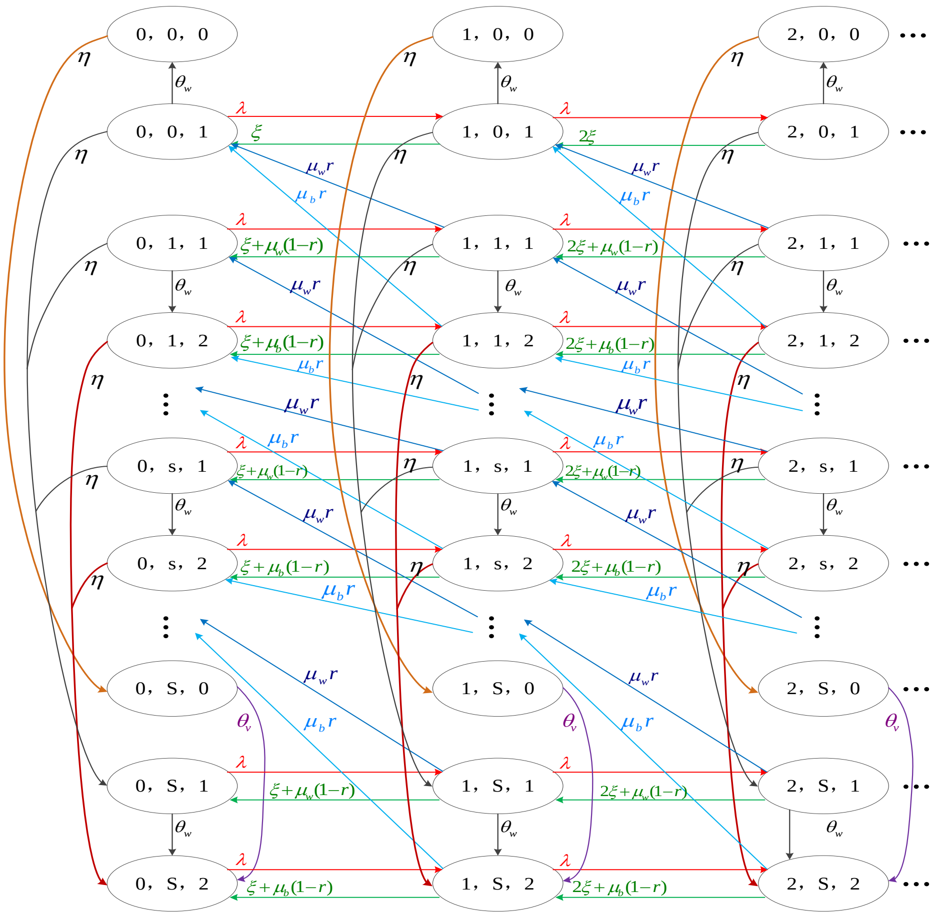

Fig. 2. The state transition diagram of the system.

$$a_{i0} = \begin{pmatrix} -\eta & 0 \\ \theta_w & -(\eta+\lambda+i\xi) \end{pmatrix},$$

$$a_{i1} = \begin{pmatrix} -(\eta+\lambda+\theta_w+i\xi+\mu_w) & \theta_w \\ 0 & -(\eta+\lambda+\mu_b) \end{pmatrix},$$

$$a_{i2} = \begin{pmatrix} -(\lambda+\theta_w+i\xi+\mu_w) & \theta_w \\ 0 & -(\lambda+\mu_b) \end{pmatrix},$$

$$a_{i3} = \begin{pmatrix} -\theta_v & 0 & \theta_v \\ 0 & -(\lambda+\theta_w+i\xi+\mu_w) & \theta_w \\ 0 & 0 & -(\lambda+\mu_b) \end{pmatrix},$$

$$B_i\,(i \geq 1) = \begin{pmatrix} b_{i0} & & & & \\ U_0 & b_{i1} & & & \\ & U_1 & b_{i1} & & \\ & & \ddots & \ddots & \\ & & & 0 & 0 \\ & & & & U_1 \quad b_{i1} \end{pmatrix},$$

where

$$b_{i0} = \begin{pmatrix} 0 & 0 \\ 0 & i\xi \end{pmatrix},$$

$$b_{i1} = \begin{pmatrix} i\xi+\mu_w(1-r) & 0 \\ 0 & \mu_b(1-r) \end{pmatrix},$$

$$U_0 = \begin{pmatrix} 0 & \mu_w r \\ 0 & \mu_b r \end{pmatrix},\ U_1 = \begin{pmatrix} \mu_w r & 0 \\ 0 & \mu_b r \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & & & & & \\ & \lambda & & & & \\ & & \ddots & & & \\ & & & 0 & & \\ & & & & \lambda & \\ & & & & & \lambda \end{pmatrix}.$$

$A_i(i \geq 0)$, $B_i(i \geq 1)$ and C are all square matrices of the order $2S+3$.

B. Steady-state Balance Condition

The steady-state probability vector of the system is solved according to the Neuts-Rao truncation approximation method (details can be found in the literature [22]). We assume that the $LDQBD$ process does not change anymore from a certain level, the generator matrix of this truncated-tailed can be obtained:

$$Q^* = \begin{pmatrix} A_0 & C & & & & & & \\ B_1 & A_1 & C & & & & & \\ & B_2 & A_2 & C & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & B_N & A_N & C & & \\ & & & & B_N & A_N & C & \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

From the matrix of Q^*, the process is a quasi-birth-and=birth process. Let $H = B_N + A_N + C$, then we get:

$$H = \begin{pmatrix} G_0 & & & & & & & h_0 \\ U_0 & G_1 & & & & & & h_1 \\ & U_1 & G_1 & & & & & h_1 \\ & & \ddots & \ddots & & & & \vdots \\ & & & U_1 & G_1 & & & h_1 \\ & & & & U_1 & G_2 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & U_2 & G_3 \end{pmatrix},$$

where,

$$G_0 = \begin{pmatrix} -\eta & 0 \\ \theta_w & -(\eta+\theta_w) \end{pmatrix},$$
$$G_1 = \begin{pmatrix} -(\eta+\theta_w+\mu_b r) & \theta_w \\ 0 & -(\eta+\mu_b r) \end{pmatrix},$$
$$G_2 = \begin{pmatrix} -(\theta_w+\mu_b r) & \theta_w \\ 0 & -\mu_b r \end{pmatrix},$$
$$G_3 = \begin{pmatrix} -\theta_v & 0 & \theta_v \\ 0 & -(\theta_w+\mu_b r) & \theta_w \\ 0 & 0 & -\mu_b r \end{pmatrix},$$
$$U_2 = \begin{pmatrix} 0 & 0 \\ \mu_w r & 0 \\ 0 & \mu_b r \end{pmatrix}.$$

Let $\varphi = (\varphi_{0,0}, \varphi_{0,1}, \varphi_{1,1}, \varphi_{1,2}, \cdots, \varphi_{S,0}, \varphi_{S,1}, \varphi_{S,2})$ be the steady-state probability vector,then φ satisfis the balance equations

$$\begin{cases} \pi Q = 0, \\ \pi e = 1, \end{cases} \tag{1}$$

where e is a column vector of appropriate dimension with all elements 1. According to Eq(1), we can obtain:

$$-\eta\varphi_{00} + \theta_w\varphi_{0,1} = 0, \tag{2}$$

$$-(\eta+\theta_w)\varphi_{0,1} + \mu_w r\varphi_{1,1} + \mu_b r\varphi_{1,2} = 0, \tag{3}$$

$$-(\eta+\theta_w+\mu_w r)\varphi_{i,1} + \mu_w r\varphi_{i+1,1} = 0, 1 \le i \le s, \tag{4}$$

$$\theta_w\varphi_{i,1} - (\eta+\mu_b r)\varphi_{i,2} + \mu_b r\varphi_{i+1,2} = 0, 1 \le i \le s, \tag{5}$$

$$-(\theta_w+\mu_w r)\varphi_{i,1} + \mu_w r\varphi_{i+1,1} = 0, s+1 \le i \le S-1, \tag{6}$$

$$\theta_w\varphi_{i,1} - \mu_b r\varphi_{i,2} + \mu_b r\varphi_{i+1,2} = 0, s+1 \le i \le S-1, \tag{7}$$

$$\eta\varphi_{0,0} - \theta_v\varphi_{S,0} = 0, \tag{8}$$

$$\eta\left(\sum_{i=0}^{s}\varphi_{i,1}\right) - (\theta_w+\mu_w r)\varphi_{S,1} = 0, \tag{9}$$

$$\eta\sum_{i=1}^{s}\varphi_{i,2} + \theta_v\varphi_{S,0} + \theta_w\varphi_{S,1} - \mu_b r\varphi_{S,2} = 0. \tag{10}$$

From the Eq(2) to Eq(10), we can get:

$$\varphi_{i,1} = \left(\frac{\mu_w r}{\mu_w r+\theta_w}\right)^{S-s-1}\left(\frac{\mu_w r}{\mu_w r+\theta_w+\eta}\right)^{s-i+1}\varphi_{S,1}, \quad 1 \le i \le s, \tag{11}$$

$$\varphi_{i,1} = \left(\frac{\mu_w r}{\mu_w r+\theta_w}\right)^{S-i}\varphi_{S,1}, s+1 \le i \le S-1, \tag{12}$$

$$\begin{aligned} \varphi_{i,2} = \frac{\theta_w}{\mu_b r}&\left[\left(\frac{\mu_b r}{\eta+\mu_b r}\right)^{s-i+1}\varphi_{S,1} - \varphi_{i,1}\right] \\ &+\left(\frac{\mu_b r}{\eta+\mu_b r}\right)^{s-i+1}\varphi_{S,2}, 1 \le i \le s, \end{aligned} \tag{13}$$

$$\varphi_{i,2} = \varphi_{S,2} + \frac{\theta_w}{\mu_b r}(\varphi_{S,1} - \varphi_{i,1}), s+1 \le i \le S-1, \tag{14}$$

$$\varphi_{1,1} = \left(\frac{\mu_w r}{\mu_w r+\theta_w+\eta}\right)^{s}\left(\frac{\mu_w r}{\mu_w r+\theta_w}\right)^{S-s-1}\varphi_{S,1}, \tag{15}$$

$$\varphi_{1,2} = \frac{\theta_w}{\mu_b r}\left[\left(\frac{\mu_b r}{\eta+\mu_b r}\right)^{s}\varphi_{S,1} - \varphi_{1,1}\right] + \left(\frac{\mu_b r}{\mu_b r+\eta}\right)^{s-i+1}\varphi_{S,2}, \tag{16}$$

$$\varphi_{0,1} = \frac{\mu_w r}{\eta+\theta_w}\varphi_{1,1} + \frac{\mu_b r}{\eta+\theta_w}\varphi_{1,2}, \tag{17}$$

$$\varphi_{0,0} = \frac{\theta_w}{\eta}\varphi_{0,1}, \tag{18}$$

$$\varphi_{S,0} = \frac{\theta_w}{\theta_v}\varphi_{0,1}, \tag{19}$$

$$\varphi_{S,2} = \frac{\eta}{\mu_b r}\sum_{i=1}^{s}\varphi_{i,2} + \frac{\theta_v}{\mu_b r}\varphi_{S,0} + \frac{\theta_w}{\mu_b r}\varphi_{S,1}, \tag{20}$$

$$\varphi_{S,1} = \frac{\eta}{\theta_w+\mu_w r}. \tag{21}$$

Through the iteration of the formula, it can be found that $\varphi_{0,0}, \varphi_{0,1}, \varphi_{1,1}, \varphi_{1,2}, \cdots, \varphi_{S,0}, \varphi_{S,2}$ can be expressed by $\varphi_{S,1}$, $\varphi_{S,1}$ can be solved according to $\varphi e = 1$. The system state process is normal and only if $\varphi Ce < \varphi B_N e$. Therefore, by the matrix B_N and C:

$$\varphi Ce = \lambda\sum_{i=0}^{S}\varphi_{i,1} + \lambda\sum_{i=1}^{S}\varphi_{i,2},$$
$$\varphi B_N e = N\xi\sum_{i=0}^{S}\varphi_{i,1} + \mu_w\sum_{i=1}^{S}\varphi_{i,1} + \mu_b\sum_{i=1}^{S}\varphi_{i,2}.$$

Finally, the steady-state balance condition of the system is:

$$(\lambda - N\xi)\varphi_{0,1} < (\mu_w - \lambda)\sum_{i=1}^{S}\varphi_{i,1} + (\mu_b - \lambda)\sum_{i=1}^{S}\varphi_{i,2}.$$

C. Matrix Geometric Solution

The process $\phi(t)$ is an $LDQBD$ process, and the steady-state probability distribution is defined as follows: $\pi_{n,i,j} = \lim_{t\to\infty}\pi\{N(t)=n, I(t)=i, J(t)=j\}, (n,i,j) \in \Omega$, where

$$\pi = (\pi_0, \pi_1, ...),$$

$$\pi_i = (\pi_{i,0,0}, \pi_{i,0,1}, \pi_{i,1,1}, \pi_{i,1,2}, \ldots, \pi_{i,S,0}, \pi_{i,S,1}, \pi_{i,S,2}), i \ge 0.$$

The steady-state probability vector satisfies the system of equations:

$$\left\{\begin{array}{l}\pi Q=0,\\ \pi e=1,\end{array}\right. \tag{22}$$

where e is a column vector of appropriate dimension with all elements 1.

The system state process returns normally if and only if the matrix quadratic equation:

$$R^2B_N+RA_N+C=0, \tag{23}$$

has a minimum non-negative solution R of spectral radius $sp\left(R\right)<1$, and

$$\left(\pi_0,\pi_1,\cdots,\pi_{N-1}\right)B\left[R\right]=0, \tag{24}$$

has a positive solution, where

$$B\left[R\right]=\begin{bmatrix} A_0 & C & & & \\ B_1 & A_1 & C & & \\ & \ddots & \ddots & \ddots & \\ & & B_{N-1} & A_{N-1} & C \\ & & & B_N & RB_N+A_N \end{bmatrix},$$

the system steady-state probability vector has the following matrix geometric solution form:

$$\pi_k=\pi_N R^{k-N},k\geq N, \tag{25}$$

and satisfies the following equations:

$$\left\{\begin{array}{l}\left(\pi_0,\pi_1,\cdots,\pi_N\right)B[R]=0,\\ \pi_k=\pi_N R^{k-N},k\geq N,\\ \left(\sum\limits_{i=0}^{N-1}\pi_i+\pi_N(I-R)^{-1}\right)e=1,\end{array}\right. \tag{26}$$

where e is a column vector of appropriate dimension with elements all 1, and I is a unit array of order $(N+1)\times(2S+3)$.

IV. Steady-state performance indicators

1) The mean queue length is given by

$$E_d=\sum_{n=0}^{\infty}n(\pi_{n00}+\pi_{nS0})+\sum_{n=0}^{\infty}\sum_{i=0}^{S}n\pi_{ni1}+\sum_{n=0}^{\infty}\sum_{i=1}^{S}n\pi_{ni2}. \tag{27}$$

2) The mean inventory level is given by

$$E_{inv}=\sum_{n=0}^{\infty}\sum_{i=0}^{S}i\pi_{ni1}+\sum_{n=0}^{\infty}\sum_{i=1}^{S}i\pi_{ni2}+\sum_{n=0}^{\infty}S\pi_{nS0}. \tag{28}$$

3) The mean replenishment rate of the system, which is the mean number of the replenishment per unit of time is given by

$$E_r=\eta\left(\sum_{n=0}^{\infty}\sum_{i=0}^{s}n\pi_{ni1}+\sum_{n=0}^{\infty}\sum_{i=1}^{s}n\pi_{ni2}+\sum_{n=0}^{\infty}\pi_{n00}\right). \tag{29}$$

4) The mean replenishment, which is the mean number of items replenished by the system in a single replenishment process is given by

$$E_q=\sum_{n=0}^{\infty}\sum_{i=0}^{s}\left(S-i\right)\pi_{ni1}+\sum_{n=0}^{\infty}\sum_{i=1}^{s}\left(S-i\right)\pi_{ni2}+\sum_{n=0}^{\infty}S\pi_{n00}. \tag{30}$$

5) The mean number of lost customers due to customer impatient is given by

$$E_l=\sum_{n=0}^{\infty}\sum_{i=0}^{S}n\xi\pi_{ni1}+\lambda\sum_{n=0}^{\infty}(\pi_{n00}+\pi_{nS0}). \tag{31}$$

6) The mean loss rate caused by customers not consuming products after receiving services is given by

$$E_p=\sum_{n=0}^{\infty}\sum_{i=1}^{s}\mu_w(1-r)\pi_{ni1}+\sum_{n=0}^{\infty}\sum_{i=1}^{s}\mu_b(1-r)\pi_{ni2}. \tag{32}$$

7) The mean vacation rate of server is given by

$$E_{vac}=\theta_w\sum_{n=0}^{\infty}\sum_{i=0}^{s}\pi_{ni0}+\theta_v\sum_{n=0}^{\infty}\pi_{n00}. \tag{33}$$

V. Numerical analysis

Within the following section, numerical experiments are conducted to investigate the influences of system parameters on the steady-state performance indicators, such as the mean inventory level of the system, the mean number of lost customers due to customer impatience, the mean replenishment rate per unit of time, and the mean loss rate due to products not taken away by customers. In addition, a mean cost function is constructed to analyze the effect of parameter variations on the optimal inventory strategy and the optimal cost.

A. Effects of parameters on the performance indicators

Parameter settings for each figure are shown in Table 1:

Table 1. System Parameter Settings

Figure number	Parameter settings
Figure 3, Figure 7, Figure 11, Figure 15	$(s,S,\xi,\theta_v,\theta_w,\mu_w,\mu_b,r)$ =(5, 10, 2, 12, 1, 6, 0.7)
Figure 4, Figure 8, Figure 12, Figure 16	$(s,S,\lambda,\eta,\theta_w,\mu_w,\mu_b,r)$ =(5, 10, 4, 1, 2, 1, 6, 0.7)
Figure 5, Figure 9, Figure 13, Figure 17	$(s,S,\lambda,\eta,\xi,\theta_v,\mu_b,r)$ =(5, 10, 4, 1, 2, 1, 6, 0.7)
Figure 6, Figure 10, Figure 14, Figure 18	$(s,S,\lambda,\eta,\theta_v,\theta_w,\mu_w,\mu_b)$ =(5, 10, 4, 1, 2, 1, 1, 6)

1) The mean inventory level:

Assuming the extent of variation of η is $0\leq\eta\leq 6$, the figure 3 illustrates the mean inventory level increases with η. η increases, the mean replenishment time of the system decreases. Replenishment will arrive more timely, so the average inventory level is higher. Assuming the extent of variation of λ is $2\leq\lambda\leq 6$, the mean inventory level decreases as λ increases. λ larger, more customers enter the system and there is a greater demand for inventory. The average inventory level decreases accordingly.

As seen in Figure 4, the mean inventory level decreases as θ_v increases. As θ_v increases, the average time that the server is on full vacation decreases. Consequently, the time spent in the busy period and working vacation increases. This leads to a higher demand for inventory by customers, resulting in a decrease in the mean inventory level. Conversely, the mean inventory level increases with ξ. A larger ξ indicates

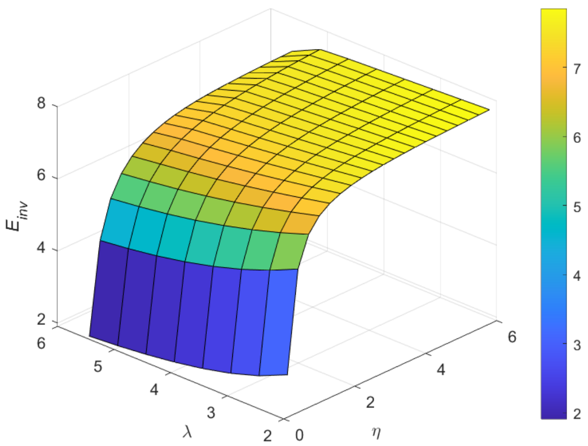

Fig. 3. Effect of λ and η on the mean inventory level.

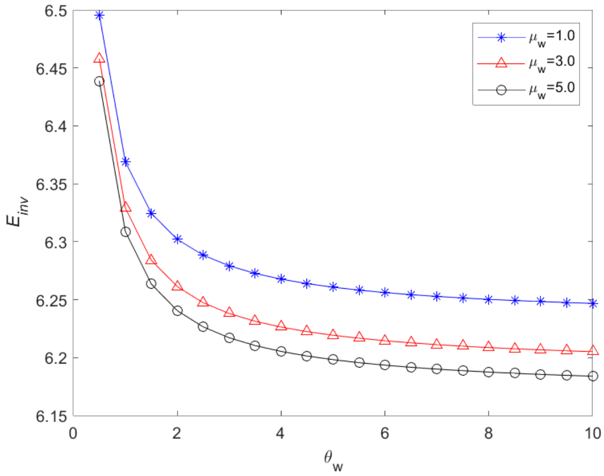

Fig. 5. Effect of μ_w and θ_w on the mean inventory level.

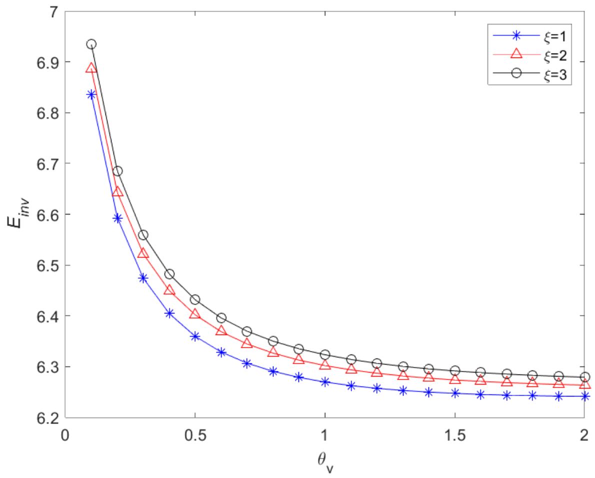

Fig. 4. Effect of θ_v and ξ on the mean inventory level.

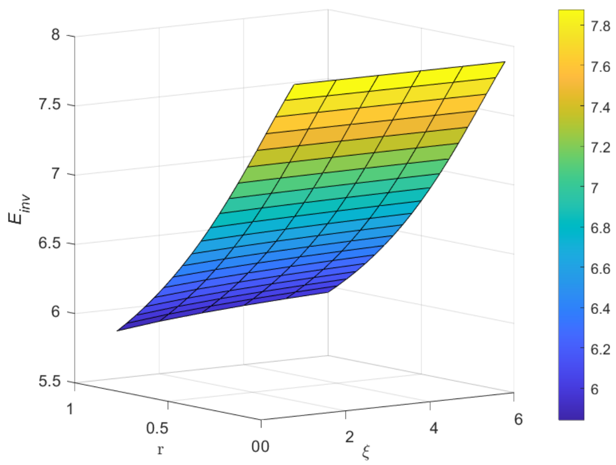

Fig. 6. Effect of r and ξ on the mean inventory level.

a shorter impatience time for customers, leading to shorter wait times in the queue and faster customer attrition. As a result, the demand for inventory decreases, causing the average inventory level to rise.

As seen in Figure 5, the mean inventory level decreases as μ_w and θ_w increase. The increase in μ_w results in an enhanced service rate during the working vacation, thereby expediting inventory depletion and reducing the average inventory level.As θ_w increases, the server's average time in working vacation shortens, while the time spent in full vacation and the busy period lengthens. A longer busy period increases customer demand for inventory, further decreasing the mean inventory level.

As shown in Figure 6, the mean inventory level decreases as r increases. A higher r indicates a greater probability that customers will choose to take a product, leading to higher demand for inventory and a lower mean inventory level. When ξ is treated as a continuous variable, the mean inventory level increases with ξ. This result is consistent with the scenario where ξ is treated as a discrete variable (see Figure 4 for details). As ξ reaches a certain threshold, representing a certain level of customer attrition, the change in average inventory level gradually stabilizes.

2) The mean number of lost customers due to customer impatient:

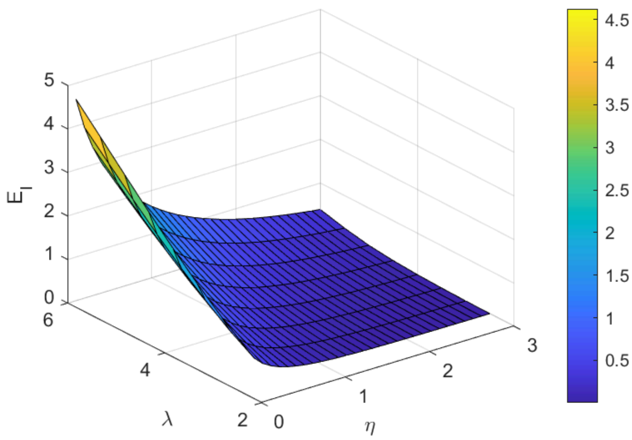

Fig. 7. Effect of λ and η on the mean customer loss.

As seen in Figure 7, we assume the extent of variation of λ is $2 \leq \lambda \leq 6$ and the extent of variation of η is $0 \leq \eta \leq 3$. The mean customer loss increases with λ. The larger λ means more customers enter the system, depleting inventory faster and triggering working vacations sooner. During working vacations, customer impatience increases, leading to higher customer loss. Conversely, the mean customer loss decreases with η. The larger η is, the shorter the mean replenishment time means the replenishment arrives timely, the slower the system enters the working vacation time represents less likely to become impatient, so the mean customer loss is smaller.

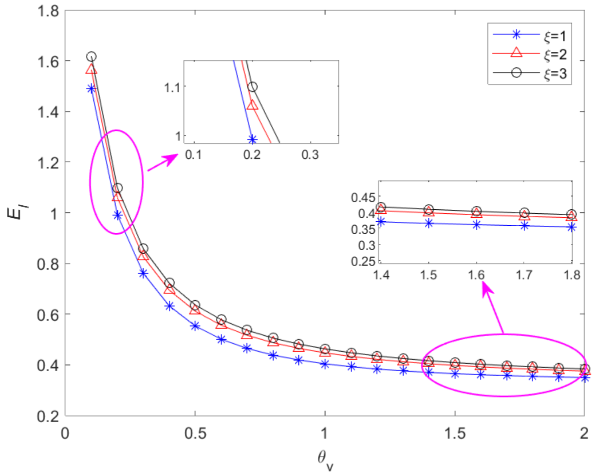

Fig. 8. Effect of θ_v and ξ on the mean customer loss.

As seen in Figure 8, the mean customer loss decreases as θ_v increases.As θ_v increases, the system spends less time in full vacation and more time in the busy period, resulting in a smaller mean customer loss. Conversely, the mean customer loss increases with ξ. ξ is larger, the mean impatience time of customers waiting to be served in the system is shorter, and customers are lost more quickly.

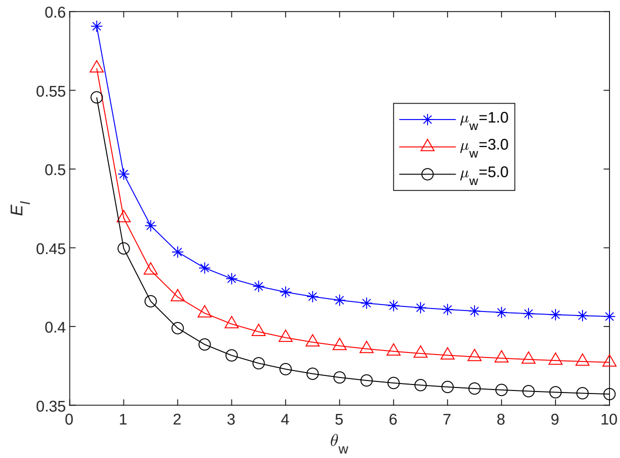

Fig. 9. Effect of μ_w and θ_w on the mean customer loss.

As seen in Figure 9, the mean customer loss decreases as μ_w and θ_w increases. The higher μ_w during a working vacation, the less likely customers are to become impatient, resulting in lower mean customer loss. The larger θ_w is, the shorter the mean working vacation time, the longer the time in full vacation and busy periods. Therefore, more customers are likely to take away one product, which makes the mean customer loss smaller.

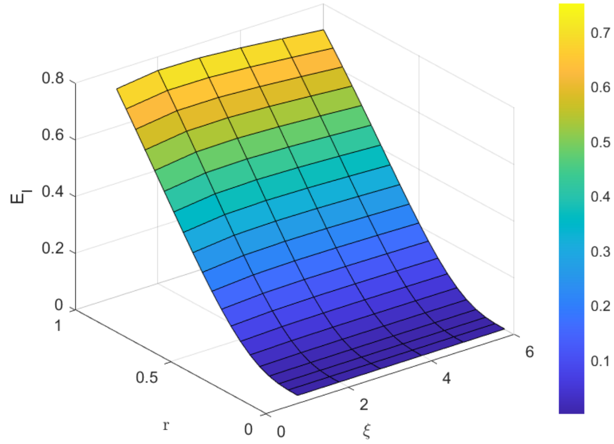

Fig. 10. Effect of r and ξ on the mean customer loss.

As seen in Figure 10, The average customer loss rate due to impatience increases with r. As r grows, the probability of customers taking products increases, leading to a faster reduction in inventory levels. Consequently, the system enters working vacation more quickly, resulting in greater customer loss. As ξ increases, the mean customer loss rate first rises and then declines. A larger ξ shortens the average impatience time of customers waiting for service, leading to faster customer attrition and a higher mean loss rate. However, when attrition reaches a certain point, new customers entering the system cannot keep pace with the loss rate, causing the mean customer loss rate to eventually decrease.

3) The mean replenishment rate:

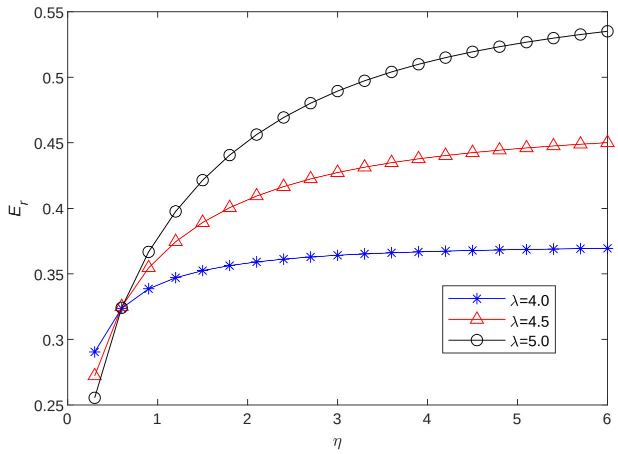

Fig. 11. Effect of λ and η on the mean replenishment rate.

As seen in Figure 11, the mean replenishment rate increases as η. As the value of η increases, the replenishment time decreases, leading to a rapid product flow into the system. What's more. This reduces the likelihood of the system entering a working vacation period, allowing customers to consume more products during regular busy periods. Consequently, there is an increase in the number of replenishments and a corresponding growth in the mean replenishment rate.

As seen in Figure 12, the mean replenishment rate in-

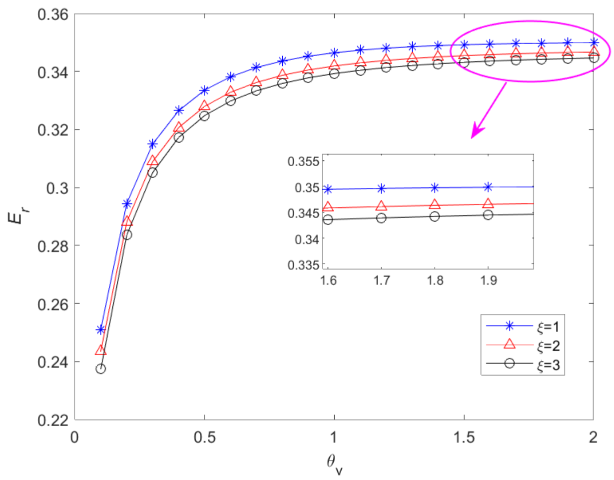

Fig. 12. Effect of θ_v and ξ on the mean replenishment rate.

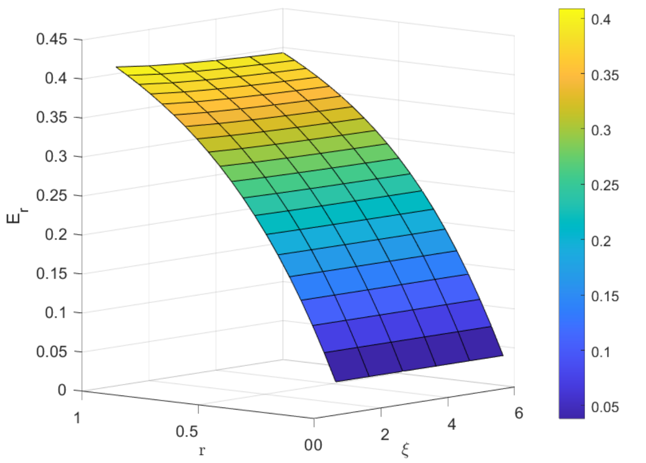

Fig. 14. Effect of r and ξ on the mean replenishment rate.

creases with θ_v. When θ_v is greater, the server has shorter periods of full vacation and enters the busy period more quickly, during which customers demand more inventory, causing the mean replenishment rate to increase. The mean replenishment rate decreases with increasing ξ. A larger ξ shortens the impatience time of customers waiting to be served in the system, leading to more customers being lost. As a result, due to reduced customer demand for inventory, the mean replenishment rate decreases.

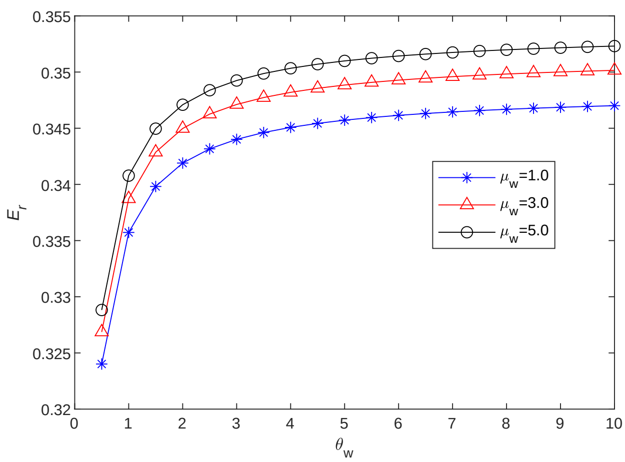

Fig. 13. Effect of μ_w and θ_w on the mean replenishment rate.

As seen in Figure 13, the mean replenishment rate increases with θ_w and μ_w. As θ_w increases, the average time the system is on working vacation decreases,while the time in the busy period increases, leading to higher inventory demand. Thus, the mean replenishment rate rises with θ_w. μ_w increases, the service rate during working vacation also rises, resulting in greater customer demand for inventory and a higher mean replenishment rate.

As seen in Figure 14, the mean replenishment rate increases as r increases. As r increases, the probability that customers take products increases, leading to higher inventory demand and a greater mean replenishment rate. Conversely, as ξ increases, the impatience time of customers waiting for service shortens, resulting in more customer loss, reduced inventory demand, and a lower mean replenishment rate.

4) The mean loss rate caused by customers not consuming products after receiving services:

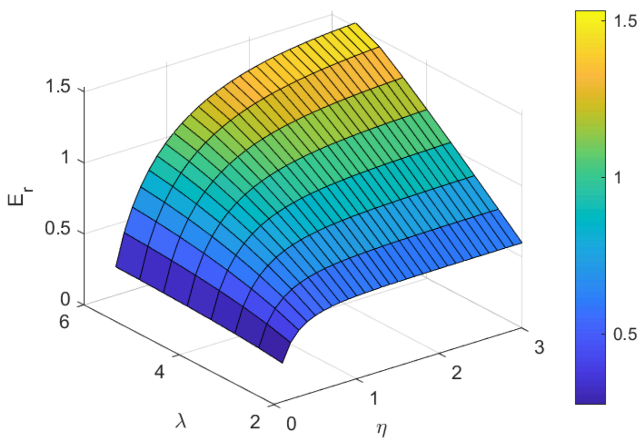

Fig. 15. Effect of λ and η on the mean loss rate.

As seen in Figure 15, we assume the extent of variation of λ is $2 \leq \lambda \leq 6$ and the extent of variation of η is $0 \leq \eta \leq 3$. The mean loss rate increases with the increase of λ and η. The larger λ is, the more customers enter the system, and the number of customers choosing not to take away one product will also increase, so the mean loss rate is greater. The larger η is, the shorter the replenishment time is, the faster the replenishment arrives, and inventory levels are more adequate. So the more likely customers are to choose not to take products.

As seen in Figure 16, the mean loss rate increases with the increase of θ_v. As θ_v increases, the system spends less time in full vacation and more time in working vacation and busy periods. During these periods, customers have the option to not take products, leading to a higher mean loss rate. The mean loss rate of customers who do not take away the products decreases with the increase of ξ. The larger ξ is, the more customers waiting to be served in the system

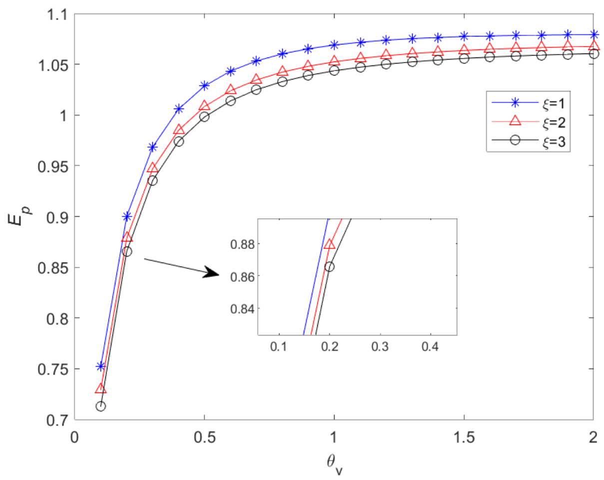

Fig. 16. Effect of θ_v and ξ on the mean loss rate.

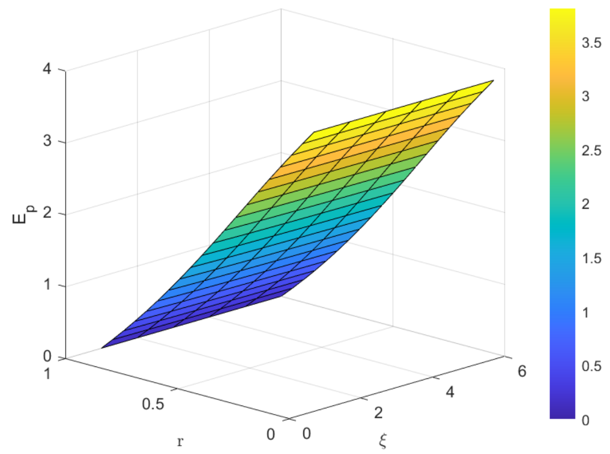

Fig. 18. Effect of r and ξ on the mean replenishment rate.

are lost due to impatience, and the number of customers who choose not to take away one product decreases as well. Consequently, the mean loss rate decreases.

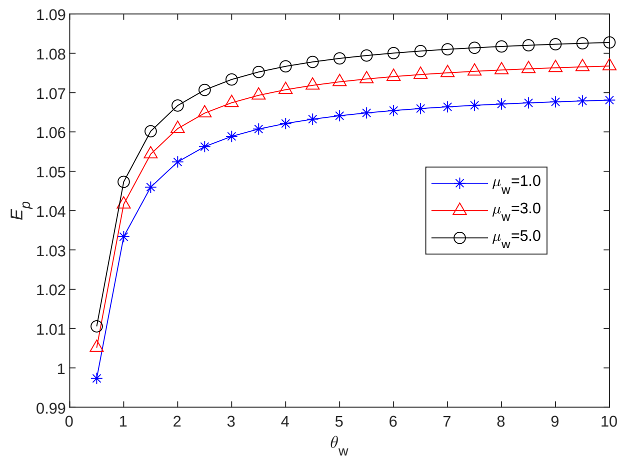

Fig. 17. Effect of μ_w and θ_w on the mean replenishment rate.

As seen in Figure 17, the mean loss rate of customers not taking products away increases with μ_w and θ_w. A higher μ_w corresponds to a higher service rate during the working vacation, reducing the likelihood of customer impatience. As a result, more customers wait in the system, leading to an increase in the number of customers who choose not to take a product, thus increasing the mean loss rate. θ_w increases, the time in working vacation becomes shorter, the busy period time may become longer. During the busy period, customers are less likely to become impatient, leading to an increase in the number of customers who choose not to take a product, thereby increasing the average loss rate.

As seen in Figure 18, the mean loss rate of products not taken away by customers decreases as r increases. The larger r is, the less likely that the customer will not take away one product, and the smaller the mean loss rate is. When $r = 1$, the customer who receives the service will certainly choose to take away a product. So at this time, the loss rate is zero.

B. *Optimal inventory strategy and optimal cost*

Based on each performance indicator of the system, an expression for the mean function of the system per unit time is established:

$$F(s,S) = C_0E_d + C_1E_i + C_2E_rE_q + C_3E_r + C_4E_l + (C_4 + C_5)E_p,$$

where C_0 is the mean waiting cost per unit of the customer, C_1 represents the inventory holding cost per unit of time, C_2 is the cost of replenishing a unit of product, C_3 is the cost of a fixed order per replenishment, C_4 is the cost of losses caused by customers dropping out of the queue, C_5 is the cost of consuming the service received by the customer, and $C_4 + C_5$ is the cost of losses that the customer does not take away with the product after receiving the service.

Genetic algorithm is used to find the optimal inventory strategy and the optimal cost of the system and analyze the effect of variation of different parameters on the optimal inventory strategy and the optimal cost function. The inventory cost is taken as $C_0 = 50$, $C_1 = 2$, $C_2 = 15$, $C_3 = 10$, $C_4 = 100$, $C_5 = 5$. The results obtained are given by Table 2 to Table 9.

Table 2. Impact of λ on optimal strategy and cost function.

λ	2.5	3	3.5	4	4.5
(s,S)	(5,13)	(6,16)	(7,18)	(8,21)	(9,23)
$F(s,S)$	141.9705	177.2169	218.1528	268.9048	338.4923

Fixed parameter $\mu_w = 1, \mu_b = 6, \eta = 1, r = 0.7, \xi = 2, \theta_w = 2, \theta_v = 1$, the effect of the change of parameter λ on the optimal strategy and cost function is shown in Table 2. The larger λ is, the greater the demand for inventory, the safety stock level s and the maximum stock level S are also gradually increased, and the optimal cost is also increased with the increase of λ. Parameter λ has a greater effect on the optimal inventory strategy and optimal cost.

Fixed parameter $\lambda = 4$, $\mu_w = 1$, $\mu_b = 6$, $r = 0.7$, $\xi = 2$, $\theta_w = 2$, $\theta_v = 1$, the influence of the change of parameter η on the optimal strategy and cost function is shown in Table 3. The larger η is, the replenishment arrives faster, both s and S decrease gradually, and the optimal cost decreases gradually.

Table 3. Impact of ξ on optimal strategy and cost function.

η	1	1.5	2	2.5	3
(s,S)	(8,21)	(7,17)	(6,15)	(5,13)	(5,12)
$F(s,S)$	268.9048	260.7964	256.3553	253.4294	251.5425

Parameter η has a significant effect on the optimal inventory strategy and optimal cost.

Table 4. Impact of r on optimal strategy and cost function.

r	0.4	0.5	0.6	0.7	0.8
(s,S)	(5,12)	(6,15)	(7,18)	(8,21)	(10,23)
$F(s,S)$	378.1632	341.7438	305.3250	268.9048	232.4222

Fixed parameter $\lambda = 4$, $\mu_w = 1$, $\mu_b = 6$, $\eta = 1$, $\xi = 2$, $\theta_w = 2$, $\theta_v = 1$, the influence of the change of parameter r on the optimal strategy and cost function is shown in Table 4. The larger r is, the higher the probability that the customer takes away the product, s and S both increase gradually, and the optimal cost increases. Parameter r has a greater impact on the optimal inventory strategy and optimal cost.

Table 5. Impact of ξ on optimal strategy and cost function.

ξ	1.5	2	2.5	3	3.5
(s,S)	(9,21)	(8,21)	(8,20)	(8,20)	(7,19)
$F(s,S)$	269.6100	268.9048	268.1765	267.5367	266.9517

Fixed parameter $\lambda = 4, \mu_w = 1, \mu_b = 6, \eta = 1, r = 0.7, \theta_w = 2, \theta_v = 1$, the influence of the change of parameter ξ on the optimal strategy and cost function is shown in Table 5. The larger parameter ξ is, customers in the queue lose faster, meanwhile s, S and the optimal cost decrease. Parameter ξ has less effect on the optimal inventory strategy and optimal cost.

Table 6. Impact of θ_w on optimal strategy and cost function.

θ_w	2	4	6	8	10
(s,S)	(8,21)	(8,21)	(9,21)	(9,21)	(9,21)
$F(s,S)$	268.9048	269.0252	269.0915	269.1128	269.1201

Fixed parameter $\lambda = 4, \mu_w = 1, \mu_b = 6, \eta = 1, r = 0.7, \xi = 2, \theta_v = 1$, the influence of the change of parameter θ_w on the optimal strategy and cost function is shown in Table 6. The larger parameter θ_w is, the slightly larger s and optimal cost are. Parameter θ_w has no significant effect on optimal inventory and optimal cost.

Table 7. Impact of θ_v on optimal strategy and cost function.

θ_v	1	3	5	7	9
(s,S)	(8,21)	(7,20)	(7,19)	(7,19)	(7,19)
$F(s,S)$	268.9048	266.8839	266.3700	266.1374	266.0081

Fixed parameter $\lambda = 4, \mu_w = 1, \mu_b = 6, \eta = 1, r = 0.7, \xi = 2, \theta_w = 2$, the change of parameter θ_v on the optimal strategy and cost function is shown in Table 7. The larger the parameter θ_v is, s decreases slightly, S and the optimal cost decrease gradually. Furthermore, the influence exerted by the parameter θ_v on the optimal inventory strategy and the optimal cost is comparatively insignificant.

Table 8. Impact of μ_w on optimal strategy and cost function.

μ_w	1	2	3	4	5
(s,S)	(8,21)	(8,21)	(8,21)	(8,20)	(8,20)
$F(s,S)$	268.9048	268.7702	268.6555	268.5546	268.4658

Fixed parameter $\lambda = 4, \mu_b = 6, \eta = 1, r = 0.7, \xi = 2, \theta_w = 2, \theta_v = 1$, the influence of the change of parameter μ_w on the optimal strategy and cost function is shown in Table 8. The larger μ_w is, the system's maximum inventory level S and optimal cost decrease slightly, and the parameter μ_w has a smaller effect on the optimal inventory strategy and a non-significant effect on the optimal cost.

Table 9. Impact of μ_b on optimal strategy and cost function.

μ_b	6	7	8	9	10
(s,S)	(8,21)	(8,21)	(8,20)	(8,20)	(8,20)
$F(s,S)$	268.9048	235.4509	218.4337	208.1193	201.2012

Fixed parameter $\lambda = 4, \mu_w = 1, \eta = 1, r = 0.7, \xi = 2, \theta_w = 2, \theta_v = 1$, the influence of the change of parameter μ_b on the optimal strategy and cost function is shown in Table 9. The larger the parameter μ_b, the larger the service rate of the attendant during the regular busy period, the slightly smaller the system maximum inventory level S, and the optimal cost gradually decreases. Parameter μ_b has a smaller impact on the optimal inventory strategy and a more significant impact on the optimal cost.

VI. Conclusion

This paper investigates the M/M/1 queuing inventory system with a hybrid vacation strategy, considering impatient customers and multiple vacations, where the first vacation is a working vacation. It is assumed that a customer selectively takes one product after receiving service. The customer's impatience time, the server's service time and vacation time, and the replenishment time follow exponential distributions. Markov's theory is used to establish the quasi-birth-and-death process, and a matrix geometric solution is utilized to obtain the system's steady-state vector. System-related performance indicators are established, and the system's mean cost function is constructed. Through numerical experiments, we examined the impact of system parameters λ, μ_w, μ_b, η, r, ξ, θ_w, θ_v on the optimal inventory strategy and the optimal cost. The results indicate that λ, η, r, ξ has a more significant impact on the optimal inventory and the optimal cost.

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