Expanding Graph Theory: Product Operations and Properties in Directed Graph Contexts

Jinta Jose, Rajesh K. Thumbakara, Sijo P. George, and Bobin George

Abstract—**A directed graph, consisting of vertices linked by directed edges or arcs, is a crucial structure utilized in analyzing various scenarios such as electrical circuits, optimal routes, and social connections. Graph theory introduces graph products, a binary operation applied to graphs. Similarly, directed graphs can undergo product operations analogous to those of standard graphs. Several authors have explored specific product operations in directed graphs, including the Cartesian, lexicographic, and strong products. In this study, we broaden the scope by extending definitions of product operations from standard graphs, such as categorical, modular, disjunctive, homomorphic, rooted, and corona products, to directed graphs. Furthermore, we delve into their properties.**

Index Terms—**graph, directed graph, graph product, directed graph product.**

I. Introduction

GRAPHS serve as invaluable data structures for depicting real-world connections, finding application across diverse domains owing to their ability to abstract complex scenarios. However, the symmetrical nature of graphs may not always align with the representation required for specific contexts, leading to the emergence of directed graphs [3], [6]. Directed graphs provide a means to analyze and address problems ranging from electrical circuits and project scheduling to identifying shortest paths and understanding social dynamics. Within graph theory, the concept of graph product, a binary operation applied to graphs, is introduced [5]. Similarly, akin to the definitions of graph products, product operations can be devised for directed graphs. Various researchers, including Bozovic [1], Changat et al. [2], Feigenbaum [4], Manion [7], Potocnik et al. [8], Thamizharasi et al. [9], and Wei et al. [10] have explored diverse product operations in directed graphs such as Cartesian, lexicographic, and strong products. This paper extends the definitions of product operations from graphs such as categorical, modular, disjunctive, homomorphic, corona, and rooted products to directed graphs and scrutinizes some of their properties.

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II. Categorical (or Tensor or Kronecker) Product of Directed Graphs

This section presents the definition of the categorical product of two directed graphs. Additionally, we provide some theorems related to the number of vertices and arcs, as well as the sum of indegrees, outdegrees and degrees in the resulting graph. Examples are included to demonstrate these results and to clarify the concepts involved.

Definition II.1. Consider two directed graphs $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. The categorical product of Δ_1 and Δ_2, denoted as $\Delta_1 \times \Delta_2$, forms a directed graph with the vertex set $T(\Delta_1 \times \Delta_2) = T_1 \times T_2$ and the arc set $R(\Delta_1 \times \Delta_2)$. Here, an arc $((v_1, v_1'), (v_2, v_2'))$ in $\Delta_1 \times \Delta_2$ exists if and only if there exists an arc (v_1, v_2) in Δ_1 and an arc (v_1', v_2') in Δ_2.

Example 1. Let $\Delta_1 = (T_1, R_1)$ represent a directed graph with a vertex set $T_1 = \{v_1, v_2, v_3, v_4, v_5\}$ and arc set $R_1 = \{(v_3, v_2), (v_3, v_1), (v_3, v_4), (v_5, v_3), (v_4, v_5)\}$ which is shown in Fig. 1. Let $\Delta_2 = (T_2, R_2)$ denote a directed

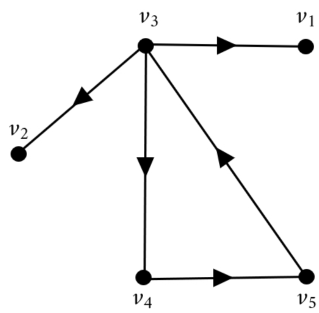

Fig. 1: Directed Graph $\Delta_1 = (T_1, R_1)$

graph with a vertex set $T_2 = \{u_1, u_2, u_3, u_4\}$ and arc set $R_2 = \{(u_3, u_1), (u_3, u_2), (u_3, u_4)\}$ which is shown in Fig. 2. The illustration in Fig. 3 showcases the categorical

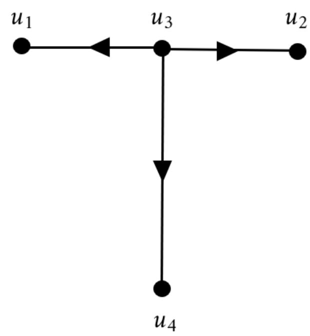

Fig. 2: Directed Graph $\Delta_2 = (T_2, R_2)$

product of the directed graphs Δ_1 and Δ_2, represented as

$D = \Delta_1 \times \Delta_2$.

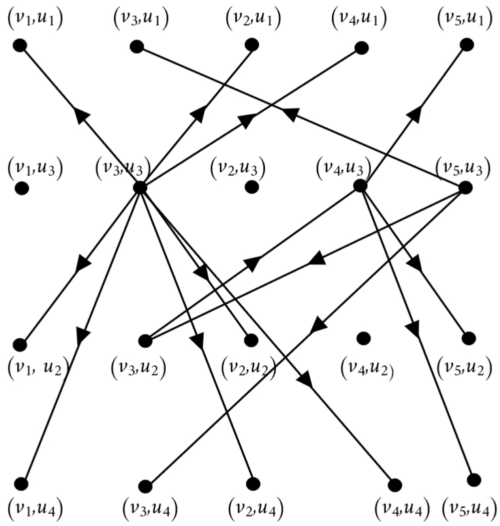

Fig. 3: $\Delta = \Delta_1 \times \Delta_2$

Theorem 1. Suppose $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ are two directed graphs. Then, the categorical product of Δ_1 and Δ_2, denoted as $\Delta_1 \times \Delta_2$, comprises $|T_1||T_2|$ vertices and $|R_1||R_2|$ arcs.

Proof: The vertex set of $\Delta_1 \times \Delta_2$ is $T_1 \times T_2$ which contains $|T_1||T_2|$ elements. Therefore total number of vertices in $\Delta_1 \times \Delta_2$ is $|T_1||T_2|$. Also we know, $((v_e, v_f), (v_k, v_l))$ is an arc in $\Delta_1 \times \Delta_2$ if and only if (v_e, v_k) is an arc in Δ_1 and (v_f, v_l) is an arc in Δ_2. There are $|R_1|$ arcs in Δ_1 and $|R_2|$ arcs in Δ_2. Thus, there are $|R_1||R_2|$ possible choices for selecting a pair of arcs a_m and a_n, where one arc belongs to Δ_1 and the other to Δ_2. Let's assume a_m represents the arc (v_e, v_k) in Δ_1, and a_n represents the arc (v_f, v_l) in Δ_2. Consequently, this selected pair of arcs forms an arc $((v_e, v_f), (v_k, v_l))$ in $\Delta_1 \times \Delta_2$. Hence $\Delta_1 \times \Delta_2$ contains totally $|R_1||R_2|$ arcs. ■

Example 2. Consider the directed graphs Δ_1 and Δ_2 given in Example 1.
Here we have, the total count of vertices in $\Delta_1 \times \Delta_2$ = 20 and $|T_1||T_2| = 5 \cdot 4 = 20$.
That is, the total count of vertices in $\Delta_1 \times \Delta_2 = |T_1||T_2|$.
Also, total count of arcs in $\Delta_1 \times \Delta_2 = 15$ and $|R_1||R_2| = 5 \cdot 3 = 15$.
That is, total count of arcs in $\Delta_1 \times \Delta_2 = |R_1||R_2|$.

Corollary 1. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs and $\Delta_1 \times \Delta_2 = (T(\Delta_1 \times \Delta_2), R(\Delta_1 \times \Delta_2))$ be their categorical product . Then

1)
$$\sum_{(u,v)\in T(\Delta_1\times\Delta_2)} ideg_{\Delta_1\times\Delta_2}(u,v) = \sum_{(u,v)\in T(\Delta_1\times\Delta_2)} odeg_{\Delta_1\times\Delta_2}(u,v) = |R_1||R_2|,$$

2)
$$\sum_{(u,v)\in T(\Delta_1\times\Delta_2)} deg_{\Delta_1\times\Delta_2}(u,v) = 2|R_1||R_2|,$$

where $ideg_{\Delta_1\times\Delta_2}(u,v)$, $odeg_{\Delta_1\times\Delta_2}(u,v)$ and $deg_{\Delta_1\times\Delta_2}(u,v)$ represent the indegree, outdegree, and degree of the vertex (u,v) respectively in $\Delta_1 \times \Delta_2$.

Proof:

1) According to Theorem 1, $\Delta_1 \times \Delta_2$ is a directed graph containing $|R_1||R_2|$ arcs. As each arc contributes 1 to the sums, $\sum_{(u,v)\in T(\Delta_1\times\Delta_2)} ideg_{\Delta_1\times\Delta_2}(u,v)$ and $\sum_{(u,v)\in T(\Delta_1\times\Delta_2)} odeg_{\Delta_1\times\Delta_2}(u,v)$, we have
$$\sum_{(u,v)\in T(\Delta_1\times\Delta_2)} ideg_{\Delta_1\times\Delta_2}(u,v) = \sum_{(u,v)\in T(\Delta_1\times\Delta_2)} odeg_{\Delta_1\times\Delta_2}(u,v) = |R_1||R_2|.$$
2) It is known that the degree of a vertex (u,v) in $\Delta_1 \times \Delta_2$ is equal to the sum of its indegree and outdegree in $\Delta_1 \times \Delta_2$, since it is a directed graph. So by part (1) of this theorem,
$$\sum_{(u,v)\in T(\Delta_1\times\Delta_2)} deg_{\Delta_1\times\Delta_2}(u,v) = 2|R_1||R_2|.$$

■

Example 3. Consider the directed graphs Δ_1 and Δ_2, and the categorical product $\Delta_1 \times \Delta_2$ given in Example 1. Here we have,
$$\sum_{u\in T(\Delta_1\times\Delta_2)} ideg\ u = 15,$$
$$\sum_{u\in T(\Delta_1\times\Delta_2)} odeg\ u = 15$$
and
$$|R_1||R_2| = 5 \cdot 3 = 15.$$

That is,
$$\sum_{u\in T(\Delta_1\times\Delta_2)} ideg\ u = \sum_{u\in T(\Delta_1\times\Delta_2)} odeg\ u = |R_1||R_2|.$$
Also,
$$\sum_{u\in T(\Delta_1\times\Delta_2)} deg\ u = 30$$
and
$$2(|R_1||R_2|) = 2(5 \cdot 3) = 30$$
That is,
$$\sum_{u\in T(\Delta_1\times\Delta_2)} deg\ u = 2(|R_1||R_2|).$$

III. Modular Product of Directed Graphs

In this section, we define the modular product of two directed graphs. The section also offers several theorems that focus on the structural properties of the resulting graph, including vertex and arc counts, indegree, outdegree, and degree. These theorems are illustrated through examples that validate and explain the findings.

Definition III.1. Consider two directed graphs $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. The modular product $\Delta_1 \bigcirc \Delta_2$ is a directed graph having vertex set $T(\Delta_1 \bigcirc \Delta_2) = T_1 \times T_2$

and arc set $R(\Delta_1 \bigcirc \Delta_2)$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $\Delta_1 \bigcirc \Delta_2$ if and only if

1) an arc (v_1, v_2) exists in Δ_1, and another arc (v_1', v_2') exists in Δ_2 or
2) (v_1, v_2) does not form an arc in Δ_1, and likewise, (v_1', v_2') does not constitute an arc in Δ_2.

Example 4. Let $\Delta_1 = (T_1, R_1)$ represent a directed graph with a vertex set $T_1 = \{v_1, v_2, v_3, v_4\}$ and arc set $R_1 = \{(v_1, v_3), (v_1, v_4), (v_4, v_1), (v_3, v_4), (v_3, v_2)\}$ which is shown in Fig. 4. Let $\Delta_2 = (T_2, R_2)$ denote a directed graph with a vertex set $T_2 = \{u_1, u_2, u_3\}$ and arc set $R_2 = \{(u_2, u_1), (u_2, u_3)\}$ which is shown in Fig. 5. The resulting modular product of these two directed graphs, Δ_1 and Δ_2, denoted as $D = \Delta_1 \bigcirc \Delta_2$, is depicted in Fig. 6.

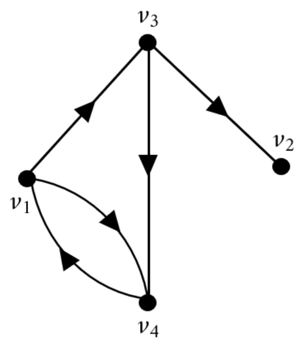

Fig. 4: Directed Graph $\Delta_1 = (T_1, R_1)$

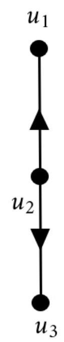

Fig. 5: Directed Graph $\Delta_2 = (T_2, R_2)$

Theorem 2. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs. Then $\Delta_1 \bigcirc \Delta_2$ contains $|T_1||T_2|$ vertices and $|R_1||R_2| + (|T_1|(|T_1|-1) - |R_1|)(|T_2|(|T_2|-1) - |R_2|)$ arcs.

Proof: The vertex set of $\Delta_1 \bigcirc \Delta_2$ is $T_1 \times T_2$ which contains $|T_1||T_2|$ elements. Therefore total number of vertices in $\Delta_1 \bigcirc \Delta_2$ is $|T_1||T_2|$. Also we know, $((v_e, v_f), (v_k, v_l))$ is an arc in $\Delta_1 \bigcirc \Delta_2$ if and only if

1) the arc (v_e, v_k) is present in Δ_1, and the arc (v_f, v_l) exists in Δ_2 or
2) the pair (v_e, v_k) does not form an arc in Δ_1, and similarly, the pair (v_f, v_l) does not constitute an arc in Δ_2.

Each arc in $\Delta_1 \bigcirc \Delta_2$ satisfies only one of the two conditions mentioned above; both cannot be fulfilled simultaneously. Hence, to ascertain the overall count of arcs in $\Delta_1 \bigcirc \Delta_2$, we must aggregate the number of arcs produced by each condition.

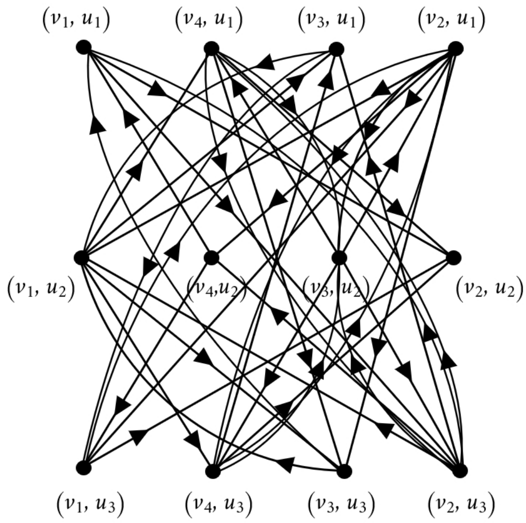

Fig. 6: $\Delta = \Delta_1 \bigcirc \Delta_2$

Let's consider the initial condition for adjacency, where (v_e, v_k) represents an arc in Δ_1, and (v_f, v_l) denotes an arc in Δ_2. There are $|R_1|$ arcs in Δ_1 and $|R_2|$ arcs in Δ_2. Hence, we can select a pair of arcs a_m and a_n such that one is from Δ_1 and the other is from Δ_2 in $|R_1||R_2|$ different ways. Suppose a_m represents the arc (v_e, v_k) in Δ_1, and a_n represents the arc (v_f, v_l) in Δ_2. This pairing of arcs results in an arc $((v_e, v_f), (v_k, v_l))$ in $\Delta_1 \bigcirc \Delta_2$. In essence, we obtain $|R_1||R_2|$ arcs satisfying the first condition of adjacency.

Now, let's consider the second condition for adjacency, where (v_e, v_k) is not an arc in Δ_1 and (v_f, v_l) is not an arc in Δ_2. We can select two distinct vertices v_e and v_k in Δ_1 such that (v_e, v_k) is not an arc in Δ_1 in $(|T_1|(|T_1|-1) - |R_1|)$ different ways. Similarly, we can choose two distinct vertices v_f and v_l in Δ_2 such that (v_f, v_l) is not an arc in Δ_2 in $(|T_2|(|T_2|-1) - |R_2|)$ different ways.

Let v_e and v_k be two vertices in Δ_1 such that (v_e, v_k) is not an arc in Δ_1, and let v_f and v_l be two vertices in Δ_2 such that (v_f, v_l) is not an arc in Δ_2. This yields an arc $((v_e, v_f), (v_k, v_l))$ in $\Delta_1 \bigcirc \Delta_2$. Therefore, the second condition for adjacency yields a total of $(|T_1|(|T_1|-1) - |R_1|)(|T_2|(|T_2|-1) - |R_2|)$ arcs in $\Delta_1 \bigcirc \Delta_2$.

Consequently, the total number of arcs in $\Delta_1 \bigcirc \Delta_2$ is $(|T_1|(|T_1|-1) - |R_1|)(|T_2|(|T_2|-1) - |R_2|) + |R_1||R_2|$. ■

Example 5. Consider the directed graphs Δ_1 and Δ_2 given in Example 4.
Here we have, the total count of vertices in $\Delta_1 \bigcirc \Delta_2$ = 12 and $|T_1||T_2| = 4 \cdot 3 = 12$.
That is, the total count of vertices in $\Delta_1 \bigcirc \Delta_2 = |T_1||T_2|$.
Also, total count of arcs in $\Delta_1 \bigcirc \Delta_2 = 38$ and $|R_1||R_2| + (|T_1|(|T_1|-1) - |R_1|)(|T_2|(|T_2|-1) - |R_2|) = 5 \cdot 2 + (4 \cdot 3 - 5)(3 \cdot 2 - 2) = 38$.
That is, total count of arcs in $\Delta_1 \bigcirc \Delta_2 = |R_1||R_2| + (|T_1|(|T_1|-1) - |R_1|)(|T_2|(|T_2|-1) - |R_2|)$.

Corollary 2. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs and $\Delta_1 \bigcirc \Delta_2 = (T(\Delta_1 \bigcirc \Delta_2), R(\Delta_1 \bigcirc \Delta_2))$ be their modular product. Then

1)
$$\sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} ideg_{\Delta_1 \bigcirc \Delta_2}(u,v)$$
$$= \sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} odeg_{\Delta_1 \bigcirc \Delta_2}(u,v)$$
$$= (|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)$$
$$-|R_2|)+|R_1||R_2|,$$

2)
$$\sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} deg_{\Delta_1 \bigcirc \Delta_2}(u,v)$$
$$= 2(|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)$$
$$-|R_2|)+2|R_1||R_2|,$$

where $ideg_{\Delta_1 \bigcirc \Delta_2}(u,v)$, $odeg_{\Delta_1 \bigcirc \Delta_2}(u,v)$ and $deg_{\Delta_1 \bigcirc \Delta_2}(u,v)$ represent the indegree, outdegree and degree of the vertex (u,v) respectively, in $\Delta_1 \bigcirc \Delta_2$.

Proof:

1) According to Theorem 2, $\Delta_1 \bigcirc \Delta_2$ forms a directed graph containing $(|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)-|R_2|)+|R_1||R_2|$ arcs. Since each arc contributes 1 to the sums $\sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} ideg_{\Delta_1 \bigcirc \Delta_2}(u,v)$ and $\sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} odeg_{\Delta_1 \bigcirc \Delta_2}(u,v)$, we have
$$\sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} ideg_{\Delta_1 \bigcirc \Delta_2}(u,v)$$
$$= \sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} odeg_{\Delta_1 \bigcirc \Delta_2}(u,v)$$
$$= (|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)$$
$$-|R_2|)+|R_1||R_2|.$$

2) The degree of a vertex (u,v) in $\Delta_1 \bigcirc \Delta_2$ equals the sum of its indegree and outdegree in $\Delta_1 \bigcirc \Delta_2$ because it is a directed graph. So by part (1) of this theorem,
$$\sum_{(u,v)\in T(\Delta_1 \bigcirc \Delta_2)} deg_{\Delta_1 \bigcirc \Delta_2}(u,v)$$
$$= 2(|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)$$
$$-|R_2|)+2|R_1||R_2|.$$

■

Example 6. Consider the directed graphs Δ_1 and Δ_2, and the modular product $\Delta_1 \bigcirc \Delta_2$ given in Example 4. Here we have,
$$\sum_{u\in T(\Delta_1 \bigcirc \Delta_2)} ideg\ u = 38,$$
$$\sum_{u\in T(\Delta_1 \bigcirc \Delta_2)} odeg\ u = 38$$
and
$$|R_1||R_2|+(|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)-|R_2|) = 38.$$

That is,
$$\sum_{u\in T(\Delta_1 \bigcirc \Delta_2)} ideg\ u = \sum_{u\in T(\Delta_1 \bigcirc \Delta_2)} odeg\ u$$
$$= |R_1||R_2| + (|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)$$
$$-|R_2|).$$

Also,
$$\sum_{u\in T(\Delta_1 \bigcirc \Delta_2)} deg\ u = 76$$
and
$$2(|R_1||R_2| + (|T_1|(|T_1|-1)-|R_1|)(|T_2|(|T_2|-1)$$
$$-|R_2|)) = 2(38) = 76$$

That is,
$$\sum_{u\in T(\Delta_1 \bigcirc \Delta_2)} deg\ u = 2(|R_1||R_2| + (|T_1|(|T_1|-1)$$
$$-|R_1|)(|T_2|(|T_2|-1)-|R_2|)).$$

IV. Disjunctive (or Co-normal Product) Product of Directed Graphs

This section gives the definition of the disjunctive product of two directed graphs. Theorems related to vertex and arc counts, as well as degree sums, are presented to explore the properties of the disjunctive product. Examples accompany the theorems to provide practical insights and demonstrate the results clearly.

Definition IV.1. Consider two directed graphs $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. The disjunctive product $\Delta_1 * \Delta_2$ forms a directed graph with the vertex set $T(\Delta_1 * \Delta_2) = T_1 \times T_2$ and the arc set $R(\Delta_1 * \Delta_2)$. Here, an arc $((v_1, v_1'), (v_2, v_2'))$ exists in $\Delta_1 * \Delta_2$ if and only if either (v_1, v_2) is an arc in Δ_1 or (v_1', v_2') is an arc in Δ_2.

Example 7. Let $\Delta_1 = (T_1, R_1)$ represent a directed graph with a vertex set $T_1 = v_1, v_2, v_3, v_4$ and an arc set $R_1 = (v_3, v_1), (v_3, v_2), (v_4, v_3), (v_2, v_1)$, as illustrated in Fig. 7. Let $\Delta_2 = (T_2, R_2)$ represent a directed

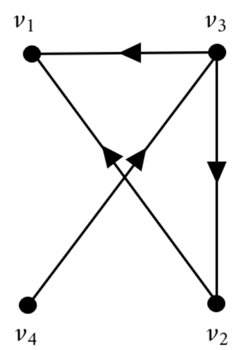

Fig. 7: Directed Graph $\Delta_1 = (T_1, R_1)$

graph with a vertex set $T_2 = u_1, u_2, u_3$ and an arc set $R_2 = (u_2, u_1), (u_3, u_2), (u_3, u_1)$, as depicted in Fig. 8. Thus, the disjunctive product of these two directed graphs, Δ_1 and Δ_2, denoted by $D = \Delta_1 * \Delta_2$, is illustrated in Fig. 9.

Theorem 3. Consider two directed graphs $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. The disjunctive product of Δ_1 and Δ_2, denoted as $\Delta_1 * \Delta_2$, contains $|T_1||T_2|$ vertices and $|R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|$ arcs.

Proof: The vertex set of $\Delta_1 * \Delta_2$ is denoted by $T_1 \times T_2$, holding a count of $|T_1||T_2|$ elements. Consequently, the

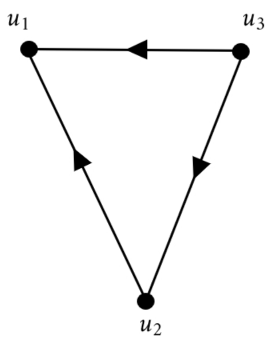

Fig. 8: Directed Graph $\Delta_2 = (T_2, R_2)$

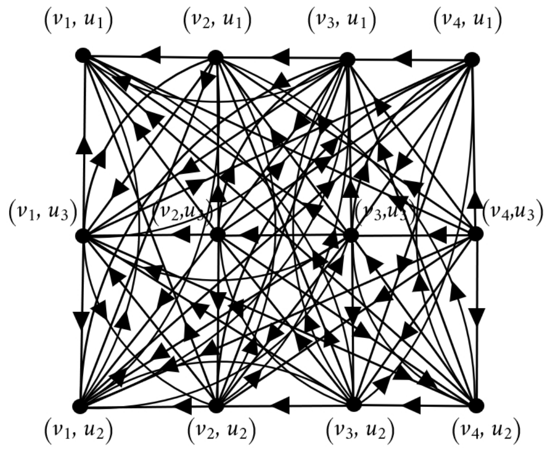

Fig. 9: $\Delta = \Delta_1 * \Delta_2$

overall count of vertices within $\Delta_1 * \Delta_2$ amounts to $|T_1||T_2|$. Additionally, an arc exists between (v_e, v_f) and (v_k, v_l) within $\Delta_1 * \Delta_2$ if and only if (v_e, v_k) forms an arc in Δ_1 or if (v_f, v_l) forms an arc in Δ_2.

Each arc in $\Delta_1 * \Delta_2$ arises from just one of these conditions, and it's possible for both conditions to hold simultaneously. To determine the overall count of arcs in $\Delta_1 * \Delta_2$, we tally the arcs generated by each condition and then subtract the count of arcs formed by both.

Let's explore the first adjacency condition, where (v_e, v_k) forms an arc in Δ_1. The number of vertices in $T_1 \times T_2$, in the form (v_e, v_p) with v_e fixed in T_1 and $v_p \in T_2$, equals $|T_2|$. Similarly, the count of vertices in $T_1 \times T_2$, in the form (v_k, v_q) with v_k fixed in T_1 and $v_q \in T_2$, is $|T_2|$. Among these $2|T_2|$ vertices, we can select 2 vertices, one in the form (v_e, v_p) and the other in the form (v_k, v_q) in $|T_2||T_2| = |T_2|^2$ distinct ways. Corresponding to each selection, an arc $((v_e, v_p), (v_k, v_q))$ emerges in $\Delta_1 * \Delta_2$. There exist $|R_1|$ distinct arcs in Δ_1, such as the arc (v_e, v_k). Therefore, the number of arcs generated by this condition corresponds to the number of arcs in the directed graph Δ_1 multiplied by $|T_2|^2$, resulting in $|R_1||T_2|^2$ arcs.

The second condition for adjacency specifies that (v_f, v_l) forms an arc in Δ_2. Like the first condition, the count of arcs formed by the second condition for adjacency is $|R_2||T_1|^2$.

Subsequently, we compute the number of arcs formed by both conditions. Δ_1 comprises $|R_1|$ arcs, and Δ_2 comprises $|R_2|$ arcs. Therefore, we can select a pair of arcs a_m and a_n, one from Δ_1 and the other from Δ_2, in $|R_1||R_2|$ distinct ways. Let's say a_m represents the arc (v_e, v_k) in Δ_1, and a_n represents the arc (v_f, v_l) in Δ_2. This chosen pair of arcs gives rise to an arc $((v_e, v_f), (v_k, v_l))$ in $\Delta_1 * \Delta_2$. Consequently, $|R_1||R_2|$ arcs satisfy both adjacency conditions.

Therefore, the total count of arcs in $\Delta_1 * \Delta_2$ is derived as $|R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|$. ■

Example 8. Consider the directed graphs Δ_1 and Δ_2 given in Example 7.
Here we have, the total count of vertices in $\Delta_1 * \Delta_2$ = 12 and $|T_1||T_2| = 4 \cdot 3 = 12$.
That is, the total count of vertices in $\Delta_1 * \Delta_2 = |T_1||T_2|$.
Also, total count of arcs in $\Delta_1 * \Delta_2 = 72$ and $|R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2| = 4 \cdot 9 + 3 \cdot 16 - 4 \cdot 3 = 72$.
That is, total count of arcs in $\Delta_1 * \Delta_2 = |R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|$.

Corollary 3. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs and $\Delta_1 * \Delta_2 = (T(\Delta_1 * \Delta_2), R(\Delta_1 * \Delta_2))$ be their disjunctive product. Then

1)
$$\sum_{(u,v)\in T(\Delta_1*\Delta_2)} ideg_{\Delta_1*\Delta_2}(u, v)$$
$$= \sum_{(u,v)\in T(\Delta_1*\Delta_2)} odeg_{\Delta_1*\Delta_2}(u, v)$$
$$= |R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|,$$

2)
$$\sum_{(u,v)\in T(\Delta_1*\Delta_2)} deg_{\Delta_1*\Delta_2}(u, v)$$
$$= 2|R_1||T_2|^2 + 2|R_2||T_1|^2 - 2|R_1||R_2|,$$

where $ideg_{\Delta_1*\Delta_2}(u, v)$, $odeg_{\Delta_1*\Delta_2}(u, v)$ and $deg_{\Delta_1*\Delta_2}(u, v)$ represent the indegree, outdegree and degree of the vertex (u, v) respectively, in $\Delta_1 * \Delta_2$.

Proof:

1) According to Theorem 3, $\Delta_1 * \Delta_2$ forms a directed graph with $|R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|$ arcs. As each arc contributes 1 to the sums $\sum_{(u,v)\in T(\Delta_1*\Delta_2)} ideg_{\Delta_1*\Delta_2}(u, v)$ and $\sum_{(u,v)\in T(\Delta_1*\Delta_2)} odeg_{\Delta_1*\Delta_2}(u, v)$, we have
$$\sum_{(u,v)\in T(\Delta_1*\Delta_2)} ideg_{\Delta_1*\Delta_2}(u, v)$$
$$= \sum_{(u,v)\in T(\Delta_1*\Delta_2)} odeg_{\Delta_1*\Delta_2}(u, v)$$
$$= |R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|.$$

2) The degree of a vertex (u, v) in $\Delta_1 * \Delta_2$ is the sum of its indegree and outdegree in $\Delta_1 * \Delta_2$ since it is a directed graph. So by part (1) of this theorem,
$$\sum_{(u,v)\in T(\Delta_1*\Delta_2)} deg_{\Delta_1*\Delta_2}(u, v)$$
$$= 2|R_1||T_2|^2 + 2|R_2||T_1|^2 - 2|R_1||R_2|.$$

■

Example 9. Consider the directed graphs Δ_1 and Δ_2, and the disjunctive product $\Delta_1 * \Delta_2$ given in Example 7. Here we have,
$$\sum_{u\in T(\Delta_1*\Delta_2)} ideg\ u = 72,$$

$$\sum_{u \in T(\Delta_1 * \Delta_2)} odeg\ u = 72$$

and

$$|R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2| = 72.$$

That is,

$$\sum_{u \in T(\Delta_1 * \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 * \Delta_2)} odeg\ u$$
$$= |R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|.$$

Also,

$$\sum_{u \in T(\Delta_1 * \Delta_2)} deg\ u = 144$$

and

$$2(|R_1||T_2|^2 + |R_2||T_1|^2 - |R_1||R_2|)$$
$$= 2(72) = 144$$

That is,

$$\sum_{u \in T(\Delta_1 * \Delta_2)} deg\ u = 2(|R_1||T_2|^2 + |R_2||T_1|^2$$
$$-|R_1||R_2|).$$

V. Homomorphic Product of Directed Graphs

This section introduces the definition of the homomorphic product of two directed graphs. We also present theorems regarding the number of vertices, arcs, and degree sums in the resulting product. Examples are provided to justify and illustrate the theoretical results discussed in this section.

Definition V.1. Consider two directed graphs $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. Then, the homomorphic product $\Delta_1 \ltimes \Delta_2$ forms a directed graph with a set of vertices $T(\Delta_1 \ltimes \Delta_2) = T_1 \times T_2$ and set of arcs $R(\Delta_1 \ltimes \Delta_2)$, where $((v_1, v_1'), (v_2, v_2'))$ is an arc in $\Delta_1 \ltimes \Delta_2$ if and only if

1) $v_1 = v_2$ or
2) the arc (v_1, v_2) exists in Δ_1, and (v_1', v_2') is not an arc in Δ_2.

Example 10. Let $\Delta_1 = (T_1, R_1)$ represent a directed graph with a vertex set $T_1 = \{v_1, v_2, v_3, v_4\}$ and arc set $R_1 = \{(v_2, v_1), (v_4, v_2), (v_2, v_3), (v_3, v_4)\}$ which is shown in Fig. 10. Let $\Delta_2 = (T_2, R_2)$ denote a directed graph with a vertex

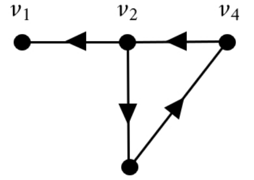

Fig. 10: Directed Graph $\Delta_1 = (T_1, R_1)$

set $T_2 = \{u_1, u_2, u_3\}$ and arc set $R_2 = \{(u_2, u_3), (u_2, u_1)\}$ which is shown in Fig. 11. Then the homomorphic product of these two directed graphs Δ_1 and Δ_2, represented by $D = \Delta_1 \ltimes \Delta_2$, is shown in Fig. 12.

Theorem 4. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs. Then the homomorphic product $\Delta_1 \ltimes \Delta_2$

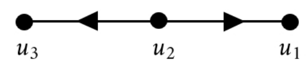

Fig. 11: Directed Graph $\Delta_2 = (T_2, R_2)$

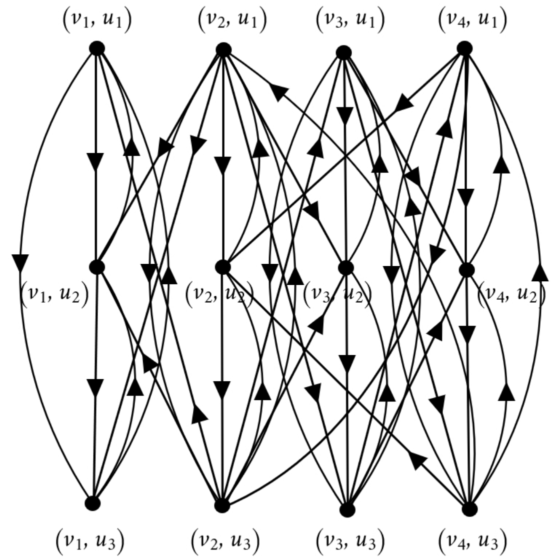

Fig. 12: $\Delta = \Delta_1 \ltimes \Delta_2$

contains $|T_1||T_2|$ vertices and $2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2| - 1) - |R_2|)$ arcs.

Proof: The vertex set of $\Delta_1 \ltimes \Delta_2$ is $T_1 \times T_2$ which contains $|T_1||T_2|$ elements. Therefore total number of vertices in $\Delta_1 \ltimes \Delta_2$ is $|T_1||T_2|$. Also we know, $((v_e, v_f), (v_k, v_l))$ is an arc in $\Delta_1 \ltimes \Delta_2$ if and only if

1) $v_e = v_k$ or
2) the arc (v_e, v_k) is present in Δ_1, and (v_f, v_l) is not an arc in Δ_2.

Each arc in $\Delta_1 \ltimes \Delta_2$ is formed by one of these two requirements; they cannot both be true simultaneously. Therefore, to determine the total number of arcs in $\Delta_1 \ltimes \Delta_2$, we sum the number of arcs generated by each condition.

Let's explore the first condition for adjacency, where $v_e = v_k$. Suppose v is any vertex in Δ_1. The directed graph Δ_2 contains $|T_2|$ vertices. We can select 2 distinct vertices v' and v'' from Δ_2 in $\binom{|T_2|}{2}$ different ways. For each choice, we obtain 2 arcs $((v, v'), (v, v''))$ and $((v, v''), (v, v'))$ in $\Delta_1 \ltimes \Delta_2$. Like v, there are a total of $|T_1|$ vertices in Δ_1. Hence, the first condition of adjacency yields $2|T_1|\binom{|T_2|}{2}$ arcs in $\Delta_1 \ltimes \Delta_2$.

Now, let's consider the second condition for adjacency, where (v_e, v_k) is an arc in Δ_1 and (v_f, v_l) is not an arc in Δ_2. We can select two different vertices v_e and v_k in Δ_1 such that there is an arc (v_e, v_k) in Δ_1, in $|R_1|$ different ways. Similarly, we can select two different vertices v_f and v_l in Δ_2 such that (v_f, v_l) is not an arc in Δ_2, which can be done in $|T_2|(|T_2| - 1) - |R_2|$ different ways. Let v_e and v_k be two vertices in Δ_1 such that (v_e, v_k) forms an arc in Δ_1, and let v_f and v_l be two vertices in Δ_2 such that (v_f, v_l) is not an arc in Δ_2. With this arrangement, an arc $((v_e, v_f), (v_k, v_l))$

is established in $\Delta_1 \ltimes \Delta_2$. Thus, the second condition for adjacency contributes $|R_1|(|T_2|(|T_2|-1)-|R_2|)$ arcs in $\Delta_1 \ltimes \Delta_2$. Consequently, the total number of arcs in $\Delta_1 \ltimes \Delta_2$ is $2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|)$. ■

Example 11. Consider the directed graphs Δ_1 and Δ_2 given in Example 10.
Here we have, the total count of vertices in $\Delta_1 \ltimes \Delta_2$ = 12 and $|T_1||T_2| = 4 \cdot 3 = 12$.
That is, the total count of vertices in $\Delta_1 \ltimes \Delta_2 = |T_1||T_2|$.
Also, total count of arcs in $\Delta_1 \ltimes \Delta_2 = 40$ and $2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|) = (2 \cdot 4 \cdot 3) + 4(3 \cdot 2 - 2) = 24 + 16 = 40$.
That is, total count of arcs in $\Delta_1 \ltimes \Delta_2 = 2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|)$.

Corollary 4. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs and $\Delta_1 \ltimes \Delta_2 = (T(\Delta_1 \ltimes \Delta_2), R(\Delta_1 \ltimes \Delta_2))$ be their homomorphic product. Then

1)
$$\sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} ideg_{\Delta_1\ltimes\Delta_2}(u,v)$$
$$= \sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} odeg_{\Delta_1\ltimes\Delta_2}(u,v)$$
$$= 2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|),$$

2)
$$\sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} deg_{\Delta_1\ltimes\Delta_2}(u,v) =$$
$$4|T_1|\binom{|T_2|}{2} + 2|R_1|(|T_2|(|T_2|-1)-|R_2|),$$

where $ideg_{\Delta_1\ltimes\Delta_2}(u,v)$, $odeg_{\Delta_1\ltimes\Delta_2}(u,v)$ and $deg_{\Delta_1\ltimes\Delta_2}(u,v)$ represent the indegree, outdegree and degree of the vertex (u,v) respectively, in $\Delta_1 \ltimes \Delta_2$.

Proof:

1) According to Theorem 4, $\Delta_1 \ltimes \Delta_2$ constitutes a directed graph with $2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|)$ arcs. As each arc contributes 1 to the sums $\sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} ideg_{\Delta_1\ltimes\Delta_2}(u,v)$ and $\sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} odeg_{\Delta_1\ltimes\Delta_2}(u,v)$, we have
$$\sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} ideg_{\Delta_1\ltimes\Delta_2}(u,v)$$
$$= \sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} odeg_{\Delta_1\ltimes\Delta_2}(u,v)$$
$$= 2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|).$$
2) The degree of a vertex (u,v) in $\Delta_1 \ltimes \Delta_2$ is the sum of its indegree and outdegree in $\Delta_1 \ltimes \Delta_2$ because it is a directed graph. So by part (1) of this theorem,
$$\sum_{(u,v)\in T(\Delta_1\ltimes\Delta_2)} deg_{\Delta_1\ltimes\Delta_2}(u,v) =$$
$$4|T_1|\binom{|T_2|}{2} + 2|R_1|(|T_2|(|T_2|-1)-|R_2|).$$

■

Example 12. Consider the directed graphs Δ_1 and Δ_2, and the homomorphic product $\Delta_1 \ltimes \Delta_2$ given in Example 10. Here we have,
$$\sum_{u\in T(\Delta_1\ltimes\Delta_2)} ideg\ u = 40,$$
$$\sum_{u\in T(\Delta_1\ltimes\Delta_2)} odeg\ u = 40$$
and
$$2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|) = 40.$$
That is,
$$\sum_{u\in T(\Delta_1\ltimes\Delta_2)} ideg\ u = \sum_{u\in T(\Delta_1\ltimes\Delta_2)} odeg\ u$$
$$= 2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|).$$
Also,
$$\sum_{u\in T(\Delta_1\ltimes\Delta_2)} deg\ u = 80$$
and
$$2(2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|))$$
$$= 2(40) = 80$$
That is,
$$\sum_{u\in T(\Delta_1\ltimes\Delta_2)} deg\ u$$
$$= 2(2|T_1|\binom{|T_2|}{2} + |R_1|(|T_2|(|T_2|-1)-|R_2|))$$
$$= 4|T_1|\binom{|T_2|}{2} + 2|R_1|(|T_2|(|T_2|-1)-|R_2|).$$

VI. Rooted Product of Directed Graphs

Here, we define the rooted product of directed graphs. The section includes several theorems that analyze the number of vertices, arcs, and degree distributions in the rooted product. To support these theoretical findings, examples are given to help visualize the construction and its properties.

Definition VI.1. Let's say $\Delta_1 = (T_1, R_1)$ represents a directed graph, and $\Delta_2 = (T_2, R_2)$ signifies a rooted directed graph with a root vertex v_r. The rooted product of Δ_1 and Δ_2, denoted as $\Delta_1 \circ_{v_r} \Delta_2$, forms a directed graph. This resulting graph has a vertex set defined as $T(\Delta_1 \circ_{v_r} \Delta_2) = \{v_{i,r} : v_i \in T_1$ and v_r is the root vertex of $\Delta_2\} \cup \{v_{i,j} : v_i \in T_1$ and v_j is a vertex in T_2 excluding the root vertex of $\Delta_2\}$, along with an arc set described as $R(\Delta_1 \circ_{v_r} \Delta_2) = \{(v_{i,r}, v_{k,r}) : v_r$ represents the root vertex in Δ_2 and (v_i, v_k) denotes an arc in $\Delta_1\} \cup \{(v_{i,k}, v_{i,l}) : v_i \in T_1$ and (v_k, v_l) forms an arc in $\Delta_2\}$.

Example 13. Consider $\Delta_1 = (T_1, R_1)$, a directed graph with a vertex set $T_1 = v_1, v_2, v_3$ and an arc set $R_1 = (v_1, v_2), (v_3, v_1), (v_2, v_3)$, as depicted in $Fig.$ 13.
Consider $\Delta_2 = (T_2, R_2)$, a rooted directed graph with a vertex set $T_2 = v_1, v_2, v_3, v_4$, a root vertex v_2, and an arc

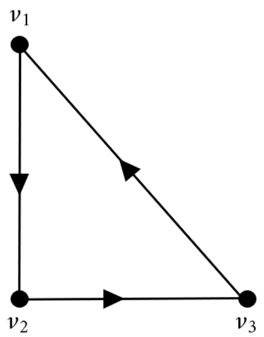

Fig. 13: Directed Graph $\Delta_1 = (T_1, R_1)$

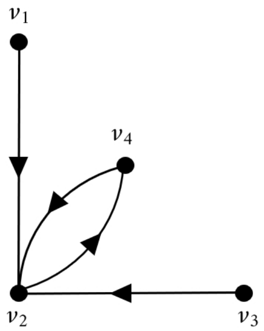

Fig. 14: Rooted Directed Graph $\Delta_2 = (T_2, R_2)$

set $R_2 = (v_1, v_2), (v_3, v_2), (v_2, v_4), (v_4, v_2)$, as illustrated in $Fig.$ 14.
The rooted product of Δ_1 and Δ_2 from these two soft directed graphs is depicted in $Fig.$ 15.

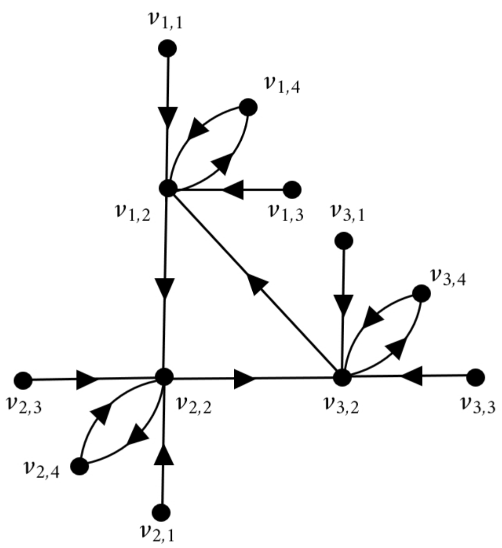

Fig. 15: $\Delta = \Delta_1 \circ_{v_2} \Delta_2$

Theorem 5. Consider $\Delta_1 = (T_1, R_1)$ as a directed graph and $\Delta_2 = (T_2, R_2)$ as a rooted directed graph with the root vertex v_r. Then, the rooted product of Δ_1 and Δ_2, denoted by $\Delta_1 \circ_{v_r} \Delta_2$, comprises $|T_1||T_2|$ vertices and $(|R_1| + |T_1||R_2|)$ arcs.

Proof: By definition, the vertex set of $\Delta_1 \circ_{v_r} \Delta_2$ is given by $T(\Delta_1 \circ_{v_r} \Delta_2) = \{v_{i,r} : v_i \in T_1$ and v_r is the root vertex of $\Delta_2\} \cup \{v_{i,j} : v_i \in T_1$ and v_j is a vertex in T_2 other than the root vertex of $\Delta_2\}$. To find the total number of vertices in $\Delta_1 \circ_{v_r} \Delta_2$, we add up the counts of vertices in these two sets which are disjoint.

Consider the set $\{v_{i,r} : v_i \in T_1$ and v_r is the root vertex of $\Delta_2\}$ which contains $|T_1|$ elements. Then, let's examine the set $\{v_{i,j} : v_i \in T_1$ and v_j is a vertex in T_2 other than the root vertex of $\Delta_2\}$ which contains $|T_1|(|T_2| - 1)$ vertices. Hence, the total number of vertices in $\Delta_1 \circ_{v_r} \Delta_2 = |T_1| + |T_1|(|T_2| - 1) = |T_1||T_2|$.

The arc set of $\Delta_1 \circ_{v_r} \Delta_2$ is given by $R(\Delta_1 \circ_{v_r} \Delta_2) = \{(v_{i,r}, v_{k,r}) : v_r$ is the root vertex in Δ_2 and (v_i, v_k) is an arc in $\Delta_1\} \cup \{(v_{i,k}, v_{i,l}) : v_i \in T_1$ and (v_k, v_l) is an arc in $\Delta_2\}$. To obtain the total count of arcs in $\Delta_1 \circ_{v_r} \Delta_2$, we sum up the number of arcs contained in these two disjoint sets.

Consider the set $\{(v_{i,r}, v_{k,r}) : v_r$ is the root vertex in Δ_2 and (v_i, v_k) is an arc in $\Delta_1\}$ which contains $|R_1|$ arcs. Then, let's examine the set $\{(v_{i,k}, v_{i,l}) : v_i \in T_1$ and (v_k, v_l) is an arc in $\Delta_2\}$ which contains $|R_2|$ arcs corresponding to each vertex $v_i, i = 1, 2, \ldots |T_1|$ and hence totally $|T_1||R_2|$ arcs. Therefore, total count of arcs in $\Delta_1 \circ_{v_r} \Delta_2$ is $|R_1| + |T_1||R_2|$. ■

Example 14. Consider the directed graphs Δ_1 and Δ_2 and the rooted product $\Delta_1 \circ_{v_2} \Delta_2$ given in Example 13.
Here we have, the total count of vertices in $\Delta_1 \circ_{v_2} \Delta_2$ = 12 and $|T_1||T_2| = 3.4 = 12$.
That is, total number of vertices in $\Delta_1 \circ_{v_2} \Delta_2 = |T_1||T_2|$.
Also, total count of arcs in $\Delta_1 \circ_{v_2} \Delta_2 = 15$ and $|R_1| + |T_1||R_2| = 3 + 3.4 = 15$.
That is, total count of arcs in $\Delta_1 \circ_{v_r} \Delta_2$= $|R_1| + |T_1||R_2|$.

Corollary 5. Consider $\Delta_1 = (T_1, R_1)$ as a directed graph and $\Delta_2 = (T_2, R_2)$ as a rooted directed graph with the root vertex v_r. Let $\Delta_1 \circ_{v_r} \Delta_2 = (T(\Delta_1 \circ_{v_r} \Delta_2), R(\Delta_1 \circ_{v_r} \Delta_2))$ represent the rooted product of Δ_1 and Δ_2. Then,

$$(i) \sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} odeg\ u = |R_1| + |T_1||R_2|$$

$$(ii) \sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} deg\ u = 2(|R_1| + |T_1||R_2|),$$

where $odeg\ u, ideg\ u$ and $deg\ u$ represent the out-degree, in-degree and degree of the vertex u respectively, in the rooted product $\Delta_1 \circ_{v_r} \Delta_2$.

Proof: (i) Considering the rooted product $\Delta_1 \circ_{v_r} \Delta_2 = (T(\Delta_1 \circ_{v_r} \Delta_2), R(\Delta_1 \circ_{v_r} \Delta_2))$, according to Theorem 5, the number of arcs in $\Delta_1 \circ_{v_r} \Delta_2$ is $|R_1| + |T_1||R_2|$. As the rooted product $\Delta_1 \circ_{v_r} \Delta_2$ is a directed graph having $|R_1| + |T_1||R_2|$ arcs, we have

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} odeg\ u = |R_1| + |T_1||R_2|,$$

since each arc in the rooted product $\Delta_1 \circ_{v_r} \Delta_2$ contributes one to each of the sums $\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} ideg\ u$ and $\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} odeg\ u$.
(ii) Since, $deg\ u = ideg\ u + odeg\ u$ and by part (i) of this corollary we have,

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} deg\ u = 2(|R_1| + |T_1||R_2|).$$

■

Example 15. Consider the directed graphs Δ_1 and Δ_2, and the rooted product $\Delta_1 \circ_{v_2} \Delta_2$ given in Example 13. Here we have,

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} ideg\ u =$$

$$0+4+1+0+4+0+1+0+4+0+0+1=15,$$

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} odeg\ u =$$

$$1+2+1+1+1+2+1+1+1+1+2+1=15$$

and

$$|R_1| + |T_1||R_2| = 3 + 3.4 = 15.$$

That is,

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} odeg\ u$$
$$= |R_1| + |T_1||R_2|.$$

Also,

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} deg\ u =$$

$$1+2+1+6+1+2+1+6+1+2+1+6=30$$

and

$$2(|R_1| + |T_1||R_2|) = 2(3 + 3.4) = 30.$$

That is,

$$\sum_{u \in T(\Delta_1 \circ_{v_r} \Delta_2)} deg\ u = 2(|R_1| + |T_1||R_2|).$$

VII. Corona Product of Directed Graphs

This section gives the definition of the corona product of two directed graphs. Theorems related to vertex and arc counts, as well as degree sums, are presented to explore the properties of the corona product. Examples accompany the theorems to provide practical insights and demonstrate the results clearly.

Definition VII.1. Consider two directed graphs, $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. The corona product of Δ_1 and Δ_2, represented as $\Delta_1 \odot \Delta_2$, is a directed graph with a vertex set $T(\Delta_1 \odot \Delta_2) = T_1 \cup \{v_{i,j} : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\}$ and arc set $R(\Delta_1 \odot \Delta_2) = R_1 \cup \{(v_i, v_{i,j}) : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\} \cup \{(v_{i,j}, v_{i,k}) : i = 1,2,\ldots,|T_1|$ and $v_j v_k$ is an arc in $\Delta_2\}$.

Example 16. Let $\Delta_1 = (T_1, R_1)$ represent a directed graph with a vertex set $T_1 = \{v_1, v_2, v_3, v_4\}$ and arc set $R_1 = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_4, v_1), (v_4, v_2), (v_2, v_3), (v_4, v_3)\}$ which is shown in Fig. 16.
Let $\Delta_2 = (T_2, R_2)$ be a directed graph with a vertex set $T_2 = v_1, v_2, v_3$ and an arc set $R_2 = (v_2, v_1), (v_2, v_3), (v_3, v_2)$, as depicted in $Fig.$ 17.
Then, the corona product of these two directed graphs, Δ_1 and Δ_2, is denoted by $\Delta = \Delta_1 \odot \Delta_2$, and it is illustrated in Fig. 18.

Theorem 6. Consider two directed graphs, $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$. Then, the corona product of Δ_1 and Δ_2, denoted by $\Delta_1 \odot \Delta_2$, comprises $|T_1| + |T_1||T_2|$ vertices and $|R_1| + |T_1||T_2| + |T_1||R_2|$ arcs.

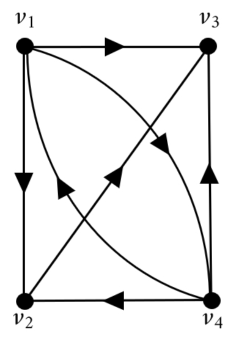

Fig. 16: Directed Graph $\Delta_1 = (T_1, R_1)$

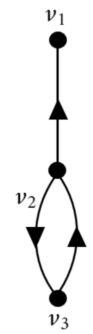

Fig. 17: Directed Graph $\Delta_2 = (T_2, R_2)$

Proof: By definition, the vertex set of $\Delta_1 \odot \Delta_2$ is $T_1 \cup \{v_{i,j} : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\}$. Here, T_1 contains $|T_1|$ vertices and $\{v_{i,j} : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\}$ contains $|T_1||T_2|$ vertices. Hence, the total number of vertices in $\Delta_1 \odot \Delta_2$ is $|T_1| + |T_1||T_2|$.

The arc set of $\Delta_1 \odot \Delta_2$ is $R(\Delta_1 \odot \Delta_2) = R_1 \cup \{(v_i, v_{i,j}) : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\} \cup \{(v_{i,j}, v_{i,k}) : i = 1,2,\ldots,|T_1|$ and $v_j v_k$ is an arc in $\Delta_2\}$. That is, the arc set of $\Delta_1 \odot \Delta_2$ is the union of three disjoint sets of arcs. So, to get the total count of arcs in $\Delta_1 \odot \Delta_2$, we add the number of arcs contained in each set. The set R_1 contains $|R_1|$ arcs. Consider the set $\{(v_i, v_{i,j}) : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\}$. This set contains $|T_2|$ arcs with v_i as their end vertex, for a particular i, and i takes $|T_1|$ different values. So, the set $\{(v_i, v_{i,j}) : i = 1,2,\ldots,|T_1|, j = 1,2,\ldots,|T_2|\}$ contains $|T_1||T_2|$ arcs. Consider the set $\{(v_{i,j}, v_{i,k}) : i = 1,2,\ldots,|T_1|$ and $v_j v_k$ is an arc in $\Delta_2\}$. This set contains $|R_2|$ arcs each, corresponding to all i, and i takes $|T_1|$ different values. So, the set $\{(v_{i,j}, v_{i,k}) : i = 1,2,\ldots,|T_1|$ and $v_j v_k$ is an arc in $\Delta_2\}$ contains $|T_1||R_2|$ arcs. Therefore, the total count of arcs in $\Delta_1 \odot \Delta_2$ is $|R_1| + |T_1||T_2| + |T_1||R_2|$. ■

Example 17. Consider the directed graphs Δ_1 and Δ_2 given in Example 16.
Here we have, the total count of vertices in $\Delta_1 \odot \Delta_2$ = 16 and $|T_1| + |T_1||T_2| = 4 + (4.3) = 16$.
That is, the total count of vertices in $\Delta_1 \odot \Delta_2 = |T_1| + |T_1||T_2|$.
Also, total count of arcs in $\Delta_1 \odot \Delta_2 = 31$ and $|R_1| + |T_1||T_2| + |T_1||R_2| = 7 + 4.3 + 4.3 = 31$.
That is, total count of arcs in $\Delta_1 \odot \Delta_2 = |R_1| + |T_1||T_2| +$

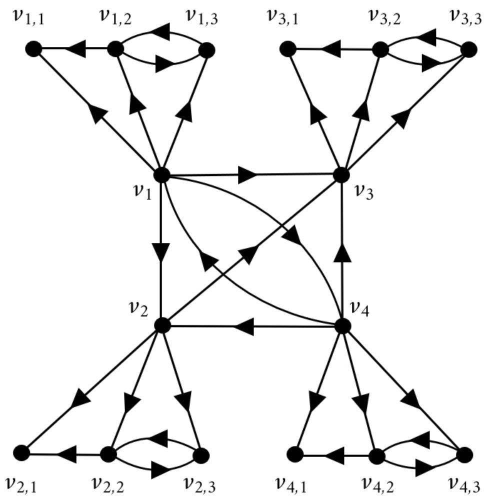

Fig. 18: $\Delta = \Delta_1 \odot \Delta_2$

$|T_1||R_2|$.

Corollary 6. Let $\Delta_1 = (T_1, R_1)$ and $\Delta_2 = (T_2, R_2)$ be two directed graphs. Let $\Delta_1 \odot \Delta_2 = (T(\Delta_1 \odot \Delta_2), R(\Delta_1 \odot \Delta_2))$ be the corona product of Δ_1 and Δ_2. Then

$$(i) \sum_{u \in T(\Delta_1 \odot \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \odot \Delta_2)} odeg\ u$$
$$= |R_1| + |T_1||T_2| + |T_1||R_2|$$
$$(ii) \sum_{u \in T(\Delta_1 \odot \Delta_2)} deg\ u$$
$$= 2(|R_1| + |T_1||T_2| + |T_1||R_2|),$$

where $ideg\ u, odeg\ u$ and $deg\ u$ represent the in-degree, out-degree and degree of the vertex u respectively, in $\Delta_1 \odot \Delta_2$.

Proof: (i) Let's consider the corona product of Δ_1 and Δ_2 denoted by $\Delta_1 \odot \Delta_2 = (T(\Delta_1 \odot \Delta_2), R(\Delta_1 \odot \Delta_2))$. According to Theorem 6, the number of arcs in $\Delta_1 \odot \Delta_2$ is given by $|R_1| + |T_1||T_2| + |T_1||R_2|$. Since the product $\Delta_1 \odot \Delta_2$ is a directed graph with $|R_1| + |T_1||T_2| + |T_1||R_2|$ arcs, we have

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \odot \Delta_2)} odeg\ u$$
$$= |R_1| + |T_1||T_2| + |T_1||R_2|,$$

since each arc in the corona product $\Delta_1 \odot \Delta_2$ contributes 1 each to the sums $\sum_{u \in T(\Delta_1 \odot \Delta_2)} ideg\ u$ and $\sum_{u \in T(\Delta_1 \odot \Delta_2)} odeg\ u$.
Hence,

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \odot \Delta_2)} odeg\ u$$
$$= |R_1| + |T_1||T_2| + |T_1||R_2|.$$

(ii) Since, $deg\ u = ideg\ u + odeg\ u$ and by part (i) of this corollary we have,

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} deg\ u = 2(|R_1| + |T_1||T_2| + |T_1||R_2|).$$

∎

Example 18. Consider the directed graphs Δ_1 and Δ_2, and the corona product $\Delta_1 \odot \Delta_2$ given in Example 16. Here we have,

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} ideg\ u = 2+2+2+1+2+2+2+3+$$
$$2+2+2+2+2+2+2+1 = 31,$$
$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} odeg\ u = 2+1+6+2+1+$$
$$3+4+2+1+6+2+1 = 31$$

and

$$|R_1| + |T_1||T_2| + |T_1||R_2| = 7 + 4.3 + 4.3 = 31.$$

That is,

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} ideg\ u = \sum_{u \in T(\Delta_1 \odot \Delta_2)} odeg\ u$$
$$= |R_1| + |T_1||T_2| + |T_1||R_2|.$$

Also,

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} deg\ u = 2+4+3+7+2+4+3+6+6+$$
$$2+4+3+7+2+4+3 = 62$$

and

$$2(|R_1| + |T_1||T_2| + |T_1||R_2|) = 2(7 + 4.3 + 4.3)$$
$$= 2.31 = 62$$

That is,

$$\sum_{u \in T(\Delta_1 \odot \Delta_2)} deg\ u = 2(|R_1| + |T_1||T_2| + |T_1||R_2|).$$

VIII. Conclusion

The graph product is a binary operation that applies to graphs. Similar to how product operations are defined in graphs, we can establish corresponding operations in directed graphs. Numerous authors have delved into product operations within directed graphs, such as the Cartesian, lexicographic, and strong products. This paper expanded the definitions of product operations from graphs like categorical, modular, disjunctive, homomorphic, corona, and rooted products adapting them to directed graphs and exploring certain properties associated with them.

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