Fixed Point Theorems for pairs of Mappings in Dislocated Cone Metric Space over Banach Algebra

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***Abstract*—This work presents a collection of fixed point theorems and introduce the idea of common fixed point theorems for a pair of weakly compatible mappings in dislocated cone metric space over Banach algebra. In addition we generalise common fixed point theorems for two pairs of self-mapping by using the α-property in cone metric space. We also provide illustrations and examples to support our results. Our findings are significant and expand upon and generalise a number of recent findings from the literature.**

***Index Terms*—cone metric space, dislocated cone metric space, α-property, weakly compatible mappings, contraction mapping.**

I. Introduction

FIXED point theory is a very effective tool in current mathematics, and its conclusions are directly applied to many existence and uniqueness theories in numerous fields. Banach Contraction Principle [1] act as the base for majority of the results obtained so far in fixed point theory. Later on, Multiple authors have derived various generalisations of it using different methods [2], [3]. S.Sessa [4] introduced the concept of weakly commuting self mapping of a complete metric space. Further, G. Jungck [5] generalised the concept of weak commutativity by introducing the notion of compatible mappings and demonstrated compatibility of weakly commuting maps but not the converse. After that, Jungck and Rhoades [7] introduced the more general idea of weak compatibility to the setting of single-valued and multi-valued mappings by replacing compatibility. Pant [9] first proposed the idea of common fixed points of incompatible mappings. Aamri $et\ al.$, [10] presented the idea of property (E.A) and Al-Thagafi $et\ al.$, [13] presented the idea of mappings that are occasionally weakly compatible. Huang and Zhang [12] introduced the idea of cone metric spaces and afterwards generalised to cone metric space over Banach algebra by Xu and Liu [21]. They established certain fixed point results for various contractive conditions by replacing the whole normed space to cone metric space over Banach algebra and showed the existence of fixed points. Muhammad Nazam $et\ al.$, [24] use the idea of Ciric-type and Hardy-Rogers type (α_s, F) - contractions based on four self-mappimgs defined on a b - metric space to α_s - complete b - metric space, ordered b - metric space and graphic b - metric space. Akash Singhal $et\ al.$, [25] present a common fixed point theorem for four self-mappings in cone metric spaces where the cone is not necessarily normal. Manoj Kumar $et\ al.$, [27] show a common fixed point theorem for four self-maps which are weakly compatible and satisfy a general contractive condition and also prove common fixed point theorem for weakly compatible maps along with E.A. and (CLR) properties. Many authors established a number of other fixed point theorems for weakly compatible mappings satisfying certain contractive condition in certain spaces [17], [26], [28]–[38]. In this paper, we generalise common fixed point theorems for two pairs of self mapping by using α - property in cone metric space and generalise common fixed point theorems for two pairs of self mapping in dislocated cone metric space over Banach algebra.

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II. Preliminaries

In this part, we outline fundamental concepts and necessary outcomes for the sequel.

Definition 1. *Given two self-mappings $J, K : X \times X$ on a metric space (X, d), then the mappings J and K are*

1) *weakly commuting if $d(JKx, KJx) \leq d(Jx, Kx)\ \forall\ x$ in X [4],*
2) *compatible if $\lim_{m\to\infty} d(JKx_m, KJx_m) = 0$ for each sequence $\{x_m\} \in X$ such that $\lim_{m\to\infty} Jx_m = \lim_{m\to\infty} Kx_m$ [5],*
3) *noncompatible if $\exists$ a sequence $\{x_m\} \in X$ such that $\lim_{m\to\infty} Jx_m = \lim_{m\to\infty} Kx_m$, but $\lim_{m\to\infty} d(JKx_m, KJx_m)$ is either nonexistent or nonzero [9],*
4) *pair is weakly compatible when it commutes to its coincidence point; that is, $JKu = KJu$ whenever $Ju = Ku$, for any $u \in X$ [7].*
5) *occasionally weakly compatible if J and K commute at a point $x \in X$ that serves as their coincidence point [13],*
6) *with the characteristic (E.A) if some $z \in X$ and a sequence $\{x_n\}$ exist in X such that $\lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Kx_n = z$ [10].*

It is evident that any two mappings meeting the condition (E.A.) do not necessarily have to be incompatible, even when noncompatible arbitrary self-mappings fulfills the property (E.A.) (see [15], Example 1). Furthermore, property (E.A) and weak compatibility are unrelated to one another (see [11], Example 2.1 and 2.2).

Definition 2. *(see [8]) Suppose (X, d) be a metric space. If $\lim_{m\to\infty} JKx_m = Ju$ and $\lim_{m\to\infty} KJx_m = Ku$ for every u in X, then the pair (J, K) of self-mappings on (X, d) is said to be reciprocally continuous.*

It is obvious that two self-mappings must be reciprocally continuous if they are continuous. However, the contrary claim cannot be true. In theory, compatible pairs of self-mappings that adhere to contractive restrictions are considered, the presence of a continuous mapping in one of the mappings implies that the mappings are reciprocally continuous, but the converse is not valid (see to Pant [18]).

Definition 3. *[14] Suppose (X, d) be a metric space. Then, the pair (J, K) of self-mappings on (X, d) is considered subcompatible if $\exists$ a sequence $\{x_n\}$ that satisfies the given condition*

$$\lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Kx_n = z,$$

for some $z \in X$ and

$$\lim_{n\to\infty} d(JKx_n, KJx_n) = 0.$$

A pair of mappings that are subcompatible meets the property (E.A). Clearly, mappings that are compatible and fulfills property (E.A) are also subcompatible. However, reverse assertion is not generally true (see [19], Example 2.3). Pairs of mappings that exhibit occasional poor compatibility are regarded as subcompatible. However, this is not always the case (see [14].

Definition 4. *[14] Let (X, d) be a metric space. Subsequentially continuous is the pair (J, K) of self-mappings on (X, d) if*

$$\exists \lim_{m\to\infty} JKx_m = Ju$$

and

$$\lim_{m\to\infty} KJx_m = Ku,$$

where

$$\lim_{m\to\infty} Jx_m = \lim_{m\to\infty} Kx_m = u$$

for some $u \in X$.

It is verifiable that two self-mappings J and K are reciprocally continuous if they are both continuous. As seen in Example 1 ([18]), J and K are not subsequentially continuous.

Definition 5. *(see [16]) For $* : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, a binary operation shall be represented, which fulfils the conditions below:*

1) *$*$ is both commutative and associative,*
2) *$*$ is continuous.*

These are a few common instances of $*$:

$$r * s = \max\{r, s\}, r * s = r + s,$$

$$r * s = rs,\ r * s = rs + r + s$$

and

$$r * s = rs/\max\{r, s, 1\},$$

for each $r, s \in \mathbb{R}^+$.

Definition 6. *(see [16]) The binary operation $*$ satisfies the α-property when $\exists\ \alpha > 0$ such that*

$$r * s \leq \alpha max\{r, s\}, \tag{1}$$

for all $r, s \in \mathbb{R}^+$.

Example 7. *(see [16])*

(1) *For $r * s = r + s$, $r, s \in \mathbb{R}^+$, thus for $\alpha \geq 2$, we get $r * s \leq \alpha \max\{r, s\}$.*
(2) *For $r*s = rs/max\{r, s, 1\}$, $r, s \in \mathbb{R}^+$, thus for $\alpha \geq 1$, we get $r * s \leq \alpha \max\{r, s\}$.*

Definition 8. *[12] Let E be a real Banach space and P a subset of E. Then P is called a cone if and only if:*

1) *P is closed, non-empty and $P \neq \{0\}$;*
2) *$ax+by \in P\ \ \forall x, y \in P$ and non negative real numbers a, b;*
3) *$P \cap (-P)$ = 0.*

Definition 9. *[12] E is a real Banach space, P is a cone in E with $intP \neq 0$ and $\leq$ is the partial ordering with respect to P. Let X be a non empty set and*

$$d : X \times X \to P$$

a mapping such that:

1) *$0 \leq d(x, y)\ \forall x, y \in X$ (non-negativity);*
2) *$d(x, y) = 0$ if and only if $x = y$;*
3) *$d(x, y) = d(y, x)\ \forall x, y \in X$ (symmetry);*
4) *$d(x, y) \leq d(x, z) + d(z, y)\ \forall x, y, z \in X$ (triangle inequality).*

Then d is called a cone metric on X and (X, d) is called a cone metric space.

III. Fixed point theorems using α-property

First, we present common fixed point theorems for two pairs of self-mapping in a cone metric space by using the α-property.

Theorem 10. *Consider a complete cone metric space (X, d) and $*$ fulfills the α-property with α greater than 0. Define $J, L, K,$ and M as self-mappings on X which satisfy the conditions below:*

1) *$M(X)$ or $K(X)$ is a closed subset of X and $J(X) \subseteq M(X), L(X) \subseteq K(X)$,*
2) *(J, K) and (L, M) are weakly compatible,*
3) *$\forall\ a, b \in X$,*

$$\begin{aligned} d(Ja, Lb) \leq\ & q_1(d(Ka, Mb) *\ d(Ja, Ka)) \\ &+ q_2(d(Ka, Mb) *\ d(Lb, Mb)) \\ &+ q_3\Bigg\{d(Ka, Mb)* \\ &\qquad \frac{d(Ka, Lb) + d(Ja, Mb)}{2}\Bigg\} \end{aligned} \tag{2}$$

where $q_1, q_2, q_3 > 0$ and $0 < \alpha(q_1 + q_2 + q_3) < 1$. Then, $J, L, K,$ and M possess a unique common fixed point within X.

Proof: Assume that a_0 is any point within the set X. By (1), we can define a sequence $\{b_m\}$ in X such that $b_{2m} = Ja_{2m} = Ma_{2m+1}$ and $b_{2m+1} = La_{2m+1} = Ka_{2m+2}$, for $m = 0, 1, 2, \ldots$

Now, we prove that $\{b_m\}$ is a Cauchy sequence.
Applying condition (3), we get

$$\begin{aligned}
&d(b_{2m}, b_{2m+1}) = d(Ja_{2m}, La_{2m+1})\\
&\le q_1(d(Ka_{2m}, Ma_{2m+1}) * d(Ja_{2m}, Ka_{2m}))\\
&\quad + q_2(d(Ka_{2m}, Ma_{2m+1}) * d(La_{2m+1}, Ma_{2m+1}))\\
&\quad + q_3\Big\{(d(Ka_{2m}, Ma_{2m}) *\\
&\qquad \frac{d(Ka_{2m}, La_{2m+1} + d(Ja_{2m}, Ma_{2m+1})}{2}\Big\}\\
&= q_1(d(b_{2m-1}, b_{2m}) * d(b_{2m}, b_{2m-1}))\\
&\quad + q_2(d(b_{2m-1}, b_{2m}) * d(b_{2m+1}, b_{2m}))\\
&\quad + q_3\left\{d(b_{2m-1}, b_{2m}) * \frac{d(b_{2m-1}, b_{2m+1}) + d(b_{2m}, b_{2m})}{2}\right\}
\end{aligned}$$

Fix $d(b_m, b_{m+1}) = d_m$. Applying the above inequality, we obtain

$$\begin{aligned}
d_{2m} &\le q_1(d_{2m-1} * d_{2m-1}) + q_2(d_{2m-1} * d_{2m})\\
&\quad + q_3\left\{d_{2m-1} * \frac{d(b_{2m-1}, b_{2m+1})}{2}\right\}
\end{aligned}$$

Therefore,

$$\begin{aligned}
d_{2m} &\le q_1\alpha \max\{d_{2m-1} * d_{2m-1}\}\\
&\quad + q_2\alpha \max\{d_{2m-1} * d_{2m}\}\\
&\quad + q_3\alpha \max\left\{d_{2m-1} * \frac{d(b_{2m-1}, b_{2m+1})}{2}\right\}
\end{aligned}$$

If $d_{2m-1} < d_{2m}$, we get

$$d_m \le q_1\alpha d_{2m} + q_2\alpha d_{2m} + q_3\alpha d_{2m} < d_{2m}$$

which contradicts itself. Hence, $d_{2m-1} \ge d_{2m}$. Similarly, $d_{2m} \ge d_{2m-1}$. Hence, $d_{m-1} \ge d_m$, for $m = 1, 2, \ldots$
Applying the above inequality, we obtain

$$d_m \le \alpha(q_1 + q_2 + q_3)d_{m-1} = qd_{m-1},$$

where $\alpha(q_1 + q_2 + q_3) = q < 1$. So

$$d_m \le qd_{m-1} \le q^2 d_{m-2} \le \ldots\ldots. \le q^m d_0$$

That is,

$$d(b_m, b_{m+1}) \le q^m d(b_0, b_1) \to 0 \; as \; m \to \infty$$

If $n > m$, then

$$\begin{aligned}
d(b_m, b_n) &\le d(b_m, b_{m+1}) + d(b_{m+1}, b_{m+2}) + \ldots\\
&\quad + d(b_{n-1}, b_n)\\
&\le q^m d(b_0, b_1) + q^{m+1} d(b_0, b_1) \ldots\\
&\quad + q^{n+1} d(b_0, b_1)\\
&= \frac{q^m}{1-q} d(b_0, b_1) \to 0
\end{aligned}$$

as $m, n \to \infty$. Hence, $\{b_m\}$ is a Cauchy sequence and by completeness property of X,$\{b_m\}$ converges to b in X. Thus,

$$\begin{aligned}
\lim_{m\to\infty} b_m &= \lim_{m\to\infty} Ja_{2m} = \lim_{m\to\infty} La_{2m+1}\\
&= \lim_{m\to\infty} Ka_{2m+2} = \lim_{m\to\infty} Ma_{2m+1} = b.
\end{aligned}$$

Suppose that $\overline{M(X)} \subseteq X$. Then $\exists\, \upsilon$ in X such that $M\upsilon = b$.
If $L\upsilon \ne b$, then applying condition (3), we get

$$\begin{aligned}
d(Ja_{2m}, L\upsilon) &\le q_1(d(Ka_{2m}, M\upsilon) * d(Ja_{2m}, Ka_{2m}))\\
&\quad + q_2(d(Ka_{2m}, M\upsilon) * d(L\upsilon, M\upsilon))\\
&\quad + q_3\Big\{d(Ka_{2m}, M\upsilon) *\\
&\qquad \frac{d(Ka_{2m}, L\upsilon) + d(Ja_{2m}, M\upsilon)}{2}\Big\}
\end{aligned}$$

For $m \to \infty$, we obtain

$$\begin{aligned}
d(b, L\upsilon) &\le q_1(d(b, M\upsilon) * d(b, b))\\
&\quad + q_2(d(b, M\upsilon) * d(L\upsilon, M\upsilon))\\
&\quad + q_3\left\{d(b, M\upsilon) * \frac{d(b, L\upsilon) + d(b, M\upsilon)}{2}\right\}\\
&\le q_1\alpha \max\{d(b, M\upsilon), 0\}\\
&\quad + q_2\alpha \max\{0, d(L\upsilon, b)\}\\
&\quad + q_3\alpha \max\left\{0, \frac{d(b, L\upsilon) + 0}{2}\right\}\\
&< d(b, L\upsilon).
\end{aligned}$$

Consequently, $L\upsilon = b = M\upsilon$. As L and M are weakly compatible, we get $LM\upsilon = ML\upsilon$ and so $Lb = Mb$
If $b \ne Bb$, by condition (3), we obtain

$$\begin{aligned}
\lim_{m\to\infty} d(Ja_{2m}, Lb) &\le \lim_{m\to\infty} q_1(d(Ka_{2m}, Mb) * d(Ja_{2m}, Ka_{2m}))\\
&\quad + q_2(d(Ka_{2m}, Mb) * d(Lb, Mb))\\
&\quad + q_3\Big\{d(Ka_{2m}, Mb) *\\
&\qquad \frac{d(Ka_{2m}, Lb) + d(Ja_{2m}, Mb)}{2}\Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
d(b, Lb) &\le q_1(d(b, Mb) * d(b, b))\\
&\quad + q_2(d(b, Mb) * d(Lb, Mb))\\
&\quad + q_3\left\{d(b, Mb) * \frac{d(b, Lb) + d(b, Mb)}{2}\right\}\\
&\le q_1\alpha \max(d(b, Mb), d(b, b))\\
&\quad + q_2\alpha \max(d(b, Mb), d(Lb, Mb))\\
&\quad + q_3\alpha \max\left\{d(b, Mb), \frac{d(b, Lb) + d(b, Mb)}{2}\right\}\\
&< d(b, Lb)
\end{aligned}$$

Therefore, $Lb = b$.
As $L(X) \subseteq K(X)$, $\exists\, w$ in X such that $Kw = b$.
If $Jw \ne b$, then applying condition (3), we get

$$\begin{aligned}
d(Jw, b) &\le q_1(d(Kw, Mb) * d(Jw, Kw))\\
&\quad + q_2(d(Kw, Mb) * d(Lb, Mb))\\
&\quad + q_3\left\{d(Kw, Mb) * \frac{d(Kw, Lb) + d(Jw, Mb)}{2}\right\}
\end{aligned}$$

consequently,

$$\begin{aligned}d(Jw,b) &\le q_1(d(Kw,b)*d(Jw,Kw))\\&\quad+q_2(d(Kw,b)*d(b,b))\\&\quad+q_3\left\{d(Kw,b)*\frac{d(Kw,b)+d(Jw,b)}{2}\right\}\\&\le q_1\alpha\max(d(Kw,b),d(Jw,Kw))\\&\quad+q_2\alpha\max(d(Kw,b),d(b,b))\\&\quad+q_3\alpha\max\left\{d(Kw,b),\frac{d(Kw,b)+d(Lw,b)}{2}\right\}\\&< d(Jw,b).\end{aligned}$$

Hence, $Jw=b$. Thus, $Jw=Kw=b$. As J and K are weakly compatible, $JKw=KJw$. Hence,

$$Jb=Kb.$$

If $Jb\neq b$, then by condition (3), we obtain

$$\begin{aligned}d(Jb,b)&=d(Jb,Lb)\\&\le q_1(d(Kb,Mb)*d(Jb,Kb))\\&\quad+q_2(d(Kb,Mb)*d(Lb,Mb))\\&\quad+q_3\left\{d(Kb,Mb)*\frac{d(Kb,Lb)+d(Jb,Mb)}{2}\right\}\\&=q_1(d(Kb,b)*d(Jb,Kb))\\&\quad+q_2(d(Kb,b)*d(b,b))\\&\quad+q_3\left\{d(Kb,b)*k_3(d(Kb,b)*\frac{d(Kb,b)+d(Jb,b)}{2}\right\}\\&\le q_1\alpha\max(d(Kb,b),d(Jb,Kb))\\&\quad+q_2\alpha\max(d(Kb,b),d(b,b))\\&\quad+q_3\alpha\max\left\{d(Kb,b),\frac{d(Kb,b)+d(Jb,b)}{2}\right\}\\&< d(Jb,b).\end{aligned}$$

So, $Jb=b$. Hence,

$$Jb=Kb=Lb=Mb=b,$$

i.e., J,L,K and M has a common fixed point b.
The proof is analogous for $\overline{K(X)}\subseteq X$.
Also, the uniqueness of b holds using condition (3). ■

Example 11. *Consider a cone metric space (X,d) with $d(a,b)=|a-b|$. Let J,L,K and M be self-mappings on X by*

$$Ja=La=\frac{1}{2},\quad Ka=\frac{3a+1}{5},\quad Ma=\frac{4a+1}{6}$$

for all $a\in X$. It is easy to verify all the conditions of Theorem 10 with $*=\max$, $\alpha=1$ and $q_1+q_2+q_3=\frac{3}{8}$. Hence, $\frac{1}{2}$ is the unique fixed point of the mappings J,L,K and M.

Now we prove our main result.

Theorem 12. *Let J,L,K, and M be four self-mappings on a cone metric space (X,d), and let the operation $*$ satisfy the α-property with $\alpha>0$. Suppose that the pairs (J,K) and (L,M) are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), fulfils the inequality (2) of Theorem 10. Then J,L,K, and M has a unique common fixed point in X.*

Proof: If the set of mappings (J,K) is both subsequentially continuous and compatible, then $\exists$ a sequence $\{a_m\}\in X$ such that

$$\lim_{m\to\infty}Ja_m=\lim_{m\to\infty}Ka_m=u, \tag{3}$$

for some $u\in X$, and

$$\lim_{m\to\infty}d(JKa_m,KJa_m)=d(Ju,Ku)=0; \tag{4}$$

i.e., $Ju=Ku$. Similarly, for the pair (L,M), $\exists$ a sequence $\{b_m\}$ in X such that

$$\lim_{m\to\infty}Lb_m=\lim_{m\to\infty}Mb_m=w, \tag{5}$$

for some $w\in X$, and

$$\lim_{m\to\infty}d(LMb_m,MLb_m)=d(Lw,Mw)=0; \tag{6}$$

i.e., $Lw=Mw$. Thus, for the pair (J,K), u is a coincidence point and for the pair (L,M), w is a coincidence point.
We now claim that $u=w$. Suppose $u\neq w$, then applying (2) with $a=a_m$ and $b=b_m$, we obtain

$$\begin{aligned}d(Ja_m,Lb_m)&\le q_1(d(Ka_m,Mb_m)*d(Ja_m,Ka_m))\\&\quad+q_2(d(Ka_m,Mb_m)*d(Lb_m,Mb_m))\\&\quad+q_3\left\{d(Ka_m,Mb_m)*\frac{d(Ka_m,Lb_m)+d(Ja_m,Mb_m)}{2}\right\}.\end{aligned} \tag{7}$$

Suppose $m\to\infty$, we obtain

$$\begin{aligned}d(u,w)&\le q_1(d(u,w)*d(u,u))\\&\quad+q_2(d(u,w)*d(w,w))\\&\quad+q_3\left\{d(u,w)*\frac{d(u,w)+d(u,w)}{2}\right\}.\end{aligned} \tag{8}$$

Given that $*$ fulfils the α-property, we get

$$\begin{aligned}d(u,w)&\le q_1\ \alpha\ \max(d(u,w),d(u,u))\\&\quad+q_2\ \alpha\ \max(d(u,w),d(w,w))\\&\quad+q_3\ \alpha\ \max\left\{d(u,w),\frac{d(u,w)+d(u,w)}{2}\right\}\\&=q_1\alpha\ \max(d(u,w),0)\\&\quad+q_2\alpha\ \max(d(u,w),0)\\&\quad+q_3\alpha\ \max(d(u,w),d(u,w))\\&=\alpha\ (q_1+q_2+q_3)d(u,w)\\&< d(u,w)\end{aligned} \tag{9}$$

which contradicts itself. Thus, $u=w$. We now show that $Ju=u$. Assume $Ju\neq u$, then applying the inequality (2) with $a = u$ and $b = b_m$, we obtain

$$\begin{aligned} d(Ju, Lb_m) \leq\ & q_1(d(Ku, Mb_m) *\ d(Ju, Ku)) \\ & + q_2(d(Ku, Mb_m) *\ d(Lb_m, Mb_m)) \\ & + q_3\Big\{(d(Ku, Mb_m) * \\ & \frac{d(Ku, Lb_m) + d(Ju, Mb_m)}{2}\Big\}. \end{aligned} \tag{10}$$

Taking the limit as $m \to \infty$, we have

$$\begin{aligned} d(Ju, w) \leq\ & q_1(d(Ku, w) *\ d(Ju, Ju)) \\ & + q_2(d(Ju, w) *\ d(w, w)) \\ & + q_3\Big\{d(Ju, w) * \\ & \frac{d(Ju, w) + d(Ju, w)}{2}\Big\}; \end{aligned} \tag{11}$$

that is,

$$\begin{aligned} d(Ju, u) \leq\ & q_1(d(Ku, u) *\ d(Ju, Ju)) \\ & + q_2(d(Ju, u) *\ d(u, u)) \\ & + q_3(d(Ju, u) * \frac{d(Ju, u) + d(Ju, u)}{2}) \\ \leq\ & q_1\ \alpha\ \max(d(Ku, u), d(Ju, Ju)) \\ & + q_2\ \alpha\ \max(d(Ju, u), d(u, u)) \\ & + q_3\ \alpha\ \max\Big\{d(Ju, u), \\ & \frac{d(Ju, u) + d(Ju, u)}{2}\Big\}. \end{aligned} \tag{12}$$

Simplifying the above inequality, we get

$$\begin{aligned} d(Ju, u) &\leq \alpha(q_1 + q_2 + q_3) d(Ju, u) \\ &< d(Ju, u), \end{aligned} \tag{13}$$

which contradicts itself. Thus, $Ju = u$. Hence, $Ju = Ku = u$. We now prove that $Lu = u$. If $Lu \neq u$ then applying condition (2) with $a = a_m$ and $b = u$, we obtain

$$\begin{aligned} d(Ja_m, Lu) \leq\ & q_1(d(Ka_m, Mu) *\ d(Ja_m, Ka_m)) \\ & + q_2(d(Ka_m, Mu) *\ d(Lu, Mu)) \\ & + q_3\Big\{d(Ka_m, Mu) * \\ & \frac{d(Ka_m, Lu) + d(Ja_m, Mu)}{2}\Big\}. \end{aligned} \tag{14}$$

When $m \to \infty$, we obtain

$$\begin{aligned} d(u, Lu) \leq\ & q_1(d(u, Lu) *\ d(u, u)) \\ & + q_2(d(u, Lu) *\ d(Lu, Lu)) \\ & + q_3\Big\{d(u, Lu) * \\ & \frac{d(u, Lu) + d(u, Lu)}{2}\Big\} \\ \leq\ & q_1\alpha\ \max(d(u, Lu), d(u, u)) \\ & + q_2\alpha\ \max(d(u, Lu), d(Lu, Lu)) \\ & + q_3\alpha\ \max\Big\{d(u, Lu), \\ & \frac{d(u, Lu) + d(u, Lu)}{2}\Big\}. \end{aligned} \tag{15}$$

Simplifying the above inequality, we get

$$\begin{aligned} d(u, Lu) &\leq \alpha(q_1 + q_2 + q_3) d(u, Lu) \\ &< d(u, Lu), \end{aligned} \tag{16}$$

which contradicts itself. Thus, $Lu = u$. Hence, $u = Ju = Ku = Lu = Mu$; i.e., J, L, K and M have a common fixed point u. The uniqueness of a common fixed point can be deduced from inequality (2).
We now assume the mappings (J, K) (as well as (L, M)) are both reciprocally continuous and subcompatible. Then $\exists$ $\{a_m\} \in X$ such that

$$\lim_{m\to\infty} Ja_m = \lim_{m\to\infty} Ka_m = u, \tag{17}$$

for some u in X, and thus

$$\lim_{m\to\infty} d(JKa_m, KJa_m) = d(Ju, Ku) = 0, \tag{18}$$

whereas in respect of the pair (L, M), there exists a sequence $\{b_m\}$ in X with

$$\lim_{m\to\infty} Lb_m = \lim_{m\to\infty} Mb_m = w, \tag{19}$$

for some w in X, and

$$\lim_{m\to\infty} d(LMb_m, MLb_m) = d(Lw, Mw) = 0. \tag{20}$$

Therefore, $Ju = Ku$ and $Lw = Mw$; i.e., u is a coincidence point of the pair (J, K), while w is a coincidence point of the pair (L, M). One can easily complete the rest of the proof. ■

Example 13. *Consider the set X defined on the interval $[0, \infty)$. Suppose d represent the usual metric on X. Let J, L, K and M be self-mappings defined as*

$$\begin{aligned} Ja = La &= \begin{cases} \frac{a}{7}, & \text{for } a \text{ in } [0, 1]; \\ \frac{a+6}{7}, & \text{for } a \text{ in } (1, \infty), \end{cases} \\ Ka = Ma &= \begin{cases} \frac{a}{6}, & \text{for } a \text{ in } [0, 1]; \\ \frac{a+5}{6}, & \text{for } a \text{ in } (1, \infty). \end{cases} \end{aligned} \tag{21}$$

Assume the sequence $\{a_m\} = \{1/m\}_{m\in\mathbb{N}} \in X$. Now

$$\begin{aligned} \lim_{m\to\infty} Ja_m &= \lim_{m\to\infty} \frac{1}{7m} = 0 \\ &= \lim_{m\to\infty} \frac{1}{6m} = \lim_{m\to\infty} Ka_m. \end{aligned} \tag{22}$$

Next

$$\begin{aligned} \lim_{m\to\infty} JKa_m &= \lim_{m\to\infty} J\left(\frac{1}{6m}\right) \\ &= \lim_{m\to\infty} \frac{1}{42m} = 0 = J0, \\ \lim_{m\to\infty} KJa_m &= \lim_{m\to\infty} K\left(\frac{1}{7m}\right) \\ &= \lim_{m\to\infty} \frac{1}{42m} = 0 = K0, \end{aligned} \tag{23}$$

$$\lim_{m\to\infty} d(JKa_m, KJa_m) = 0.$$

Assume another sequence $\{a_m\} = \{1 + 1/m\}_{m\in\mathbb{N}} \in X$. Now,

$$\begin{aligned} \lim_{m\to\infty} Ja_m &= \lim_{m\to\infty} \left(1 + \frac{1}{7m}\right) = 1 \\ &= \lim_{m\to\infty} \left(1 + \frac{1}{6m}\right) \\ &= \lim_{m\to\infty} Ka_m. \end{aligned} \tag{24}$$

However,

$$\begin{aligned}\lim_{m\to\infty} JKa_m &= \lim_{m\to\infty} J\left(1+\frac{1}{6m}\right)\\ &= \lim_{m\to\infty}\left(1+\frac{1}{42m}\right) = 1, \neq J1\\ \lim_{m\to\infty} KJa_m &= \lim_{m\to\infty} K\left(1+\frac{1}{7m}\right)\\ &= \lim_{m\to\infty}\left(1+\frac{1}{42m}\right) = 1 \neq K1,\end{aligned} \tag{25}$$

but $\lim_{m\to\infty} d(JKa_m, KJa_m) = 0$. Therefore, the pair (J,K) is not reciprocally continuous but is compatible and subsequentially continuous (holds for (L,M) as well). It is straightforward to verify that inequality (2) is fulfilled using $* = max$, $\alpha = 1$, $q_1+q_2+q_3 = 6/7$. Thus, all criteria of Theorem 12 are fulfilled. Hence, the mappings J, L, K and M have 0 as both the coincidence and the unique common fixed point.
It should be noted that fixed point theorems that require criteria of closedness of respective ranges or that require both reciprocal continuity and compatibility cannot apply to this example. In fact, neither $J(X)$ nor $K(X)$ are closed in this case since $J(X) = [0, 1/7] \cup (1,\infty)$ and $K(X) = [0, 1/6] \cup (1,\infty)$.

Example 14. *Let d be the usual metric on X and let $X = \mathbb{R}$ be a collection of real numbers. Define J, L, K and M self-mappings by*

$$\begin{aligned} Ja = La &= \begin{cases} \frac{a}{4}, & \text{if } a \in (-\infty, 1);\\ 4a-3, & \text{if } a\in[1,\infty),\end{cases}\\ Ka = Ma &= \begin{cases} a+3, & \text{if } a \in (-\infty,1);\\ 5a-5, & \text{if } a \in [1,\infty].\end{cases}\end{aligned} \tag{26}$$

Assume the sequence $\{a_m\} = \{1 + 1/m\}_{m\in\mathbb{N}}$ in X. Then

$$\begin{aligned}\lim_{m\to\infty} Ja_m &= \lim_{m\to\infty}\left(1+\frac{4}{m}\right) = 1\\ &= \lim_{m\to\infty}\left(1+\frac{5}{m}\right) = \lim_{m\to\infty} Ka_m.\end{aligned} \tag{27}$$

Also,

$$\begin{aligned}\lim_{m\to\infty} JKa_m &= \lim_{m\to\infty} J\left(1+\frac{5}{m}\right)\\ &= \lim_{m\to\infty}\left(1+\frac{20}{m}\right) = 1 = J1,\\ \lim_{m\to\infty} KJa_m &= \lim_{m\to\infty} K\left(1+\frac{4}{m}\right)\\ &= \lim_{m\to\infty}\left(1+\frac{20}{m}\right) = 1 = K1,\\ &\lim_{m\to\infty} d(JKa_m, KJa_m) = 0.\end{aligned} \tag{28}$$

Assume another sequence

$$\{a_m\} = \{(1/m) - 4\}_{m\in\mathbb{N}}$$

$\in X$. *Wehave*

$$\begin{aligned}\lim_{m\to\infty} J(a_m) &= \lim_{m\to\infty}\left(\frac{1}{4m}-1\right) = -1\\ &= \lim_{m\to\infty}\left(\frac{1}{m}-1\right)\\ &= \lim_{m\to\infty} K(a_m).\end{aligned} \tag{29}$$

Next,

$$\begin{aligned}\lim_{m\to\infty} JKa_m &= \lim_{m\to\infty} J\left(\frac{1}{m}-1\right)\\ &= \lim_{m\to\infty}\left(\frac{1}{4m}-\frac{1}{4}\right) = -\frac{1}{4} = J(-1),\\ \lim_{m\to\infty} KJa_m &= \lim_{m\to\infty} K\left(\frac{1}{4m}-1\right)\\ &= \lim_{m\to\infty}\left(\frac{1}{4m}-1+3\right) = 2 = K(-1),\end{aligned} \tag{30}$$

also, $\lim_{m\to\infty} d(JKa_m, KJa_m) \neq 0$. Hence, the pair (J,K) is not compatible but is subcompatible and reciprocally continuous (holds for (L,M) as well). It is straightforward to verify inequality (2) is fulfilled using $* = max$, $\alpha = 1$, $q_1+q_2+q_3 = 4/5$. Thus, criteria of Theorem 12 are fulfilled. Here, the pair (J,K) have 1 as both the coincidence and unique common fixed point.
This example does not satisfy the conditions of fixed point theorems that need both compatibility and reciprocal continuity. Also, $J(X) = (-\infty, \frac{1}{4}) \cup [1,+\infty)$ is not closed. It is important to mention that the mappings J and K have $(-1$ and $1)$ as points of coincidence, which not weakly compatible but occasionally weakly compatible.
In the following example (see [14], Example 1.4), we illustrate a scenario where the criteria of Theorem 12 are not met, where the pairs have no common fixed points.

Example 15. *Assume $X = [0,+\infty)$ with the standard metric d. Suppose $J, L, K, M : X \to X$ be given by*

$$\begin{aligned} Ja = La &= \begin{cases} a+1, & 0\le a\le 1,\\ 2a-1, & a>1,\end{cases}\\ Ka = Ma &= \begin{cases} 1-a, & 0\le a<1,\\ 3a-2, & a\ge 1.\end{cases}\end{aligned} \tag{31}$$

Furthermore, as seen in ([14], Example 1.4), the pairs (J,K) and (L,M) are both subcompatible and subsequentially continuous. They are not, however, compatible or reciprocally continuous — not even occassionally weakly compatible. We see that the pair (J,K) doesn't have common fixed point, however it does have a single point of coincidence at $u = 1$. We provide an alternative example that was motivated by ([22], Example 3)

Example 16. *Define a metric d on X ,where $X = \{0,1,2,.....,10\}$, by*

$$d(a,b) = \begin{cases} 0, & a=b,\\ \max\{a,b\}, & a\neq b.\end{cases} \tag{32}$$

Let the mappings of $J, L, K, M : X \to X$ as

$$\begin{aligned} Ja = La &= \begin{cases} 0, & a=0,\\ a-1, & a\ge 1\end{cases};\\ Ka = Ma &= \begin{cases} 0, & a=0,\\ a+1, & 1\le a\le 9,\\ 10, & a=10\end{cases}.\end{aligned} \tag{33}$$

Consider $* = max$, and fulfils the α-condition for $\alpha = 1$. Now, (1)

1) (J, K) is both subsequentially continuous and compatible, as is the pair (L, M),
2) with $q_1 = q_2 = q_3 = 0.3$ inequality (2) is satisfied.

Indeed, to demonstrate (1), assume that $a_m = 0 \ \forall \ m$ except for a finite number of them, as this is the sole method by which the same limit can be obtained for (Ja_m) and (Ka_m). Now, $d(Ja_m, 0) \to 0$ and $d(Ka_m, 0) \to 0$; $JKa_m \to 0 = J0$ and $KJa_m \to 0 = K0$. Thus, (J, K) is both subsequentially continuous and compatible.

To show (2), for a, b in X, $a \neq b$ (for $a = b$ is trivial). With $J = L, K = M$ and $q_1 = q_2$, inequality (2) shows symmetric in a, b; Therefore, we can assume that $a \geq b$ without losing generality. Now, the following cases arises.

Case 1. If $a = 1$ and $b = 0$. Then, $Ja = Lb = 0, d(Ja, Lb) = 0$, and inequality (2) is satisfied.

Case 2. If $2 \leq a \leq 9$ and $b \in \{0, 1\}$. Then, $Bb = 0, Ja = a - 1$ and $d(Ja, Lb) = a - 1$. The solution to inequality (2) on the right-hand side is now $(t \in \{0, 2\})$.

$$\begin{aligned} T =& q_1 \max\{a+1, t\} + q_2 \max\{a+1, a+1\} \\ &+ q_3 \max\left\{a+1, \frac{1}{2}[\max\{Ka, Lb\} + a + 1]\right\} \\ =&(q_1 + q_2 + q_3)(a+1) = 0.9(a+1) \\ \geq& \ 0.9.\frac{10}{8}(a-1) > a - 1 = d(Ja, Lb). \end{aligned} \tag{34}$$

Case 3. If $a = 10$ and $b \in \{0, 1\}$. Then,

$$d(Ja, Lb) = 9 = (q_1 + q_2 + q_3) \cdot 10 = T. \tag{35}$$

Case 4. If $2 \leq b < a \leq 9$. Then

$$d(Ja, Lb) = a - 1 = d(a-1, b-1)$$

also,

$$\begin{aligned} T =& \ q_1 \max\{a+1, a+1\} + q_2 max\{a+1, a+1\} \\ &+ q_3 \max\left\{a+1, \frac{1}{2}[a+1+\max\{Ja, Mb\}]\right\} \\ =& \ (q_1 + q_2 + q_3)(a+1) = 0.9(a+1) \\ \geq& \ 0.9 \cdot \frac{10}{8}(a-1) \\ >& \ d(Ja, Lb) = a - 1. \end{aligned} \tag{36}$$

Case 5. If $2 \leq b < b < a = 10$. Again inequality (2) reduces to $(q_1 + q_2 + q_3) \cdot 10 = 9$.

All criteria of Theorem 12 are satisfied, and J, L, K, and M possess a unique common fixed point, $u = 0$.

We can derive corollaries for two or three self-mappings by selecting J, L, K and M appropriately in Theorem 12. We derive the subsequent corollary for two self-mappings as a sample.

Corollary 17. *Let J and K be two self-mappings on a cone metric space (X, d) that satisfy the α-property, where α is a positive constant. Let the pair (J, K) be both subsequentially continuous and compatible, or alternatively, reciprocally continuous and subcompatible, satisfying*

$$\begin{aligned} d(Ja, Jb) \leq& \ q_1(d(Ka, Kb) * d(Ja, Ka)) \\ &+ q_2(d(Ka, Kb) * d(Jb, Kb)) \\ &+ q_3\left\{d(Ka, Kb) * \right. \\ &\left. \frac{d(Ka, Jb) + d(Ja, Kb)}{2}\right\}, \end{aligned} \tag{37}$$

$\forall \ a, b$ in X, where $q_1, q_2, q_3 > 0$ and

$$0 < \alpha(q_1 + q_2 + q_3) < 1,$$

then, $\exists$ a unique common fixed point in X for both J and K.

Remark 18. *The result of Theorem 12 remains valid when inequality (2) is substituted with the following:*

$$\begin{aligned} d(Ja, Lb) \leq& \ q_1(d(Ka, Mb) + d(Ja, Ka)) \\ &+ q_2(d(Ka, Mb) + d(Lb, Mb)) \\ &+ q_3\left\{d(Ka, Mb) + \right. \\ &\left. \frac{d(Ka, Lb) + d(Ja, Mb)}{2}\right\}, \end{aligned} \tag{38}$$

$\forall \ a, b$ in X, where $q_1, q_2, q_3 > 0$ and

$$0 < q_1 + q_2 + q_3 < 1/2.$$

Likewise, different contractive conditions can be derived by defining operation $*$.

Remark 19. *Similar findings can be achieved by replacing inequality (2) with the following one:*
With an appropriate function $\psi : [0, +\infty) \to [0, +\infty)$,

$$\begin{aligned} & d(Ja, Lb) \leq \psi(z) \text{ for some} \\ & z \in \{d(Ka, Mb), d(Ja, Ka), \\ & \quad d(Lb, Mb), d(Ka, Lb), d(Ja, Mb)\}. \end{aligned} \tag{39}$$

IV. Fixed point results in Banach space

Second, we introduce the notion of common fixed point theorems for a pair of weakly compatible mappings in dislocated cone metric space over Banach algebra.

Definition 20. *[6] A Banach algebra $\mathcal{A}$ is a real Banach space with a multiplication operation is defined as $\forall \ a, b, c$ in $\mathcal{A}$, α in $\mathbb{R}$*

1) $(ab)c = a(bc)$,
2) $a(b + c) = ab + ac$ *and* $(a + b)c = ac + bc$,
3) $\alpha(ab) = (\alpha a)b = a(\alpha b)$,
4) $||ab|| \leq ||a|| \ ||b||$

Then, $\mathcal{B} \subseteq \mathcal{A}$ is called a cone if

1) *$\mathcal{B}$ is non-empty, closed and $\{\theta, e\} \subset \mathcal{B}$;*
2) *$\beta\mathcal{B} + \gamma\mathcal{B} \subset \mathcal{B} \ \forall$ non-negative $\beta, \gamma \in \mathbb{R}$;*
3) *$\mathcal{B}^2 = \mathcal{B}\mathcal{B} \subset \mathcal{B}$;*
4) *$\mathcal{B} \cap (-\mathcal{B}) = \{\theta\}$,*

where the unit and zero elements of the Banach algebra $\mathcal{A}$ are denoted by e and θ, respectively. For $\mathcal{B} \subset \mathcal{A}$, we write $b - c \in \mathcal{B}$ iff $c \preceq b$, where $\preceq$ is a partial ordering on $\mathcal{B}$. Furthermore, $a \ll b$ will be denoted for $b - a \in int\mathcal{B}$, where $int\mathcal{B}$ represents the interior of $\mathcal{B}$. Also, $\mathcal{B}$ is called a solid cone if $int\mathcal{B} \neq \phi$.

Definition 21. *[23] Let X be a non-empty set. Suppose that $d : X \times X \to \mathcal{A}$ be a mapping satisfying the following conditions:*

1) *$\theta \preceq d(m, n) \ \forall \ a, b$ in X and $d(m, n) = \theta \Rightarrow m = n$;*
2) *$d(m, n) = d(n, m) \ \forall \ m, n$ in X;*
3) *$d(m, n) \preceq d(m, o) + d(o, n) \ \forall \ m, n, o$ in X*

Then, d is called a dislocated cone metric space on X and (X, d) is called a dislocated cone metric space over Banach algebra $\mathcal{A}$.

Remark 22. *[23] For each element $a \in X$ in a dislocated cone metric space (X, d), $d(a, a)$ does not necessarily have to be zero. Therefore, any metric space that is on a Banach algebra is also a dislocated cone metric space on the same algebra. However, the reverse is not always true.*

Example 23. *[23] Let $\mathcal{A} = \{p = (p_{mn})_{3\times 3} : p_{mn} \in R, 1 \leq m, n \leq 3\}$, $||p|| = \sum_{1\leq m,n\leq 3} |p_{m,n}|$, $\mathcal{B} = \{p \in \mathcal{A} : p_{m,n} \geq 0, 1 \leq m, n \leq 3\}$ be a cone in $\mathcal{A}$. Let $X = \mathbb{R}^+ \cup \{0\}$. Let a mapping $d : X \times X \to \mathcal{A}$ be define by*

$$d(a,b) = \begin{pmatrix} a+b & a+b & a+b \\ 2a+2b & 2a+2b & 2a+2b \\ 3a+3b & 3a+3b & 3a+3b \end{pmatrix} \forall\ a, b\ in\ X$$

Then, over a Banach algebra $\mathcal{A}, (X, d)$ is a dislocated cone metric space but not a cone metric space because

$$d\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix} \neq \theta$$

Definition 24. *[23] Let $a \in X$ and $\{a_m\}$ be a sequence in a dislocated cone metric space (X, d) over Banach algebra $\mathcal{A}$, then*

1) *$\{a_m\}$ converges to a whenever for each $b \in \mathcal{A}$ with $\theta \ll b$, $\exists N \in \mathbb{N}$ such that $d(a_m, a_n) \ll b\ \forall\ m, n \geq N$.*
2) *$\{a_i\}$ is a Cauchy sequence whenever for each $b \in \mathcal{A}$ with $\theta \ll b$, $\exists N \in \mathbb{N}$ such that $d(a_m, a_n) \ll b\ \forall\ m, n \geq N$.*
3) *(X, d) is considered complete if every Cauchy sequence in X is convergent.*

Definition 25. *[20] Let $\mathcal{B}$ be a solid cone in a Banach algebra $\mathcal{A}$. A sequence $\{a_m\} \subset \mathcal{B}$ is defined as a c-sequence if for every $\theta \ll b$, there exists a N in $\mathbb{N}$ such that $a_m \ll b$ for all $m > N$.*

Lemma 26. *[20] Consider a solid cone $\mathcal{B}$ in a Banach algebra $\mathcal{A}$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $\mathcal{B}$. If $\{\alpha_n\}$ and $\{\beta_n\}$ are c-sequences and $q_1, q_2 \in \mathcal{B}$, then $\{q_1\alpha_n + q_2\beta_n\}$ is also a c-sequence.*

Lemma 27. *[20] Consider a real Banach space E with a solid cone $\mathcal{B}$:*

1) *If a, b, c are in E and $a \preceq b \preceq c$ implies $a \preceq c$.*
2) *If $a \in \mathcal{B}$ and $\theta \preceq a \preceq c$ implies $a = \theta$.*

Definition 28. *Consider two self-mappings, J and K, represented on a set X. A coincidence of J and K is defined as $Ja = Ka$ for some $a \in X$.*

Definition 29. *Assume J, K be two self-mappings on set X. Mappings J, K are said to be commuting if $JKx = KJx$ for all $x \in X$.*

Lemma 30. *Consider a complete dislocated cone metric space (X, d) over Banach algebra $\mathcal{A}$, suppose $R : X \to X$ be a contraction mapping. Then, R has a unique fixed point.*

Definition 31. *Let J and K be mappings from a dislocated cone metric space (X, d) over Banach algebra $\mathcal{A}$ into itself. Then, J and K are said to be weakly compatible if they commute at their coincident point, that is, $Jx = Kx$ for some $x \in X$ implies $JKx = KJx$.*

Definition 32. *Consider a dislocated cone metric space (X, d) over Banach algebra $\mathcal{A}$, then a mapping $T : X \to X$ is said to be contraction if $\exists$ a number λ with $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda d(x, y)$.*

Definition 33. *A function ϕ defined on $\mathcal{B}$ over Banach algebra $\mathcal{A}$ is said to be upper semi-continuous if*

$$\lim_{n\to\infty} \phi(t_n) \leq \phi(t)$$

for every sequence $\{t_n\} \in X$ with $t_n \to t$ as $n \to \infty$.

Definition 34. *A function $\phi : \mathcal{B} \to \mathcal{B}$ is said to be contractive modulus if $\phi(t) < t$ for $t > 0$.*

Theorem 35. *Consider a complete dislocated cone metric space (X, d) over Banach algebra $\mathcal{A}$, where $\mathcal{B}$ be the underlying solid cone and e is a unit. Suppose that J, L, K and M be four self-mappings of X fulfilling the conditions below:*

1) *$M(X) \subseteq J(X)$ and $K(X) \subseteq L(X)$*
2) *$d(Ka, Mb) \leq \phi(m(a, b))$, ϕ is upper semi-continuous contractive modules and*
 $m(a, b) = \max\{d(Ja, Lb), d(Ja, Ka), d(Lb, Mb), \frac{1}{2}d(Ja, Mb), \frac{1}{2}d(Lb, Ka)\}$
3) *(K, J) and (M, L) are weakly compatible,*

then J, L, K and M have a unique common fixed point.

Proof: Consider an arbitrary point a_0 in X. Define a sequence $\{b_n\} \in X$ such that

$$b_n = Ka_n = La_{n+1}$$

and

$$b_{n+1} = Ma_{n+1} = Ja_{n+2}$$

Now by condition (2), we have

$$d(a_n, b_{n+1}) = d(Ka_n, Ma_{n+1}) \leq \phi(m(a_n, a_{n+1}))$$

where

$$\begin{aligned} m(a_n, a_{n+1}) &= \max\{d(Ja_n, La_{n+1}), \\ &\quad d(JKa_n, KJa_n), d(La_n, Ma_{n+1}), \\ &\quad \frac{1}{2}d(Ja_n, Ma_{n+1}), \frac{1}{2}d(La_{n+1}, Ka_n)\} \\ &= \max\{d(Ma_{n-1}, Ka_n), \\ &\quad d(Ma_{n-1}, Ka_n), d(Ka_n, Ma_{n+1}), \\ &\quad \frac{1}{2}d(Ma_{n-1}, Ma_{n+1}), \frac{1}{2}d(Ka_n, Ka_n)\} \\ &= \max\{d(b_{n-1}, b_n), d(b_n, b_{n+1}), \\ &\quad \frac{1}{2}d(b_{n-1}, b_{n+1}), \frac{1}{2}d(b_n, b_n)\} \\ &= \max\{d(b_{n-1}, b_n), d(b_n, b_{n-1})\} \\ m(a_n, a_{n+1}) &= d(b_n, b_{n+1}) \end{aligned}$$

is impossible because ϕ is contractive modulus, therefore

$$d(b_n, b_{n+1}) \leq (d(b_{n-1}, b_n)) \quad (40)$$

According to equation (40), because ϕ is upper semi-continuous contractive modulus, sequence $\{d(b_{n+1}, b_n)\}$ is continuous and monotonic decreasing. Therefore, $\exists\ t \in \mathbb{R}$, $t \geq 0$ such that

$$\lim_{n\to\infty} d(b_{n+1}, b_n) = t.$$

By limiting in (40), we get, $t \leq \phi(t)$, possible only when $t = 0$, because ϕ is contractive modulus, therefore

$$\lim_{n\to\infty} d(b_{n+1}, b_n) = 0.$$

Next, we show $\{b_n\}$ is a Cauchy sequence.
Suppose $\{b_n\}$ is not a Cauchy sequence. Then, $\exists$ a real number $\varepsilon > 0$, also subsequences q_i and p_i such that $p_i < q_i < p_{i+1}$ and

$$d(b_{p_i}, b_{q_{i-1}}) \geq \varepsilon \text{ and } d(b_{p_i}, b_{q_{i-1}}) < \varepsilon \tag{41}$$

so that,

$$\begin{aligned}\varepsilon \leq& d(b_{p_i}, b_{q_i})\\ \leq& d(b_{p_i}, b_{q_{i-1}}) + d(b_{q_{i-1}}, b_{q_i})\\ <& \varepsilon + d(b_{q_{i-1}}, b_{q_i})\end{aligned}$$

Hence,

$$\lim_{n\to\infty} d(b_{p_i}, b_{q_i}) = \varepsilon$$

Now,

$$d(b_{p_{i-1}}, b_{q_{i-1}}) \leq d(b_{p_{i-1}}, b_{p_i}) + d(b_{p_i}, b_{q_i}) + d(b_{q_i}, b_{q_{i-1}})$$

Taking limit as $n \to \infty$ we have

$$\lim_{i\to\infty} d(b_{p_i}, b_{q_i}) = \varepsilon$$

So by contractive condition (2) and equation (41),

$$\varepsilon \leq d(b_{p_i}, b_{q_i}) = d(Ka_{p_i}, Ma_{q_i}) \leq \phi(m(a_{p_i}, a_{q_i})) \tag{42}$$

where

$$\begin{aligned}m(a_{p_i}, a_{q_i}) =& max\{d(Ja_{p_i}, La_{q_i}), d(Ja_{p_i}, Ka_{p_i}),\\ & d(La_{q_i}, Ma_{q_i}), \frac{1}{2}d(Ja_{p_i}, Ma_{q_i}),\\ & \frac{1}{2}d(La_{q_i}, Ka_{q_i})\}\\ =& \max\{d(Ma_{p_{i-1}}, Ka_{q_{i-1}}),\\ & d(Ma_{p_{i-1}}, Ka_{p_i}),\\ & d(Ka_{q_{i-1}}, Ma_{q_i}), \frac{1}{2}d(Ma_{p_{i-1}}, Ma_{q_i}),\\ & \frac{1}{2}d(Ka_{q_{i-1}}, Ka_{q_i})\}\\ =& \max\{d(b_{p_{i-1}}, b_{q_{i-1}}), d(b_{p_{i-1}}, b_{p_i}),\\ & d(b_{q_{i-1}}, b_{q_i}), \frac{1}{2}d(b_{p_{i-1}}, b_{q_i}),\\ & \frac{1}{2}d(b_{q_{i-1}}, b_{p_i})\}\end{aligned}$$

Now, taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} m(a_{p_i}, a_{q_i}) = max\{\varepsilon, 0, 0, \frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon\} = \varepsilon$$

Hence, from (42), we get $\varepsilon \leq \phi(\varepsilon)$, a contradiction, because ϕ is contractive modulus. Thus, $\{b_n\}$ is a Cauchy sequence. Because X is complete, $\exists$ a point $u \in X$ such that

$$\lim_{n\to\infty} b_n = u.$$

Therefore,

$$\lim_{n\to\infty} Ka_n = \lim_{n\to\infty} La_{n+1} = u$$

and

$$\lim_{n\to\infty} Ma_{n+1} = \lim_{n\to\infty} Ja_{n+2} = u.$$

Hence,

$$\frac{1}{2}d(b_{q_{i-1}}, b_{p_i})\}$$

$$\begin{aligned}\lim_{n\to\infty} Ka_n =& \lim_{n\to\infty} La_{n+1}\\ =& \lim_{n\to\infty} Ma_{n+1}\\ =& \lim_{n\to\infty} Ja_{n+2} = u.\end{aligned}$$

Since, $M(X) \subseteq J(X)$, $\exists$ a point v in X such that $u = Jv$. So, by condition (2)

$$\begin{aligned}d(Kv, u) \leq& d(Kv, Ma_{n+1}) + d(Ma_{n+1}, u)\\ \leq& \phi(m(v, a_{n+1})) + d(Ma_{n+1}, u)\end{aligned}$$

where

$$\begin{aligned}m(v, a_{n+1}) =& \max\{d(Jv, La_{n+1}), d(Jv, Kv),\\ & d(La_n, Ma_{n+1}), \frac{1}{2}d(Jv, Ma_{n+1}),\\ & \frac{1}{2}d(La_{n+1}, Kv)\}\\ =& \max\{d(u, Ka_n), d(u, Kv),\\ & d(Ka_n, Ma_{n+1}), \frac{1}{2}d(u, Ma_{n+1}),\\ & \frac{1}{2}d(Ka_n, Kv)\}.\end{aligned}$$

Taking limit $n \to \infty$, we get

$$m(v, a_{n+1}) = \max\{d(u, Kv), \frac{1}{2}d(u, Kv) = d(u, Kv)\}$$

For $n \to \infty \Rightarrow d(u, Kv) \leq \phi(d(u, Kv))$, a contradiction because ϕ is contractive modulus. Thus, $Kv = u$ and $Jv = Kv = u$ represents v is the coincidence point of J and K. Since the pair (K, J) are weakly compatible, so $KJv = JKv \Rightarrow Ku = Ju$. Also, $K(X) \subseteq L(X)$ then $\exists$ a point w in X such that $u = Lw$. Thus, by condition (2), we get,

$$d(u, Mw) = d(Kv, Mw) \leq (m(v, w))$$

where

$$\begin{aligned}m(v, w) =& \max\{d(Jv, Lw), d(Jv, Kv), d(Lw, Mw),\\ & \frac{1}{2}d(Jv, Mw), \frac{1}{2}d(Lw, Kv)\}\\ =& \max\{d(u, u), d(u, u), d(u, Mw), \frac{1}{2}d(u, Mw),\\ & \frac{1}{2}d(u, u)\}\\ =& \max\{d(u, u), d(u, Mw)\}.\end{aligned}$$

If $d(u, u) = m(v, u)$ then, we have $m(v, w) \leq 2d(u, Mw)$ implies that

$$d(u, Mw) \leq \phi(2d(u, Mw)) < 2d(u, Mw)$$

which is a contradiction because ϕ is a contractive modulus. Also, if $m(v, w) = d(u, Mw)$ then, we have

$$d(u, Mw) \leq \phi(d(u, Mw)) < d(u, Mw)$$

which is a contradiction. Thus,

$$d(u, Mw) = 0$$

implies that $u = Mw$. Thus, $Mw = Lw = u$. Hence, w is the coincidence point of L and M.
As the pair (L, M) is weakly compatible, $LMw = MLw \Rightarrow Lu = Mu$. We shall now prove that u is the fixed point of K. From condition (2), we get

$$d(Ku, u) = d(Ku, Mw) \leq \phi(m(u, w))$$

where

$$\begin{aligned} m(v, w) &= \max\{d(Ju, Lw), d(Ju, Ku), d(Lw, Mw), \\ &\quad \frac{1}{2}d(Ju, Mw), \frac{1}{2}d(Lw, Ku)\} \\ &= \max\{d(Ku, u), d(Ku, Ku), d(u, u), \\ &\quad \frac{1}{2}d(Ku, u), \frac{1}{2}d(u, Ku)\} \\ &= \max\{d(Ku, u), d(Ku, Ku), d(u, u)\} \end{aligned}$$

If $d(Ku, u) = m(u, w)$, we have

$$d(Ku, u) \leq \phi(m(u, u)) = \phi(d(Ku, u)) < d(Ku, u)$$

which is a contradiction because ϕ is contractive modulus. Also, if $d(Ku, Ku) = m(u, w)$ or $d(u, u) = m(u, w)$, which is a contradictions in both the cases. Thus, $d(Ku, u) = 0$ implies that $Ku = u$. Hence, $Ku = Ju = u$.
We now prove u is fixed point of M. From (2), we have

$$d(u, Mu) = d(Ku, Mu) \leq \phi(m(u, u))$$

where

$$\begin{aligned} m(u, u) &= \max\{d(Ju, Lu), d(Ju, Ku), d(Lu, Mu), \\ &\quad \frac{1}{2}d(Ju, Mu), \frac{1}{2}d(Lu, Ku)\} \\ &= \max\{d(u, Mu), d(u, u), d(Mu, Mu), \\ &\quad \frac{1}{2}d(u, Mu), \frac{1}{2}d(Mu, u)\} \\ &= \max\{d(u, Mu), d(u, u), d(Mu, Mu)\} \end{aligned}$$

If $m(u, u) = d(u, Mu)$ then,

$$d(u, Mu) \leq \phi(m(u, u)) = \phi(d(u, Mu)) < d(u, Mu)$$

which is a contradiction.
If $d(u, u) = m(u, u)$ or $d(Mu, Mu) = m(u, w)$, a contradictions in both the cases. Thus, $d(u, Mu)$ = 0 implies that $Mu = u$. Thus, $Mu = Lu = u$.
Therefore, $Ju = Lu = Ku = Mu = u$, that is, J, L, K and M have a common fixed point u.

Uniqueness:

Suppose the mappings J, L, K and M have two common fixed points u and $z(u \neq z)$. Then, from condition (2), we get

$$d(u, z) = d(Ku, Mz) \leq \phi(m(u, z))$$

where

$$\begin{aligned} m(u, z) &= \max\{d(Ju, Lz), d(Ju, Ku), d(Lz, Mz), \\ &\quad \frac{1}{2}d(Ju, Mz), \frac{1}{2}d(Lz, Ku)\} \\ &= \max\{d(u, z), d(u, u), d(z, z), \\ &\quad \frac{1}{2}d(u, z), \frac{1}{2}d(z, u)\} \\ &= \max\{d(u, z), d(u, u), d(z, z)\} \end{aligned}$$

If $d(u, z) = m(u, z)$ implies that

$$d(u, z) \leq \phi(m(du, z)) < d(u, z)$$

which is a contradiction because ϕ is a contractive modulus. Again, if $d(u, u) = m(u, z)$ or $d(z, z) = m(u, z)$, we can see that it is a contradiction in both the cases. Thus, $d(u, z) = 0 \Rightarrow u = z$.
Hence, J, L, K and M have a unique common fixed point u. ■

Example 36. *Consider the set $X = (0, 1]$ equipped with the usual metric defined by*

$$d(a, b) = |a - b| \quad \forall\, a, b \in X.$$

Self-maps J, L, K, M of X are defined as follows:

$$Ja = La = \begin{cases} \frac{1}{2}, & if\ 0 < a \leq \frac{1}{2} \\ \frac{2}{3}, & if\ \frac{1}{2} < a \leq 1. \end{cases}$$

and

$$Ka = Ma = \begin{cases} 1 - a, & if\ 0 < a \leq \frac{1}{2} \\ a, & if\ \frac{1}{2} < a \leq 1. \end{cases}$$

Then, $K(X) = M(X) = [\frac{1}{2}, 1]$ and $J(X) = L(X)$ = $\{\frac{1}{2}, \frac{2}{3}\}$

Clearly, $J(X) \subseteq M(X)$ and $L(X) \subseteq K(X)$. Next, Consider a sequence $\{a_n\}$, where

$$a_n = \frac{1}{2} - \frac{1}{5n}$$

for $n \geq 1$. Then,

$$J(\frac{1}{2}) = K(\frac{1}{2}) = \frac{1}{2}$$

so that

$$JK(\frac{1}{2}) = \frac{1}{2}$$

and

$$KJ(\frac{1}{2}) = \frac{1}{2}.$$

Also,

$$L(\frac{1}{2}) = M(\frac{1}{2}) = \frac{1}{2}$$

so that

$$LM(\frac{1}{2}) = \frac{1}{2}$$

and

$$ML(\frac{1}{2}) = \frac{1}{2}.$$

Further,

$$d(Ja, La) =\mid \frac{1}{2} - \frac{2}{3} \mid = \frac{1}{6}$$

and

$$d(Ka, Ma) =\mid 1 - a - a \mid = \mid 1 - 2a \mid \leq d(Ja, La)$$

$\forall\ 0 < a \leq 1$. *Taking ϕ = 1, the contractive result holds. Hence, J, L, K, M have a unique common fixed point $\frac{1}{2}$.*

Example 37. *Assume $X = [0, \infty)$ with the usual metric $d(a, b) = |a - b|\ \forall\ a, b\ \in X$. We define self-maps J, L, K and M of X by*

$$Ja = \frac{a}{3}, La = \frac{a}{6}, Ka = \frac{a}{24}, Ma = \frac{a}{36}.$$

Clearly, all the conditions of Theorem 35 are satisfied with ϕ *= 1. Hence, 0 is unique common fixed point of* J, L, K *and* M *in* X.

Presented below are the corollaries:

Corollary 38. *Consider a complete dislocated cone metric space* (X, d) *over Banach algebra* $\mathcal{A}$ *where* $\mathcal{B}$ *be the underlying solid cone and* e *is a unit. Suppose that* J, K *and* M *are three self-mappings of* X *fulfilling the conditions below:*

1) $M(X) \subseteq J(X)$ *and* $K(X) \subseteq J(X)$
2) $d(Ka, Mb) \leq \phi(m(a, b))$, ϕ *is upper semi-continuous contractive modulus and*
$$m(a, b) = \max\{d(Ja, Jb), d(Ja, Ka), d(Jb, Mb), \tfrac{1}{2}d(Ja, Mb), \tfrac{1}{2}d(Jb, Ka)\}$$
3) (K, J) *and* (M, J) *are weakly compatible, then* J, K *and* M *have a unique common fixed point.*

Proof: Take $J = L$ in Theorem (35) then we get the desired result. ■

Corollary 39. *Consider a complete dislocated cone metric space* (X, d) *over Banach algebra* $\mathcal{A}$ *where* $\mathcal{B}$ *be the underlying solid cone and* e *is a unit. Let* J *and* K *be two self mappings of* X *such that:*

1) $K(X) \subseteq J(X)$.
2) $d(Ka, Kb) \leq \phi(m(a, b))$, ϕ *is upper semi-continuous contractive modulus and*
$$m(a, b) = \max\{d(Ja, Jb), d(Ja, Ka), d(Jb, Kb), \tfrac{1}{2}d(Ja, Kb), \tfrac{1}{2}d(Jb, Ka)\}$$
3) *The pairs* (K, J) *is weakly compatible, then* J *and* K *have an unique common fixed point.*

Proof: Take $J = L$ and $K = M$ in Theorem (35) then we get the desired result. ■

Corollary 40. *Consider a complete dislocated cone metric space* (X, d) *over Banach algebra* $\mathcal{A}$ *where* $\mathcal{B}$ *be the underlying solid cone and* e *is a unit. Let* K *and* M *be two self mappings in* X *such that:*

1) $d(Ka, Mb) \leq \phi(m(a, b))$, ϕ *is upper semi-continuous contractive modulus and*
$$m(a, b) = \max\{d(a, b), d(a, Kb), d(a, Mb), \tfrac{1}{2}d(a, Mb), \tfrac{1}{2}d(b, Ka)\}$$
2) (K, I) *and* (M, I) *are weakly compatible. Then* K *and* M *have a unique common fixed point.*

Proof: Take $J = L = I$ in Theorem (35) then we get the desired result. ■

Corollary 41. *Consider a complete dislocated cone metric space* (X, d) *over Banach algebra* $\mathcal{A}$ *where* $\mathcal{B}$ *be the underlying solid cone and* e *is a unit. Consider a mapping* $K : X \to X$ *such that*
$d(Ka, Kb) \leq \phi(m(a, b))$, ϕ *is upper semi-continuous contractive modulus and*
$$m(a, b) = \max\{d(a, b), d(a, Kb), d(a, Mb), \tfrac{1}{2}d(a, Kb), \tfrac{1}{2}d(a, Kb)\}.$$
Then, K *has a unique fixed point.*

Proof: Take $M = K$ in Theorem (35) then we get the desired result. ■

V. Conclusion

The goal of this study is to look into common fixed point theorems and fixed point theorems for weakly compatible mappings in dislocated cone metric space over Banach algebra. In this study, we successfully established the unique common fixed point for four self-mappings that satisfy certain contractive conditions over a Banach algebra. We also had successfully dicovered the unique common fixed point for four self-mappings that satisfy certain contractive conditions and α-property in a cone metric space. The inclusion of illustrative examples bolstered the validity of our findings. We anticipate that our findings will aid in ascertaining the presence of solutions to mathematical representations of real-life scenarios.

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