Tripolar Fuzzy Ideals in Semigroups

Tanaphong Prommai, Aiyared Iampan, Thiti Gaketem*

Abstract—**The tripolar fuzzy set is a generalization of fuzzy set, bipolar fuzzy sets, and intuitionistic fuzzy sets, which was introduced by Rao in 2018. In this paper, we defined the concept of tripolar ideals and fuzzy tripolar ideals of semigroups. We discuss the properties of tripolar fuzzy ideals of semigroups and the among of ideals semigroups and a class of tripolar fuzzy ideals. Finally, we characterized regular and intra-regular semigroups in terms of tripolar fuzzy ideals.**

Index Terms—**Regular, Intra-regular, Tripolar fuzzy set, Tripolar fuzzy ideal**

I. Introduction

THE THEORY of fuzzy sets is the most appropriate theory for dealing with uncertainty and was introduced by Zadeh [1] in 1965. After the concept of fuzzy sets, several researched the generalizations of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research and many branches of pure and applied mathematics. In 1979, N. Kuroki [2] first applied fuzzy set theory in semigroups. He investigated the properties of fuzzy semigroups. In 1986, K. T. Attsnsov [3] gave the concept and studied properties of intuitionistic fuzzy sets. Later in 1994, Zhang [4] studied the concept of bipolar fuzzy sets. In 2000, K. M. Lee [5] developed knowledge of bipolar fuzzy sets extension to algebraic systems. In addition, Gaketem and Khamrot [6] studied bipolar weakly interior ideals in semigroups. Gaketem et al. [7] expand cubic bipolar fuzzy subsemigroups and ideals in semigroups. In 2018, M. M. K. Rao [8] was introduced to the concept of tripolar fuzzy set, which is a generalization of fuzzy sets, bipolar fuzzy sets, and intuitionistic fuzzy sets. In the same year, M. M. K. Rao and B. Venkateswarlu [9] studied tripolar fuzzy ideals Γ-semirings. In 2020, M. M. K. Rao and B. Venkateswarlu [10] studied tripolar fuzzy soft interior ideals Γ-semirings. In 2022, N. Wattansiripong et al. [11] present properties of tripolar fuzzy pure ideals in ordered semigroups. In the same year N. Wattansiripong et al.[12] gave the concept of tripolar fuzzy interior ideals in ordered semigroups and characterized semisimple ordered semigroups in terms of tripolar fuzzy interior ideals.

In this paper, we give the definition of tripolar fuzzy ideals in semigroups. We discuss the properties of tripolar fuzzy ideals in semigroups together we will prove the among ideals and tripolar fuzzy ideals. Moreover, we characterized regular and intra-regular semigroups in terms of tripolar fuzzy ideals.

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T. Prommai is a lecturer at the School of Science, University of Phayao, Phayao, Thailand. (e-mail:tanaphong.pr@up.ac.th).

A. Iampan is a lecturer at the Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand. (e-mail: aiyared.ia@up.ac.th).

T. Gaketem is a lecturer at the Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Phayao, Thailand.(corresponding author to provide email: thiti.ga@up.ac.th).

II. Preliminaries

In this section, we will recall some concepts and results, which help us study the next sections.

Definition 2.1. *A non-empty subset $\mathcal{B}$ of a semigroup $\mathcal{S}$ is called*

(1) *a subsemigroup (SG) of $\mathcal{S}$ if $\mathcal{B}^2 \subseteq \mathcal{B}$,*
(2) *a left ideal (LI) of $\mathcal{S}$ if $\mathcal{S}\mathcal{B} \subseteq \mathcal{B}$*
(3) *a right ideal (RI) of $\mathcal{S}$ if $\mathcal{B}\mathcal{S} \subseteq \mathcal{B}$.*
(4) *By an* ideal *(ID) $\mathcal{B}$ of $\mathcal{S}$ we mean a LI and a RI of $\mathcal{S}$.*

For any $h_i \in [0,1]$, $i \in \mathcal{F}$, define

$$\bigvee_{i\in\mathcal{F}} h_i := \sup_{i\in\mathcal{F}}\{h_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{F}} h_i := \inf_{i\in\mathcal{F}}\{h_i\}.$$

We see that for any $h, r \in [0,1]$, we have

$$h \vee r = \max\{h, r\} \quad \text{and} \quad h \wedge r = \min\{h, r\}.$$

A *fuzzy set (FS)* of a non-empty set $\mathcal{E}$ is a function $\rho : \mathcal{E} \to [0,1]$.

For any two FSs ρ and ν of a non-empty set $\mathcal{E}$, define the symbol as follows:

(1) $\rho \leq \nu \Leftrightarrow \rho(h) \leq \nu(h)$ for all $h \in \mathcal{E}$,
(2) $\rho = \nu \Leftrightarrow \rho \subseteq \nu$ and $\nu \subseteq \rho$,
(3) $(\rho \wedge \nu)(h) = \rho(h) \wedge \nu(h)$ and $(\rho \vee \nu)(h) = \rho(h) \vee \nu(h)$ for all $h \in \mathcal{E}$, For the symbol $\rho \leq \nu$, we mean $\nu \leq \rho$.

Let k be an element of a semigroup $\mathcal{S}$. Then $\mathcal{F}_k := \{(m,n) \in \mathcal{S} \times \mathcal{S} \mid k = mn\}$.

For any two FSs ρ and ν of a semigroup $\mathcal{S}$. The product of FSs ρ and ν of $\mathcal{S}$ is defined as follows, for all $h \in \mathcal{S}$

$$(\rho \circ \nu)(h) = \begin{cases} \bigvee_{(m,n)\in\mathcal{F}_k} \{\rho(k) \wedge \nu(r)\} & \text{if} \quad h = kr, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic function of a subset $\mathcal{B}$ of a non-empty set $\mathcal{E}$ is a fuzzy set of $\mathcal{E}$

$$\lambda_{\mathcal{B}}(h) = \begin{cases} 1 & \text{if} \quad h \in \mathcal{B}, \\ 0 & \text{if} \quad h \notin \mathcal{B}. \end{cases}$$

for all $h \in \mathcal{S}$.

Lemma 2.2. *[2] Let $\mathcal{B}$ and $\mathcal{L}$ be non-empty subsets of a semigroup $\mathcal{S}$. Then the following holds.*

(1) *If $\mathcal{B} \subseteq \mathcal{L}$, then $\lambda_{\mathcal{B}} \subseteq \lambda_{\mathcal{L}}$*
(2) $\lambda_{\mathcal{B}} \wedge \lambda_{\mathcal{L}} = \lambda_{\mathcal{B}\cap\mathcal{L}}$.
(3) $\lambda_{\mathcal{B}} \circ \lambda_{\mathcal{L}} = \lambda_{\mathcal{B}\mathcal{L}}$.

Definition 2.3. *[2] A FS ρ of a semigroup $\mathcal{S}$ is said to be*

(1) *a fuzzy subsemigroup of $\mathcal{S}$ if $\rho(h) \wedge \rho(r) \leq \rho(hr)$, for all $h, r \in \mathcal{S}$.*
(2) *a fuzzy left ideal of $\mathcal{S}$ if $\rho(r) \leq \rho(hr)$, for all $h, r \in \mathcal{S}$.*

(3) *a fuzzy right ideal of* $\mathcal{S}$ *if* $\rho(h) \leq \rho(hr))$, *for all* $h, r \in \mathcal{S}$.
(4) *a fuzzy ideal of* $\mathcal{S}$ *if it is both a fuzzy left ideal and a fuzzy right ideal of* $\mathcal{S}$.

Definition 2.4. *[8] The* tripolar fuzzy set *(TFS)* $\mathcal{TF}$ *on a non-empty set* $\mathcal{E}$ *if*

$$\mathcal{TF} := \{(h, \rho(h), \nu(h), \delta(h)) \mid h \in \mathcal{E}\},$$

where $\rho(h) : \mathcal{E} \to [0,1]$, $\nu(h) : \mathcal{E} \to [0,1]$ *and* $\rho(h) : \mathcal{E} \to [-1,0]$, *such that* $0 \leq \rho(h) + \nu(h) \leq 1$ *for all* $h \in \mathcal{E}$. *The membership degree* $\rho(h)$ *characterizes the extent that the element* $\mathcal{E}$ *satisfies the property corresponding to TFS* $\mathcal{TF}$ $\nu(h)$ *characterizes the extent to the element* $\mathcal{E}$ *satisfies the not property (irrelevant) corresponding to tripolar fuzzy set* ρ, *and* $\delta(h)$ *characterizes the extent that the element* $\mathcal{E}$ *satisfies the implicit counter property corresponding to TFS* $\mathcal{TF}$. *For simplicity* $\mathcal{TF} := (\rho, \nu, \delta)$ *has been used for* $\mathcal{TF} := \{(h, \rho(h), \nu(h), \delta(h)) \mid h \in \mathcal{E}\}$ *such that* $0 \leq \rho(h) + \nu(h) \leq 1$.

The *characteristic tripolar fuzzy set* (CTFS) $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ of a non-empty subset $\mathcal{B}$ of $\mathcal{E}$ is defined as follows:

$$\rho_{\mathcal{B}}(k) = \begin{cases} 1 & \text{if } k \in \mathcal{B}, \\ 0 & \text{if } k \notin \mathcal{B}, \end{cases}$$

$$\nu_{\mathcal{B}}(k) = \begin{cases} 0 & \text{if } k \in \mathcal{B}, \\ 1 & \text{if } k \notin \mathcal{B}, \end{cases}$$

$$\delta_{\mathcal{B}}(k) = \begin{cases} -1 & \text{if } k \in \mathcal{B}, \\ 0 & \text{if } k \notin \mathcal{B} \end{cases}$$

for all $k \in \mathcal{E}$. In this case of $\mathcal{B} = \mathcal{E}$ defined $\mathcal{TF}_{\mathcal{B}} = (1, 0, -1)$.

III. Tripolar fuzzy ideals in semigroups

In this section, we define the notions of tripolar fuzzy ideals in semigroups and some properties of them are investigated.

Definition 3.1. *A TFS is called a* $\mathcal{TF} = (\rho, \nu, \delta)$ *of a semigroup* $\mathcal{S}$ *is called a tripolar fuzzy subsemigroup (TFSG) of* $\mathcal{S}$ *if*
(1) $\rho(hk) \geq \rho(h) \wedge \rho(k)$
(2) $\nu(hk) \leq \nu(h) \vee \nu(k)$
(3) $\delta(hk) \leq \delta(h) \vee \delta(k)$
for all $h, k \in \mathcal{S}$.

Definition 3.2. *A TFS is called a* $\mathcal{TF} = (\rho, \nu, \delta)$ *of a semigroup* $\mathcal{S}$ *is called a tripolar fuzzy left ideal (TFLI) of* $\mathcal{S}$ *if*
(1) $\rho(hk) \geq \rho(k)$
(2) $\nu(hk) \leq \nu(k)$
(3) $\delta(hk) \leq \delta(k)$
for all $h, k \in \mathcal{S}$.

Example 3.3. *Let* $\mathcal{S} = \{w, x, y, z\}$ *be semigroup with the following Cayley table:*

$\cdot$	w	x	y	z
w	w	w	w	w
x	w	w	w	w
y	w	w	x	w
z	w	w	x	x

Define $\mathcal{TF} = (\rho, \nu, \delta)$ *by* $\rho(w) = 0.4$, $\rho(x) = 0.7$, $\rho(y) = 0.8$, $\rho(z) = 0.3$; $\nu(w) = 0.5$, $\rho(x) = 0.2$, $\rho(y) = 0.1$, $\rho(z) = 0.4$ *and* $\delta(w) = -0.7$, $\delta(x) = -0.5$, $\delta(y) = -0.3$, $\delta(z) = -0.3$. *Then* $\mathcal{TF}$ *is a TFLI of* $\mathcal{S}$.

Definition 3.4. *A TFS is called a* $\mathcal{TF} = (\rho, \nu, \delta)$ *of a semigroup* $\mathcal{S}$ *is called a tripolar fuzzy right ideal (TFRI) of* $\mathcal{S}$ *if*
(1) $\rho(hk) \geq \rho(h)$
(2) $\nu(hk) \leq \nu(h)$
(3) $\delta(hk) \leq \delta(h)$
for all $h, k \in \mathcal{S}$.

Definition 3.5. *A TFS is called a* $\mathcal{TF} = (\rho, \nu, \delta)$ *of a semigroup* $\mathcal{S}$ *is called a tripolar fuzzy ideal (TFI) of* $\mathcal{S}$ *if it is both TFLI and TFRI of* $\mathcal{S}$.

Theorem 3.6. *Let* $\mathcal{TF} = (\rho, \nu, \delta)$ *be a TFS in a semigroup* $\mathcal{S}$. *Then the following statements hold.*
(1) $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ *is a TFSG of* $\mathcal{S}$.
(2) $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ *is a TFLI of* $\mathcal{S}$.
(3) $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ *is a TFRI of* $\mathcal{S}$.
(4) $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ *is a TFI of* $\mathcal{S}$.

Proof: Let $\overline{\rho} = 1 - \rho$ and $h, k \in \mathcal{S}$. Then
(1) $\overline{\rho}(hk) = 1 - \rho(hk) \leq 1 - (\rho(h) \wedge \rho(k)) = 1 - \rho(h) \vee 1 - \rho(k) = \overline{\rho}(h) \vee \overline{\rho}(k)$. Thus, $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ is a TFSG of $\mathcal{S}$.
(2) $\overline{\rho}(hk) = 1 - \rho(hk) \leq 1 - \rho(k) = \overline{\rho}(k)$. Thus, $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ is a TFLI of $\mathcal{S}$.
(3) $\overline{\rho}(hk) = 1 - \rho(hk) \leq 1 - \rho(h) = \overline{\rho}(h)$. Thus, $\mathcal{TF} = (\rho, \overline{\rho}, \delta)$ is a TFRI of $\mathcal{S}$.
(4) By (2) and (3), we have (4) is true.

■

The following theorems show the connection between ideals and TFIs in semigroups.

Theorem 3.7. *Let* $\mathcal{B}$ *be a nonempty subset of a semigroup* $\mathcal{S}$. *Then the following statement holds;*
(1) $\mathcal{B}$ *is a SG of* $\mathcal{S}$ *if and only if* $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ *is a TFSG of* $\mathcal{S}$.
(2) $\mathcal{B}$ *is a LI of* $\mathcal{S}$ *if and only if* $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ *is a TFLI of* $\mathcal{S}$.
(3) $\mathcal{B}$ *is a RI of* $\mathcal{S}$ *if and only if* $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ *is a TFRI of* $\mathcal{S}$.
(4) $\mathcal{B}$ *is an ID of* $\mathcal{S}$ *if and only if* $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ *is a TFI of* $\mathcal{S}$.

Proof:
(1) Suppose that $\mathcal{B}$ is a SG of $\mathcal{S}$ and let $h, k \in \mathcal{S}$. Then $\mathcal{B}^2 \subseteq \mathcal{B}$.
If $h, k \in \mathcal{B}$, then $hk \in \mathcal{B}$. Thus, $\rho_{\mathcal{B}}(h) = \rho_{\mathcal{B}}(k) = \rho_{\mathcal{B}}(hk) = 1$, $\nu_{\mathcal{B}}(h) = \nu_{\mathcal{B}}(k) = \nu_{\mathcal{B}}(hk) = 0$ and $\delta_{\mathcal{B}}(h) = \delta_{\mathcal{B}}(k) = \delta_{\mathcal{B}}(hk) = -1$. Hence, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(k)$.
If $h \notin \mathcal{B}$ or $k \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(k)$. Therefore, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TFSG of $\mathcal{S}$.
For the converse, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TFSG of $\mathcal{S}$, let $h, k \in \mathcal{S}$ with $h, k \in \mathcal{B}$. Then $\rho_{\mathcal{B}}(h) = \rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(h) = \nu_{\mathcal{B}}(k) = 0$ and $\delta_{\mathcal{B}}(h) = \delta_{\mathcal{B}}(k) = -1$. By assumption, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(h) \wedge \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq$

$\nu_{\mathcal{B}}(h) \vee \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(h) \vee \delta_{\mathcal{B}}(k)$. Thus, $hk \in \mathcal{B}$. Therefore $\mathcal{B}$ is a SG of $\mathcal{S}$.

(2) Suppose that $\mathcal{B}$ is a LI of $\mathcal{S}$ and let $h, k \in \mathcal{S}$. Then $\mathcal{SB} \subseteq \mathcal{B}$.
If $k \in \mathcal{B}$, then $hk \in \mathcal{B}$. Thus, $\rho_{\mathcal{B}}(k) = \rho_{\mathcal{B}}(hk) = 1$, $\nu_{\mathcal{B}}(k) = \nu_{\mathcal{B}}(hk) = 0$ and $\delta_{\mathcal{B}}(k) = \delta_{\mathcal{B}}(hk) = -1$. Hence, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$.
If $k \notin \mathcal{B}$, then $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$.
Therefore, $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TFLI of $\mathcal{S}$.
For the converse, assume that $\mathcal{TF}_{\mathcal{B}} = (\rho_{\mathcal{B}}, \nu_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a TFLI of $\mathcal{S}$, let $h, k \in \mathcal{S}$ and $k \in \mathcal{B}$. Then $\rho_{\mathcal{B}}(k) = 1$, $\nu_{\mathcal{B}}(k) = 0$ and $\delta_{\mathcal{B}}(k) = -1$. By assumption, $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$. Thus, $hk \in \mathcal{B}$. Therefore $\mathcal{B}$ is a LI of $\mathcal{S}$.

(3) It follows from (2).

(4) By (2) and (3), we have (4) is true.

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The *support* of $\mathcal{TF} := (\rho, \nu, \delta)$ tripolar fuzzy set instead of $\text{supp}(\mathcal{TF}) = \{h \in \mathcal{E} \mid \rho(h) \neq 0, \nu(h) \neq 1, \delta(h) \neq 0\}$.

Theorem 3.8. *Let ρ, ν and δ be nonzero fuzzy sets of a semigroup $\mathcal{S}$. Then the following statement holds;*

(1) *$\mathcal{TF} := (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$ if and only if $\text{supp}(\mathcal{TF})$ is a SG of $\mathcal{S}$.*

(2) *$\mathcal{TF} := (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$ if and only if $\text{supp}(\mathcal{TF})$ is a LI of $\mathcal{S}$.*

(3) *$\mathcal{TF} := (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$ if and only if $\text{supp}(\mathcal{TF})$ is a RI of $\mathcal{S}$.*

(4) *$\mathcal{TF} := (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$ if and only if $\text{supp}(\mathcal{TF})$ is an ID of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$ and $h, k \in \mathcal{S}$ with $h, k \in \text{supp}(\mathcal{TF})$. Then $\rho(h) \neq 0, \rho(k) \neq 0$, $\nu(h) \neq 1, \nu(k) \neq 1$ and $\delta(h) \neq 0$, $\delta(k) \neq 0$. By assumption, $\rho(hk) \geq \rho(h) \wedge \rho(k)$, $\nu(hk) \leq \nu(h) \vee \nu(k)$ and $\delta(hk) \leq \delta(h) \vee \delta(k)$. Thus, $\rho(hk) \neq 0$, $\nu(hk) \neq 1$ and $\rho(hk) \neq 0$. So $hk \in \text{supp}(\mathcal{TF})$. Hence, $\text{supp}(\mathcal{TF})$ is a SG of $\mathcal{S}$.
For the converse, suppose that $\text{supp}(\mathcal{TF})$ is a SG of $\mathcal{S}$ and let $\mathcal{TF} = (\rho, \nu, \delta)$ is not a TFSG of $\mathcal{S}$. Then there exist $h, k \in \mathcal{S}$ such that $\rho(hk) < \rho(h) \wedge \rho(k)$, $\nu(hk) > \nu(h) \vee \nu(k)$ and $\delta(hk) > \delta(h) \vee \delta(k)$. Since $\text{supp}(\mathcal{TF})$ is a SG of $\mathcal{S}$ we have $hk \in \text{supp}(\mathcal{TF})$. Thus, $\rho(hk) \neq 0$, $\nu(hk) \neq 1$ and $\delta(hk) \neq 0$.
If $h, k \in \text{supp}(\mathcal{TF})$, then $\rho(h) \neq 0$, $\rho(k) \neq 0$, $\nu(h) \neq 1$, $\nu(k) \neq 1$ and $\delta(h) \neq 0$, $\delta(k) \neq 0$. Thus, $\rho(hk) \geq \rho(h) \wedge \rho(k)$, $\nu(hk) \leq \nu(h) \vee \nu(k)$ and $\delta(hk) \leq \delta(h) \vee \delta(k)$. It is a contradiction.
If $h \notin \text{supp}(\mathcal{TF})$ or $k \notin \text{supp}(\mathcal{TF})$, then $\rho(hk) \geq \rho(h) \wedge \rho(k)$, $\nu(hk) \leq \nu(h) \vee \nu(k)$ and $\delta(hk) \leq \delta(h) \vee \delta(k)$. It is a contradiction.
Therefore, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$.

(2) Suppose that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$ and $h, k \in \mathcal{S}$ with $k \in \text{supp}(\mathcal{TF})$. Then $\rho(k) \neq 0$, $\nu(k) \neq 1$ and $\rho(k) \neq 0$. By assumption, $\rho(hk) \geq \rho(k)$, $\nu(hk) \leq \nu(k)$ and $\delta(hk) \leq \delta(k)$. Thus, $\rho(hk) \neq 0$, $\nu(hk) \neq 1$ and $\rho(hk) \neq 0$. So $hk \in \text{supp}(\mathcal{TF})$. Hence, $\text{supp}(\mathcal{TF})$ is a LI of $\mathcal{S}$.
For the converse, suppose that $\text{supp}(\mathcal{TF})$ is a LI of $\mathcal{S}$ and let $\mathcal{TF} = (\rho, \nu, \delta)$ is not a TFLI of $\mathcal{S}$. Then there exist $h, k \in \mathcal{S}$ such that $\rho(hk) < \rho(k)$, $\nu(hk) > \nu(k)$ and $\delta(hk) > \delta(k)$. Since $\text{supp}(\mathcal{TF})$ is a LI of $\mathcal{S}$ we have $hk \in \text{supp}(\mathcal{TF})$. Thus, $\rho(hk) \neq 0$, $\nu(hk) \neq 1$ and $\delta(hk) \neq 0$.
If $k \in \text{supp}(\mathcal{TF})$, then $\rho(k) \neq 0$, $\nu(k) \neq 1$ and $\delta(k) \neq 0$. Thus, $\rho(hk) \geq \rho(k)$, $\nu(hk) \leq \nu(k)$ and $\delta(hk) \leq \delta(k)$. It is a contradiction.
If $k \notin \text{supp}(\mathcal{TF})$, then $\rho_{\mathcal{B}}(hk) \geq \rho_{\mathcal{B}}(k)$, $\nu_{\mathcal{B}}(hk) \leq \nu_{\mathcal{B}}(k)$ and $\delta_{\mathcal{B}}(hk) \leq \delta_{\mathcal{B}}(k)$. It is a contradiction.
Therefore, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$.

(3) It follows from (2).

(4) By (2) and (3), we have (4) is true.

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Next, we give the definition of a ρ-level β-cut, ν-level β-cut and δ-level β-cut. And we prove the set ρ-level β-cut, ν-level β-cut and δ-level β-cut are ideals of semigroups.

Definition 3.9. *Let $\mathcal{TF} = (\rho, \nu, \delta)$ be a TFS of a semigroup $\mathcal{E}$ and $\beta \in [0, 1]$. Then the set $\rho_\beta = \{h \in \mathcal{E} : \rho(h) \geq \beta\}, \nu_\beta = \{h \in \mathcal{E} : \nu(h) \leq \beta$, and $\delta_\beta = \{h \in \mathcal{E} : \delta(h) \leq -\beta$ are called a ρ-level β-cut, ν-level β-cut and δ-level β-cut of $\mathcal{E}$ respectively.*

Theorem 3.10. *Let $\mathcal{TF} = (\rho, \nu, \delta)$ be a TFS of a semigroup $\mathcal{S}$. Then the following statement holds;*

(1) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$, then the ρ-level β-cut, ν-level β-cut and δ-level β-cut SGs of $\mathcal{S}$, for every $\beta \in Im(\rho) \cap Im(\nu) \subseteq [0, 1]$ and $-\beta \in Im(\delta)$.*

(2) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$, then the ρ-level β-cut, ν-level β-cut and δ-level β-cut LIs of $\mathcal{S}$, for every $\beta \in Im(\rho) \cap Im(\nu) \subseteq [0, 1]$ and $-\beta \in Im(\delta)$.*

(3) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$, then the ρ-level β-cut, ν-level β-cut and δ-level β-cut RIs of $\mathcal{S}$, for every $\beta \in Im(\rho) \cap Im(\nu) \subseteq [0, 1]$ and $-\beta \in Im(\delta)$.*

(4) *If $\mathcal{TF} = (\rho, \nu, \delta)$ be a TFI of $\mathcal{S}$, then the ρ-level β-cut, ν-level β-cut and δ-level β-cut IDs of $\mathcal{S}$, for every $\beta \in Im(\rho) \cap Im(\nu) \subseteq [0, 1]$ and $-\beta \in Im(\delta)$.*

Proof: Let $\beta \in \text{Im}(\rho) \cap \text{Im}(\nu) \subseteq [0, 1]$ and $-\beta \in \text{Im}(\delta)$ with $h, k \in \mathcal{TF} := (\rho, \nu, \delta)$

(1) If $h, k \in \rho_\beta$, then $\rho(h) \geq \beta$, $\rho(k) \geq \beta$. Thus, $\rho(hk) \geq \rho(h) \wedge \rho(k) \geq \beta$. Hence, $hk \in \rho_\beta$.
If $h, k \in \nu_\beta$, then $\nu(h) \leq \beta$, $\nu(k) \leq \beta$. Thus, $\nu(hk) \leq \nu(h) \vee \nu(k) \leq \beta$. Hence, $hk \in \nu_\beta$.
If $h, k \in \delta_\beta$, then $\delta(h) \leq -\beta$, $\delta(k) \leq -\beta$. Thus, $\delta(hk) \leq \delta(h) \vee \delta(k) \leq -\beta$. Hence, $hk \in \delta_\beta$.
Therefore, ρ-level β-cut, ν-level β-cut and δ-level β-cut SGs of $\mathcal{S}$.

(2) If $k \in \rho_\beta$, then $\rho(k) \geq \beta$. Thus, $\rho(hk) \geq \rho(k) \geq \beta$. Hence, $hk \in \rho_\beta$.
If $k \in \nu_\beta$, then $\nu(k) \leq \beta$. Thus, $\nu(hk) \leq \nu(k) \leq \beta$. Hence, $hk \in \nu_\beta$.
If $k \in \delta_\beta$, then $\delta(k) \leq -\beta$. Thus, $\delta(hk) \leq \delta(k) \leq -\beta$. Hence, $hk \in \delta_\beta$.
Therefore, ρ-level β-cut, ν-level β-cut and δ-level β-cut IDs of $\mathcal{S}$.

(3) It follows from (2).

(4) By (2) and (3), we have (4) is true.

■

For $\mathcal{TF}_1 = (\rho, \nu, \delta)$ and $\mathcal{TF}_2 = (\lambda, \mu, \omega)$ be a TFSs. Defined the product $\mathcal{TF}_1 \circ \mathcal{TF}_2$ of a semigroup $\mathcal{S}$ as follows:

$$(\rho \circ \lambda)(k) = \begin{cases} \bigvee_{(m,n)\in\mathcal{F}_k} \{\rho(m) \wedge \lambda(n)\} & \text{if} \quad \mathcal{F}_k \neq \emptyset, \\ 0 & \text{if} \quad \mathcal{F}_k = \emptyset, \end{cases}$$

$$(\nu \circ \mu)(k) = \begin{cases} \bigwedge_{(m,n)\in\mathcal{F}_k} \{\nu(m) \vee \mu(n)\} & \text{if} \quad \mathcal{F}_k \neq \emptyset, \\ 1 & \text{if} \quad \mathcal{F}_k = \emptyset, \end{cases}$$

$$(\delta \circ \omega)(k) = \begin{cases} \bigwedge_{(m,n)\in\mathcal{F}_k} \{\delta(m) \vee \omega(n)\} & \text{if} \quad \mathcal{F}_k \neq \emptyset, \\ 0 & \text{if} \quad \mathcal{F}_k = \emptyset, \end{cases}$$

for all $k \in \mathcal{E}$. It is easy to verify that the structure $(\mathcal{TF}_1, \circ)$ is a semigroup. In the set of all TFSs of $\mathcal{S}$ we define the order relation as follows: $\mathcal{TF}_1 \subseteq \mathcal{TF}_2$ if and only if $\rho(h) \leq \lambda(h)$, $\nu(h) \geq \mu(h)$ and $\delta(h) \geq \omega(h)$ for all $h \in \mathcal{S}$. Finally, we define a binary operation $\cap$ on $\mathcal{TF}$ as follows:

$$\mathcal{TF}_1 \cap \mathcal{TF}_2 := (\rho \wedge \lambda, \nu \vee \mu, \delta \vee \omega),$$

where $(\rho \wedge \lambda)(h) := \rho(h) \wedge \lambda(h), (\nu \vee \mu)(h) := \nu(h) \vee \mu(h)$ and $(\delta \vee \omega)(h) := \delta(h) \vee \omega(h)$ for all $h \in \mathcal{S}$.

Theorem 3.11. *Let $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFS of a semigroup $\mathcal{S}$. Then the following statement holds;*

(1) *If $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFSG of $\mathcal{S}$, then the TFS $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ of $\mathcal{S}$ is a TFSG of $\mathcal{S}$.*
(2) *If $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFLI of $\mathcal{S}$, then the TFS $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ of $\mathcal{S}$ is a TFLI of $\mathcal{S}$.*
(3) *If $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFRI of $\mathcal{S}$, then the TFS $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ of $\mathcal{S}$ is a TFRI of $\mathcal{S}$.*
(4) *If $\{\mathcal{TF}_i \mid i \in I\}$ be a family of TFI of $\mathcal{S}$, then the TFS $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ of $\mathcal{S}$ is a TFI of $\mathcal{S}$.*

Proof: Note that we defined $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ as follows:

$$\left(\bigcap_{i\in I} \rho\right)(h) := \bigcap_{i\in I} \rho_i(h) : \inf\{\rho_i(h) \in I\},$$

$$\left(\bigcup_{i\in I} \nu\right)(h) := \bigcup_{i\in I} \nu_i(h) : \sup\{\nu_i(h) \in I\}$$

and

$$\left(\bigcup_{i\in I} \delta\right)(h) := \bigcup_{i\in I} \delta_i(h) : \sup\{\delta_i(h) \in I\},$$

for all $h \in \mathcal{S}$.

(1) Let $h, k \in \mathcal{S}$. Then

$$\begin{aligned} \left(\bigcap_{i\in I} \rho\right)(hk) &= \bigcap_{i\in I} \rho_i(hk) \\ &= \inf\{\rho_i(hk) \mid i \in I\} \\ &\geq \inf\{\rho(h) \wedge \rho_i(k) \mid i \in I\} \\ &= \bigcap_{i\in I} (\rho_i(h) \wedge \rho_i(k)) \\ &= \left(\bigcap_{i\in I} \rho\right)(h) \wedge \left(\bigcap_{i\in I} \rho\right)(k) \end{aligned}$$

$$\begin{aligned} \left(\bigcup_{i\in I} \nu\right)(hk) &= \bigcup_{i\in I} \nu_i(hk) \\ &= \sup\{\nu_i(hk) \mid i \in I\} \\ &\geq \sup\{\nu_i(h) \vee \nu_i(k) \mid i \in I\} \\ &= \bigcup_{i\in I} (\nu_i(k) \vee \nu_i(k)) \\ &= \left(\bigcup_{i\in I} \nu\right)(h) \vee \left(\bigcup_{i\in I} \nu\right)(k) \end{aligned}$$

and

$$\begin{aligned} \left(\bigcup_{i\in I} \delta\right)(hk) &= \bigcup_{i\in I} \delta_i(hk) \\ &= \sup\{\delta_i(hk) \mid i \in I\} \\ &\geq \sup\{\delta_i(h) \vee \delta_i(k) \mid i \in I\} \\ &= \bigcup_{i\in I} (\delta_i(h) \vee \delta_i(k)) \\ &= \left(\bigcup_{i\in I} \delta\right)(h) \vee \left(\bigcup_{i\in I} \delta\right)(k) \end{aligned}$$

Thus, $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ of $\mathcal{S}$ is a TFSG of $\mathcal{S}$.

(2) Let $h, k \in \mathcal{S}$. Then

$$\begin{aligned} \left(\bigcap_{i\in I} \rho\right)(hk) &= \bigcap_{i\in I} \rho_i(hk) \\ &= \inf\{\rho_i(hk) \mid i \in I\} \\ &\geq \inf\{\rho_i(k) \mid i \in I\} \\ &= \bigcap_{i\in I} \rho_i(k) \\ &= \left(\bigcap_{i\in I} \rho\right)(k) \end{aligned}$$

$$\begin{aligned} \left(\bigcup_{i\in I} \nu\right)(hk) &= \bigcup_{i\in I} \nu_i(hk) \\ &= \sup\{\nu_i(hk) \mid i \in I\} \\ &\geq \sup\{\nu_i(k) \mid i \in I\} \\ &= \bigcup_{i\in I} \nu_i(k) \\ &= \left(\bigcup_{i\in I} \nu\right)(k) \end{aligned}$$

and

$$\begin{aligned} \left(\bigcup_{i\in I} \delta\right)(hk) &= \bigcup_{i\in I} \delta_i(hk) \\ &= \sup\{\delta_i(hk) \mid i \in I\} \\ &\geq \sup\{\delta_i(k) \mid i \in I\} \\ &= \bigcup_{i\in I} \delta_i(k) \\ &= \left(\bigcup_{i\in I} \delta\right)(k) \end{aligned}$$

Thus, $\bigcap_{i\in I} \mathcal{TF}_i := (\bigcap_{i\in I} \rho, \bigcup_{i\in I} \nu, \bigcup_{i\in I} \delta)$ of $\mathcal{S}$ is a TFLI of $\mathcal{S}$.

(3) It follows from (2).
(4) By (2) and (3), we have (4) is true.

■

Theorem 3.12. *Let $\mathcal{TF} = (\rho, \nu, \delta)$ be TFS of a semigroup $\mathcal{S}$. Then the following statement holds;*

(1) *$\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$ if and only if $\mathcal{TF} \circ \mathcal{TF} \subseteq \mathcal{TF}$.*
(2) *$\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{S}} \circ \mathcal{TF} \subseteq \mathcal{TF}$.*
(3) *$\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{S}$ if and only if $\mathcal{TF} \circ \mathcal{TF}_{\mathcal{S}} \subseteq \mathcal{TF}$.*
(4) *$\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{S}} \circ \mathcal{TF} \subseteq \mathcal{TF}$ and $\mathcal{TF} \circ \mathcal{TF}_{\mathcal{S}} \subseteq \mathcal{TF}$, where $\mathcal{TF}_{\mathcal{S}} = (\rho_{\mathcal{S}}, \nu_{\mathcal{S}}, \delta_{\mathcal{S}})$.*

Proof:

(1) Assume that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$ and let $h, k \in \mathcal{S}$.
If $\mathcal{F}_k = \emptyset$. Then $(\rho_{\mathcal{S}} \circ \rho)(hk) = 0 \leq \rho(h) \wedge \rho(k)$, $(\nu_{\mathcal{S}} \circ \nu)(k) = 1 \geq \nu(h) \vee \nu(k)$ and $(\delta_{\mathcal{S}} \circ \delta)(k) = 0 \geq \delta(h) \vee \delta(k)$.
If $\mathcal{F}_k \neq \emptyset$. Then $(\rho \circ \rho)(hk) = \bigvee_{(m,n) \in \mathcal{F}_{hk}} \{\rho \wedge \rho(n)\} \leq \rho(h) \wedge \rho(k)$,
$(\nu \circ \nu)(k) = \bigwedge_{(m,n) \in \mathcal{F}_{hk}} \{\nu(n) \vee \nu(m) \vee \nu(n)\} \geq \nu(h) \vee \nu(k)$, and
$(\delta \circ \delta)(k) = \bigwedge_{(m,n) \in \mathcal{F}_{hk}} \{\delta(m) \vee \delta(n)\} \geq \delta(h) \vee \delta(k)$.
Hence, $\mathcal{TF} \circ \mathcal{TF} \subseteq \mathcal{TF}$.
Conversely, assume that $\mathcal{TF} \circ \mathcal{TF} \subseteq \mathcal{TF}$ and $h, k \in \mathcal{S}$. Then

$$\begin{aligned}\rho(hk) &\geq (\rho \circ \rho)(hk) \\ &= \bigvee_{(m,n) \in \mathcal{F}_{hk}} \rho(m) \wedge \rho(n) \\ &\geq \rho(h) \wedge \rho(k),\end{aligned}$$

$$\begin{aligned}\nu(hk) &\leq (\nu \circ \nu)(hk) \\ &= \bigwedge_{(m,n) \in \mathcal{F}_{hk}} \nu(m) \vee \nu(n) \\ &\leq \nu(h) \vee \nu(k),\end{aligned}$$

$$\begin{aligned}\delta(hk) &\leq (\delta \circ \delta)(hk) \\ &= \bigwedge_{(m,n) \in \mathcal{F}_{hk}} \delta(m) \vee \delta(n) \\ &\leq \delta(h) \vee \delta(k).\end{aligned}$$

Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{S}$.

(2) Assume that $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$ and let $k \in \mathcal{S}$.
If $\mathcal{F}_k = \emptyset$. Then $(\rho_{\mathcal{S}} \circ \rho)(k) = 0 \leq \rho(k)$, $(\nu_{\mathcal{S}} \circ \nu)(k) = 1 \geq \nu(k)$ and $(\delta_{\mathcal{S}} \circ \delta)(k) = 0 \geq \delta(k)$.
If $\mathcal{F}_k \neq \emptyset$. Then

$$\begin{aligned}(\rho_{\mathcal{S}} \circ \rho)(k) &= \bigvee_{(m,n) \in \mathcal{F}_k} \{\rho_{\mathcal{S}}(m) \wedge \rho(n)\} \\ &\leq \bigvee_{(m,n) \in \mathcal{F}_k} \rho(n) \\ &= \rho(k),\end{aligned}$$

$$\begin{aligned}(\nu_{\mathcal{S}} \circ \nu)(k) &= \bigwedge_{(m,n) \in \mathcal{F}_k} \{\nu_{\mathcal{S}}(m) \vee \nu(n)\} \\ &\geq \bigwedge_{(m,n) \in \mathcal{F}_k} \nu(n) \\ &= \nu(k),\end{aligned}$$

and

$$\begin{aligned}(\delta_{\mathcal{S}} \circ \delta)(k) &= \bigwedge_{(m,n) \in \mathcal{F}_k} \{\delta(m) \vee \delta(n)\} \\ &\geq \bigwedge_{(m,n) \in \mathcal{F}_k} \delta(n) \\ &= \delta(k).\end{aligned}$$

Hence, $\mathcal{TF}_{\mathcal{S}} \circ \mathcal{TF} \subseteq \mathcal{TF}$. is a TFLI of $\mathcal{S}$.
Conversely, assume that $\mathcal{TF}_{\mathcal{S}} \circ \mathcal{TF} \subseteq \mathcal{TF}$ is a TFLI of $\mathcal{S}$ and $h, k \in \mathcal{S}$. Then

$$\begin{aligned}\rho(hk) &\geq (\rho_{\mathcal{S}} \circ \rho)(hk) \\ &= \bigvee_{(m,n) \in \mathcal{F}_{hk}} \rho_{\mathcal{S}}(m) \wedge \rho(n) \\ &\geq \rho(k),\end{aligned}$$

$$\begin{aligned}\nu(hk) &\leq (\nu_{\mathcal{S}} \circ \nu)(hk) \\ &= \bigwedge_{(m,n) \in \mathcal{F}_{hk}} \nu_{\mathcal{S}}(m) \vee \nu(n) \\ &\leq \nu(k),\end{aligned}$$

$$\begin{aligned}\delta(hk) &\leq (\delta_{\mathcal{S}} \circ \delta)(hk) \\ &= \bigwedge_{(m,n) \in \mathcal{F}_{hk}} \delta_{\mathcal{S}}(m) \vee \delta(n) \\ &\leq \delta(k).\end{aligned}$$

Thus, $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{S}$.

(3) It follows from (2).
(4) By (2) and (3), we have (4) is true.

■

Definition 3.13. *Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a map and let $\mathcal{TF}_1 = (\rho, \nu, \delta)$ and $\mathcal{TF}_2 = (\lambda, \mu, \omega)$ be a TFSs in $\mathcal{X}$ and $\mathcal{Y}$, respectively. The pre-image of $\mathcal{TF}_2$ under φ, denoted by $\varphi^{-1}(\mathcal{TF}_2)$ is a TFS in $\mathcal{X}$ defined by:*

$$\varphi^{-1}(\mathcal{TF}_2) := (\varphi^{-1}(\lambda), \varphi^{-1}(\mu), \varphi^{-1}(\omega)),$$

whrer $\varphi^{-1}(\lambda) = \lambda(\varphi)$, $\varphi^{-1}(\mu) = \mu(\varphi)$, $\varphi^{-1}(\omega) = \omega(\varphi)$.

Theorem 3.14. *Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a homomorphism of semigroups. Then the following statement holds;*

(1) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFSG of $\mathcal{Y}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\rho), \varphi^{-1}(\nu), \varphi^{-1}(\delta))$ is a TFSG of $\mathcal{X}$.*
(2) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFLI of $\mathcal{Y}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\rho), \varphi^{-1}(\nu), \varphi^{-1}(\delta))$ is a TFLI of $\mathcal{X}$.*
(3) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFRI of $\mathcal{Y}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\rho), \varphi^{-1}(\nu), \varphi^{-1}(\delta))$ is a TFRI of $\mathcal{X}$.*
(4) *If $\mathcal{TF} = (\rho, \nu, \delta)$ is a TFI of $\mathcal{Y}$, then $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\rho), \varphi^{-1}(\nu), \varphi^{-1}(\delta))$ is a TFI of $\mathcal{X}$.*

Proof:

(1) Assume that $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFSG of $\mathcal{Y}$ and $h, k \in \mathcal{X}$. Then

$$\begin{aligned}\varphi^{-1}(\rho(hk)) &= \rho(\varphi(hk)) = \rho(\varphi(h)\varphi(k)) \\ &\geq \rho(\varphi(h)) \wedge \rho(\varphi(k)) \\ &= \varphi^{-1}(\rho\varphi(h)) \wedge \varphi^{-1}(\rho\varphi(k)),\end{aligned}$$

$$\begin{aligned}\varphi^{-1}(\nu(hk)) &= \nu(\varphi(hk)) = \nu(\varphi(h)\varphi(k)) \\ &\leq \nu(\varphi(h)) \vee \nu(\varphi(k)) \\ &= \varphi(\nu(\varphi(h))) \vee \varphi^{-1}(\nu(\varphi(k))),\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\delta(hk)) &= \delta(\varphi(hk)) = \delta(\varphi(h)\varphi(k)) \\ &\geq \delta(\varphi(h)) \vee \delta(\varphi(k)) \\ &= \varphi^{-1}(\delta(\varphi(h))) \vee \varphi^{-1}(\delta(\varphi(k))).\end{aligned}$$

Thus, $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\rho),\, \varphi^{-1}(\nu),\, \varphi^{-1}(\delta))$ is a TFSG of $\mathcal{X}$.

(2) Assume that $\mathcal{TF} = (\rho,\, \nu,\, \delta)$ is a TFLI of $\mathcal{Y}$ and $h, k \in \mathcal{X}$. Then

$$\begin{aligned}\varphi^{-1}(\rho(hk)) &= \rho(\varphi(hk)) = \rho(\varphi(h)\varphi(k)) \\ &\geq \rho(\varphi(k)) = \varphi^{-1}(\rho\varphi(k)),\end{aligned}$$

$$\begin{aligned}\varphi^{-1}(\nu(hk)) &= \nu(\varphi(hk)) = \nu(\varphi(h)\varphi(k)) \\ &\leq \nu(\varphi(k)) = \varphi^{-1}(\nu(\varphi(k))),\end{aligned}$$

and

$$\begin{aligned}\varphi^{-1}(\delta(hk)) &= \delta(\varphi(hk)) = \delta(\varphi(h)\varphi(k)) \\ &\geq \delta(\varphi(k)) = \varphi^{-1}(\delta(\varphi(k))).\end{aligned}$$

Thus, $\varphi^{-1}(\mathcal{TF}) = (\varphi^{-1}(\rho), \varphi^{-1}(\nu), \varphi^{-1}(\delta))$ is a TFLI of $\mathcal{X}$.

(3) It follows from (2).

(4) By (2) and (3), we have (4) is true.

■

Definition 3.15. *[13] A semigroup $\mathcal{S}$ is called a* regular *if for each $h \in \mathcal{S}$, there exists $v \in \mathcal{S}$ such that $h = hvh$.*

Lemma 3.16. *[13] A semigroup $\mathcal{S}$ is regular if and only if $\mathcal{RL} = \mathcal{R} \cap \mathcal{L}$, for every right ideal $\mathcal{R}$ and left ideal $\mathcal{L}$ of $\mathcal{S}$.*

The following theorem show an equivalent conditional statement for a regular semigroup.

Theorem 3.17. *A semigroup $\mathcal{S}$ is regular if and only if $\mathcal{TF}_1 \circ \mathcal{TF}_2 = \mathcal{TF}_1 \cap \mathcal{TF}_2$ for every TFRI $\mathcal{TF}_1 = (\rho,\, \nu,\, \delta)$ and TFLI $\mathcal{TF}_2 = (\lambda,\, \mu,\, \omega)$ of $\mathcal{S}$.*

Proof: ($\Rightarrow$) Let $\mathcal{TF}_1 = (\rho,\, \nu,\, \delta)$ and $\mathcal{TF}_2 = (\lambda,\, \mu,\, \omega)$ be a TFRI and a TFLI of $\mathcal{S}$ respectively. Then by Theorem 3.12 (2), $\mathcal{TF}_1 \circ \mathcal{TF}_2 \subseteq \mathcal{TF}_1 \circ \mathcal{TF}_\mathcal{S} \subseteq \mathcal{TF}_1$ and $\mathcal{TF}_1 \circ \mathcal{TF}_2 \subseteq \mathcal{TF}_\mathcal{S} \circ \mathcal{TF}_2 \subseteq \mathcal{TF}_2$. Hence, $\mathcal{TF}_1 \circ \mathcal{TF}_2 \subseteq \mathcal{TF}_1 \cap \mathcal{TF}_2$.

Let $h \in \mathcal{S}$. Then there exists $x \in \mathcal{S}$ such that $h = hvh$. Thus,

$$\begin{aligned}(\rho \circ \lambda)(h) &= \bigvee_{(m,n)\in F_h} \{\rho(m) \wedge \lambda(n)\} \\ &= \bigvee_{(m,n)\in F_{hvh}} \{\rho(m) \wedge \lambda(n)\} \\ &\geq (\rho(hv) \wedge \lambda(h)) \\ &\geq (\rho(h) \wedge \lambda(h)) \\ &= (\rho \wedge \lambda)(h),\end{aligned}$$

$$\begin{aligned}(\nu \circ \mu)(h) &= \bigwedge_{(m,n)\in F_h} \{\nu(m) \vee \mu(n)\} \\ &= \bigwedge_{(m,n)\in F_{hvh}} \{\nu(m) \vee \mu(n)\} \\ &\leq (\nu(hv) \vee \mu(h)) \\ &\leq (\nu(h) \vee \mu(h)) \\ &= (\nu \vee \mu)(h)\end{aligned}$$

and

$$\begin{aligned}(\delta \circ \omega)(h) &= \bigwedge_{(m,n)\in F_h} \{\delta(m) \vee \omega(n)\} \\ &= \bigwedge_{(m,n)\in F_{hvh}} \{\delta(m) \vee \omega(n)\} \\ &\leq (\delta(hv) \vee \omega(h)) \\ &\leq (\delta(h) \vee \omega(h)) \\ &= (\delta \vee \omega)(h).\end{aligned}$$

Hence, $(\rho \circ \lambda)(h) \geq (\rho \wedge \lambda)(h)$, $(\nu \circ \mu)(h) \leq (\nu \vee \mu)(h)$ and $(\delta \circ \omega)(h) \leq (\delta \vee \omega)(h)$. and so $\mathcal{TF}_1 \cap \mathcal{TF}_2 \subseteq \mathcal{TF}_1 \circ \mathcal{TF}_2$. Therefore, $\mathcal{TF}_1 \circ \mathcal{TF}_2 = \mathcal{TF}_1 \cap \mathcal{TF}_2$.

($\Leftarrow$) Let $\mathcal{R}$ and $\mathcal{L}$ be a RI and a LI of $\mathcal{S}$ respectively. Then by Theorem 3.7 (2) and (3), $\mathcal{TF}_\mathcal{R}$ and $\mathcal{TF}_\mathcal{L}$ is a TFRI and a TFLI of $\mathcal{S}$ respectively. By supposition and Lemma 2.2, we have

$$\begin{aligned}\mathcal{TF}_{\mathcal{RL}} &= (\mathcal{TF}_R) \circ (\mathcal{TF}_\mathcal{L}) \\ &= (\mathcal{TF}_\mathcal{R}) \sqcap (\mathcal{TF}_\mathcal{L}) \\ &= \mathcal{TF}_{\mathcal{R}\cap\mathcal{L}}\end{aligned}$$

Thus $h \in \mathcal{RL}$, and so $\mathcal{RL} = \mathcal{R} \cap \mathcal{L}$. It follows that by Lemma 3.16, $\mathcal{S}$ is regular. ■

The following definition and lemma will be used to prove theorem 3.20.

Definition 3.18. *[13] A semigroup $\mathcal{S}$ is called an* intra-regular *if for each $uh \in \mathcal{S}$, there exist $h, v \in \mathcal{S}$ such that $h = vh^2k$.*

Lemma 3.19. *[13] A semigroup $\mathcal{S}$ is intra-regular if and only if $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{LR}$, for every LI $\mathcal{L}$ and every RI $\mathcal{R}$ of $\mathcal{S}$.*

Theorem 3.20. *A semigroup $\mathcal{S}$ is intra-regular if and only if $\mathcal{TF}_1 \cap \mathcal{TF}_2 \subseteq \mathcal{TF}_1 \circ \mathcal{TF}_2$, for every TFLI $\mathcal{TF}_1 = (\rho,\, \nu,\, \delta)$ and every TFRI $\mathcal{TF}_2 = (\lambda,\, \mu,\, \omega)$ of $\mathcal{S}$.*

Proof: ($\Rightarrow$) Let $\mathcal{TF}_1$ and $\mathcal{TF}_2$ be a TFLI and a TFRI of $\mathcal{S}$ respectively. Let $h \in \mathcal{S}$. Then there exist $v, k \in \mathcal{S}$ such that $h = vh^2k$. Thus,

$$\begin{aligned}(\rho \circ \lambda)(h) &= \bigvee_{(m,n)\in F_h} \{\rho(m) \wedge \lambda(n)\} \\ &= \bigvee_{(m,n)\in F_{vh^2k}} \{\rho(m) \wedge \lambda(n)\} \\ &\geq (\rho(vh) \wedge \lambda(hk)) \\ &\geq (\rho(h) \wedge \lambda(h)) \\ &= (\rho \wedge \lambda)(h),\end{aligned}$$

$$\begin{aligned}(\nu \circ \mu)(h) &= \bigwedge_{(m,n)\in F_h} \{\nu(m) \vee \mu(n)\} \\ &= \bigwedge_{(m,n)\in F_{vh^2k}} \{\nu(m) \vee \mu(n)\} \\ &\leq (\nu(vh) \vee \mu(hk)) \\ &\leq (\nu(h) \vee \mu(h)) \\ &= (\nu \vee \mu)(h)\end{aligned}$$

and

$$\begin{aligned}(\delta \circ \omega)(h) &= \bigwedge_{(m,n)\in F_h} \{\delta(m) \vee \omega(n)\} \\ &= \bigwedge_{(m,n)\in F_{vh^2k}} \{\delta(m) \vee \omega(n)\} \\ &\leq (\delta(vh) \vee \omega(hk)) \\ &\leq (\delta(h) \vee \omega(h)) \\ &= (\delta \vee \omega)(h).\end{aligned}$$

Hence, $(\rho \circ \lambda)(h) \geq (\rho \wedge \lambda)(h)$, $(\nu \circ \mu)(h) \leq (\nu \vee \mu)(h)$ and $(\delta \circ \omega)(h) \leq (\delta \vee \omega)(h)$. Therefore, $\mathcal{TF}_1 \cap \mathcal{TF}_2 \subseteq \mathcal{TF}_1 \circ \mathcal{TF}_2$.

($\Leftarrow$) Let $\mathcal{R}$ and $\mathcal{L}$ be a RI and a LI of $\mathcal{S}$ respectively. Then by Theorem 3.7 (2) and (3), $\mathcal{TF}_{\mathcal{R}}$ and $\mathcal{TF}_{\mathcal{L}}$ are a TFRI and a TFLI of $\mathcal{S}$ respectively. By supposition and Lemma 2.2, we have

$$\begin{aligned}\mathcal{TF}_{\mathcal{R}\cap\mathcal{L}} &= (\mathcal{TF}_{\mathcal{R}}) \sqcap (\mathcal{TF}_{\mathcal{L}}) \\ &\subseteq (\mathcal{TF}_{R}) \circ (\mathcal{TF}_{\mathcal{L}}) \\ &= \mathcal{TF}_{\mathcal{RL}}\end{aligned}$$

Thus $h \in \mathcal{RL}$, and so $\mathcal{RL} \subseteq \mathcal{R} \cap \mathcal{L}$. It follows that by Lemma 3.19, $\mathcal{S}$ is intra-regular. ■

IV. Minimal and Maximal Tripolar Fuzzy Ideal

Definition 4.1. *An ID $\mathcal{B}$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{B}$, we have $\mathcal{J} = \mathcal{B}$,*

(2) *a* maximal *if for every ideal of $\mathcal{J}$ of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{J}$, we have $\mathcal{B} = \mathcal{I}$,*

Definition 4.2. *A TRFI $\mathcal{TF} = (\rho, \nu, \delta)$ of a semigroup $\mathcal{S}$ is called*

(1) *a* minimal *if for every TRFI of $\mathcal{TF}_1 = (\lambda, \mu, \omega)$ of $\mathcal{S}$ such that $\mathcal{TF}_1 \subseteq \mathcal{TF}$, we have* $\text{supp}(\mathcal{TF}_1) = \text{supp}(\mathcal{TF})$,

(2) *a* maximal *if for every TRFI of $\mathcal{TF}_1 = (\lambda, \mu, \omega)$ of $\mathcal{S}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_1$, we have* $\text{supp}(\mathcal{TF}_1) = \text{supp}(\mathcal{TF})$.

Theorem 4.3. *Let $\mathcal{B}$ be a non-empty subset of a semigroup $\mathcal{S}$. Then the following statement holds.*

(1) *$\mathcal{B}$ is a minimal ID of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a minimal IRFI of $\mathcal{S}$,*

(2) *$\mathcal{B}$ is a maximal ID of $\mathcal{S}$ if and only if $\mathcal{TF}_{\mathfrak{B}}$ is a maximal TRFI of $\mathcal{S}$.*

Proof:

(1) Suppose that $\mathcal{B}$ is a minimal ID of $\mathcal{S}$. Then $\mathcal{B}$ is an ID of $\mathcal{S}$. By Theorem 3.7, $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a TRFI of $\mathcal{S}$. Let $\mathcal{TF} = (\lambda, \mu, \omega)$ be a TRFI of $\mathcal{S}$ such that $\mathcal{TF} \subseteq \mathcal{TF}_{\mathcal{B}}$. Then $\text{supp}(\mathcal{TF}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Thus, $\text{supp}(\mathcal{TF}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathcal{B}$. Hence, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B}$. Since $\mathcal{TF} = (\lambda, \mu, \omega)$ is a TRFI of $\mathcal{S}$ we have $\text{supp}(\mathcal{TF})$ is an ID of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Hence, $\mathcal{TF}_{\mathcal{B}}$ is a minimal TRFI of $\mathfrak{S}$.
Conversely, $\mathcal{TF}_{\mathcal{B}}$ is a minimal TRFI of $\mathfrak{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a TRFI of $\mathcal{S}$. By Theorem 3.7, $\mathcal{B}$ is an ID of $\mathcal{S}$. Let $\mathcal{J}$ be an ID of $\mathcal{S}$ such that $\mathcal{J} \subseteq \mathcal{B}$. Then by Theorem 3.7, $\mathcal{TF}_{\mathfrak{J}} = (\lambda_{\mathcal{J}}, \mu_{\mathcal{J}}\, \omega_{\mathcal{J}})$ is a TRFI of $\mathfrak{S}$ such that $\mathcal{TF}_{\mathcal{J}} \subseteq \mathcal{TF}_{\mathcal{B}}$. Hence, $\mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathcal{B}$. By assumption, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. So, $\mathcal{B} = \chi_{\mathcal{I}}$. Hence, $\mathcal{B}$ is a minimal ID of $\mathfrak{S}$.

(2) Suppose that $\mathcal{B}$ is a maximal ID of $\mathcal{S}$. Then $\mathcal{B}$ is an ID of $\mathcal{S}$. By Theorem 3.7, $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a TRFI of $\mathcal{S}$. Let $\mathcal{TF} = (\lambda, \mu, \omega)$ be a TRFI of $\mathcal{S}$ such that $\mathcal{TF}_{\mathcal{B}} \subseteq \mathcal{TF}$. Then $\text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF})$. Thus, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF})$. Hence, $\mathcal{B} \subseteq \text{supp}(\mathcal{TF})$. Since $\mathcal{TF} = (\lambda, \mu, \omega)$ is a TRFI of $\mathcal{S}$ we have $\text{supp}(\mathcal{TF})$ is an ID of $\mathcal{S}$. By assumption, $\text{supp}(\mathcal{TF}) \subseteq \mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}})$. Hence, $\mathcal{TF}_{\mathcal{B}}$ is a maximal TRFI of $\mathfrak{S}$.
Conversely, $\mathcal{TF}_{\mathcal{B}}$ is a maxiimal TRFI of $\mathfrak{S}$. Then $\mathcal{TF}_{\mathcal{B}} = (\lambda_{\mathcal{B}}, \mu_{\mathcal{B}}\, \omega_{\mathcal{B}})$ is a TRFI of $\mathcal{S}$. By Theorem 3.7, $\mathcal{B}$ is an ID of $\mathcal{S}$. Let $\mathcal{J}$ be an ID of $\mathcal{S}$ such that $\mathcal{B} \subseteq \mathcal{J}$. Then by Theorem 3.7, $\mathcal{TF}_{\mathfrak{J}} = (\lambda_{\mathcal{J}}, \mu_{\mathcal{J}}\, \omega_{\mathcal{J}})$ is a TRFI of $\mathfrak{S}$ such that $\mathcal{TF}_{\mathcal{J}} \subseteq \mathcal{TF}_{\mathcal{B}}$. Hence, $\mathfrak{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) \subseteq \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. By assumption, $\mathcal{B} = \text{supp}(\mathcal{TF}_{\mathcal{B}}) = \mathfrak{J} = \text{supp}(\mathcal{TF}_{\mathcal{J}}) = \mathcal{J}$. So, $\mathcal{B} = \chi_{\mathcal{I}}$. Hence, $\mathcal{B}$ is a minimal ID of $\mathfrak{S}$.

■

V. Conclusion

In paper, we study concpet tripolar fuzzy ideals in semigroup and connection between ideals and tripolar fuzzy ideals in semigroups. In the important results, regular and intra-regular semigroups are characterized in terms of tripolar fuzzy ideals are provided. In the future work, we can study tripolar fuzzy interior ideal in semigroups and their fuzzifications in other algebraic structures.

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