Fixed Point Theorems in Bicomplex Partial $\mathcal{S}$ Metric Spaces and Applications to Boundary Value Problems

M Madhuri and M V R Kameswari

Abstract—**In this article, we extend the concept of bicomplex partial metric space to bicomplex partial $\mathcal{S}$ metric spaces and apply this extension to determine fixed points for generalized contractions. Our findings extend and enhance existing results in this field. To illustrate the significance and applicability of our main result, we present several examples. Furthermore, our results are employed to explore the existence of solutions for two-point boundary value problems.**

Index Terms—**Bicomplex partial $\mathcal{S}$ metric space, Boundary value problems, Fixed points, Generalized contractions.**

I. Introduction and Preliminaries

In the advancement of special algebra, Serge[11] created a commutative generalization of complex numbers, bicomplex numbers, tricomplex numbers, and so on, as members of an infinite set of algebra. Various researchers, such as Choi[2], Jebril[5], Beg[1], and Datta[3], have made significant contributions by formulating fixed point theorems in bicomplex-valued metric spaces. Recently, Gu et al.[4] introduced the concept of bicomplex partial metric spaces and established several fixed point theorems in this new context.

In this paper, $\mathcal{R}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ represents the set of real numbers, complex numbers and bicomplex numbers respectively.

In [9,6] the set of bicomplex numbers defined as follows:

$\mathcal{C}_2 = \{\chi : \chi = \eta_0 + \eta_1\ i_1 + \eta_2 i_2 + \eta_3 i_1 i_2$, where $\eta_0, \eta_1, \eta_2, \eta_3 \in \mathcal{R}\}$,

that is $\mathcal{C}_2 = \{\chi : \chi = \zeta_1 + i_2\zeta_2,\ \zeta_1, \zeta_2 \in \mathcal{C}_1\}$,

where $\zeta_1 = \eta_0 + i_1\ \eta_1$ and $\zeta_2 = \eta_2 + i_1\eta_3$ and i_1, i_2 are an imaginary independent units such that $i_1^2 = -1 = i_2^2$, $i_1 i_2 = i_2 i_1$.

The norm of a bicomplex number $\parallel \chi \parallel$ is defined by

$\parallel \chi \parallel = \parallel \zeta_1 + i_2\zeta_2 \parallel = (\parallel \zeta_1 \parallel^2 + \parallel \zeta_2 \parallel |^2)^{\frac{1}{2}}$

$= (\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2)^{\frac{1}{2}}$.

A bicomplex number $\chi = \eta_0 + \eta_1\ i_1 + \eta_2 i_2 + \eta_3 i_1 i_2$ is degenerated [9] if the matrix $\begin{pmatrix} \eta_0 & \eta_1 \\ \eta_2 & \eta_3 \end{pmatrix}$ is degenerated.

For any two complex numbers $\rho, \tau \in \mathcal{C}_2$, we have

(i) $0 \prec_{i_2} \rho \prec_{i_2} \tau \Rightarrow \parallel \rho \parallel \leq \parallel \tau \parallel$

(ii) $\parallel \rho + \tau \parallel \leq \parallel \rho \parallel + \parallel \tau \parallel$

(iii) $\parallel \alpha\rho \parallel \leq \alpha \parallel \rho \parallel$, if $\alpha \in \mathcal{R}$

(iv) $\parallel \frac{\rho}{\tau} \parallel = \frac{\|\rho\|}{\|\tau\|}$, if τ is a degenerated bicomplex number.

Also, for $\rho, \tau \in \mathcal{C}_2$, we have

(i) $\parallel \rho\tau \parallel \leq \sqrt{2} \parallel \rho \parallel\parallel \tau \parallel$.

(ii) $\parallel \rho\tau \parallel = \parallel \rho \parallel\parallel \tau \parallel$ whenever at least one of ρ and τ.

(iii) $\parallel \rho^{-1} \parallel = \parallel \rho \parallel^{-1}$ holds for any degenerated bicomplex number.

In [2], the partial order relation on $\preceq_{i_2}$ is defined as follows:

Let $\rho = \rho_1 + i_2\rho_2 \in \mathcal{C}_2$ and $\tau = \tau_1 + i_2\tau_2 \in \mathcal{C}_2$, we define a partial order relation on $\mathcal{C}_2$ as $\rho \preceq_{i_2} \tau$ if and only if $\rho_1 \preceq_{i_1} \tau_1$ and $\rho_2 \preceq_{i_1} \tau_2$, where $\preceq_{i_1}$ is a partial order relation in $\mathcal{C}_1$. Then

(1) $Re(\rho_1) = Re(\tau_1)$ and $\Im m(\rho_1) = \Im m(\tau_1)$
$Re(\rho_2) = Re(\tau_2)$ and $\Im m(\rho_2) = \Im m(\tau_2)$

(2) $Re(\rho_1) < Re(\tau_1)$ and $\Im m(\rho_1) < \Im m(\tau_1)$
$Re(\rho_2) = Re(\tau_2)$ and $\Im m(\rho_2) = \Im m(\tau_2)$

(3) $Re(\rho_1) = Re(\tau_1)$ and $\Im m(\rho_1) = \Im m(\tau_1)$
$Re(\rho_2) < Re(\tau_2)$ and $\Im m(\rho_2) < \Im m(\tau_2)$

(4) $Re(\rho_1) < Re(\tau_1)$ and $\Im m(\rho_1) < \Im m(\tau_1)$
$Re(\rho_2) < Re(\tau_2)$ and $\Im m(\rho_2) < \Im m(\tau_2)$.

We write $\rho \prec_{i_2} \tau$ and $\rho \neq \tau$ if any one of (1), (2) and (3) is satisfied and $\rho \prec_{i_2} \tau$ if condition (4) is satisfied.

Throughout this paper, we denote:

(a) SMS as $\mathcal{S}$ - metric space;

(b) PSMS as partial $\mathcal{S}_p$ - metric space;

(c) BCSMS as bicomplex $\mathcal{S}_{bs}$ metric space;

(d) BCPSMS as bicomplex partial $\mathcal{S}_{bps}$ metric space;

(e) POSET as partial order set.

Definition I.1: ([10,7]) Let $\mathcal{D}$ be a non-empty set. If a function $\mathcal{S}$: $\mathcal{D}^3 \to [1, \infty)$ satisfies the following:

1) $\mathcal{S}(\varrho, \sigma, \varsigma) \geq 0$;

2) $\mathcal{S}(\varrho, \sigma, \varsigma) = 0$ iff $\varrho = \sigma = \varsigma$;

Manuscript received June 03, 2024; revised November 10, 2024.

M. Madhuri is a Research scholar in the Department of Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam-530045, Andhra Pradesh, India. (email: mmudunur@gitam.in)

M. V. R. Kameswari is an Associate Professor in the Department of Mathematics, School of Science, GITAM Deemed to be University, Rushikonda, Visakhapatnam-530045, Andhra Pradesh, India. (Corresponding Author; e-mail: kmukkavi@gitam.edu)

3) $\mathcal{S}(\varrho,\sigma,\varsigma) \le \mathcal{S}(\varrho,\varrho,\kappa)+\mathcal{S}(\sigma,\sigma,\kappa)+\mathcal{S}(\varsigma,\varsigma,\kappa)$,

for each ϱ,σ,ς, and $\kappa \in \mathcal{D}$. Then the pair $(\mathcal{D},\mathcal{S})$ is referred as SMS.

In several studies, comparisons between metrics and SMS have been explored [13, 14, 15]. As noted in [15], an S-metric can arise from the standard metric d. However, there also exists an S-metric that does not arise from any metric [15].

Definition I.2: ([12]) Let $\mathcal{D}$ be a non-empty set. If a function $\mathcal{S}_p$: $\mathcal{D}^3 \to [1,\infty)$ satisfies the following:

1) $\mathcal{S}_p(\varrho,\varrho,\varrho) = \mathcal{S}_p(\sigma,\sigma,\sigma) = \mathcal{S}_p(\varrho,\varrho,\sigma) = \mathcal{S}_p(\varrho,\sigma,\varsigma)$ if and only if $\varrho=\sigma=\varsigma$;
2) $\mathcal{S}_p(\varrho,\varrho,\varrho) \le \mathcal{S}_p(\varrho,\varrho,\varsigma)$;
3) $\mathcal{S}_p(\varrho,\sigma,\varsigma) \le \mathcal{S}_p(\varrho,\varrho,\varrho)+\mathcal{S}_p(\sigma,\sigma,\varrho)+\mathcal{S}_p(\varsigma,\varsigma,\varrho)$
$$-2\mathcal{S}_p(\varrho,\varrho,\varrho),$$

for each ϱ,σ,ς, and $\varrho \in \mathcal{D}$. Then the pair $(\mathcal{D},\mathcal{S}_p)$ is referred as a PSMS. Clearly, each SMS is a PSMS with zero self distance, but the converse of this fact need not be true [12].

Definition I.3: ([16]) Let $\mathcal{D}$ be a non-empty set. If a function

$\mathcal{S}_{bs}$: $\mathcal{D}^3 \to \mathcal{C}_2^+$ satisfies the following:

1) $\mathcal{S}_{bs}(\varrho,\sigma,\varsigma) \succeq_{i_2} 0$;
2) $\mathcal{S}_{bs}(\varrho,\sigma,\varsigma)=0$ iff $\varrho=\sigma=\varsigma$;
3) $\mathcal{S}_{bs}(\varrho,\sigma,\varsigma)\preceq_{i_2} \mathcal{S}_{bs}(\varrho,\varrho,\kappa)+\mathcal{S}_{bs}(\sigma,\sigma,\kappa)+\mathcal{S}_{bs}(\varsigma,\varsigma,\kappa)$,

for each ϱ,σ,ς, and $\kappa \in \mathcal{D}$. Then the pair $(\mathcal{D},\mathcal{S}_{bs})$ is referred as BCSMS.

Definition I.4: Let $\mathcal{D}$ be a non-empty set. If a function $\mathcal{S}_{bps}$: $\mathcal{D}^3 \to \mathcal{C}_2^+$ satisfies the following:

1) $\varrho=\sigma=\varsigma$ if and only if $\mathcal{S}_{bps}(\varrho,\sigma,\varsigma) = \mathcal{S}_{bps}(\varrho,\varrho,\varrho)=\mathcal{S}_{bps}(\sigma,\sigma,\sigma)=\mathcal{S}_{bps}(\varsigma,\varsigma,\varsigma)$;
2) $\mathcal{S}_{bps}(\varrho,\varrho,\varrho) \preceq_{i_2} \mathcal{S}_{bps}(\varrho,\varrho,\varsigma)$;
3) $\mathcal{S}_{bps}(\varrho,\sigma,\varsigma) \preceq_{i_2} \mathcal{S}_{bps}(\varrho,\varrho,\varrho)+\mathcal{S}_{bps}(\sigma,\sigma,\varrho)$
$$+\mathcal{S}_{bps}(\varsigma,\varsigma,\varrho)-2.\mathcal{S}_{bps}(\varrho,\varrho,\varrho),$$

for each ϱ,σ,ς, and $\varrho \in \mathcal{D}$. Then the pair $(\mathcal{D},\mathcal{S}_{bps})$ is referred as BCPSMS.

A BCSMS is obviously a BCPSMS with self distance. A BCPSMS does not have to be a BCSMS.

Example I.5: Let $\mathcal{D}$ = $\mathcal{R}^+$ and we define $\mathcal{S}_{bps}$: $\mathcal{D}^3 \to \mathcal{C}_2^+$ by $\mathcal{S}_{bps}(\varrho,\sigma,\varsigma) = 2+|\varrho-\sigma|+|\sigma-\varsigma|+|\varsigma-\varrho|$ $+i_2(2+|\varrho-\sigma|+|\sigma-\varsigma|+|\varsigma-\varrho|)$, for each $\varrho,\sigma,\varsigma \in \mathcal{D}$.

Therefore $\mathcal{S}_{bps}$ is a BCPSMS, but it is does not have to be a BCSMS, since $\mathcal{S}_{bs}(\varrho,\varrho,\varrho)=2(1+i_2)\neq 0$.

Example I.6: Let $\mathcal{D}$ = $\mathcal{R}^+$ and we define $\mathcal{S}_{bps}$: $\mathcal{D}^3 \to \mathcal{C}_2^+$ by $\mathcal{S}_{bps}(\varrho,\sigma,\varsigma) = \frac{1}{2}(max\{\varrho,\sigma,\varsigma\}+|\varrho-\sigma|+|\sigma-\varsigma|+|\varsigma-\varrho|)+\frac{i_2}{2}(max\{\varrho,\sigma,\varsigma\}+|\varrho-\sigma|+|\sigma-\varsigma|+|\varsigma-\varrho|)$, for each $\varrho,\sigma,\varsigma \in \mathcal{D}$.

Therefore $\mathcal{S}_{bps}$ is a BCPSMS, but it is does not have to be a BCSMS, since $\mathcal{S}_{bs}(\varrho,\varrho,\varrho)=(\frac{1+i_2}{2})\varrho, \varrho\neq 0$.

Definition I.7: In a BCPSMS $(\mathcal{D},\mathcal{S}_{bps})$, a sequence $\{\varrho_n\}$ converges to $\varrho \in \mathcal{D}$ if and only if $\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$ $= \lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho) = \lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n)$.

i.e. for each $0 \preceq_{i_2} \varepsilon \in \mathcal{C}_2^+$ then there exist $n_0 \in \mathcal{N}$ such that

$||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)|| < \varepsilon$, for all $n \ge n_0$.

Definition I.8: Let $(\mathcal{D},\mathcal{S}_{bps})$ be a BCPSMS. If $\mathcal{K} \succeq_{i_2} 0$ then the ball $B_{s_b}(\varrho,\mathcal{K})$ with centre $\varrho \in \mathcal{D}$ and radius $\mathcal{K}$ is known as an open ball, where

$B_{s_b}(\varrho,\mathcal{K})$=$\{\sigma \in X : \mathcal{S}_{bps}(\varrho,\varrho,\sigma) \preceq_{i_2} \mathcal{S}_{bps}(\varrho,\varrho,\varrho)+\mathcal{K}\}$, for $0 \preceq_{i_2} \mathcal{K} \in \mathcal{C}_2^+$.

Lemma I.9: Let $(\mathcal{D},\mathcal{S}_{bps})$ be a BCPSMS. A sequence $\{\varrho_n\} \in \mathcal{D}$ converges to $\varrho \in \mathcal{D}$ if and only if $\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$ $= \lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho) = \lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n)$.

Proof. Let $\{\varrho_n\}$ converges to ϱ.

Given any $\varepsilon>0$, let $\mathcal{K}=\frac{\epsilon}{2}+i_1\frac{\epsilon}{2}+i_2\frac{\epsilon}{2}+i_1i_2\frac{\epsilon}{2}$, then $0 \preceq_{i_2} \mathcal{K} \in \mathcal{C}_2^+$.

For every $\mathcal{K}$, then there exists $n_0 \in \mathcal{N}$ such that $\varrho_n \in B_{\mathcal{S}_{bps}}(\varrho,\mathcal{K})$, for all $n \ge n_0$.

i.e., $\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho) \preceq_{i_2} \mathcal{K}+\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$

$\Rightarrow ||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)|| < \epsilon$

$\Rightarrow ||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)|| < \epsilon$,

for all $n \ge n_0$.

Thus $\lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho)$= $\lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n)$
$$= \mathcal{S}_{bps}(\varrho,\varrho,\varrho).$$

Conversly, suppose that $\lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho)$ $= \lim_{n\to\infty} \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n) = \mathcal{S}_{bps}(\varrho,\varrho,\varrho)$.

i.e., for $0 \preceq_{i_2} \mathcal{K} \in \mathcal{C}_2^+$, then there exists a real number $\varepsilon>0$ such that for all $\zeta \in \mathcal{C}_2^+$, $||\zeta||<\epsilon$ implies $\zeta \preceq_{i_2} \mathcal{K}$.

For $\varepsilon>0$, then there exists $n_0 \in \mathcal{N}$ such that $||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)|| <\varepsilon$ and $||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)|| < \varepsilon$, for all $n \ge n_0$.

$\Rightarrow \mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho) \preceq_{i_2} \mathcal{K}+\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$, $n \ge n_0$.

Hence $\{\varrho_n\}$ is converges to a point ϱ.

Lemma I.10: For BCPSMS, we have

(i) $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)=\mathcal{S}_{bps}(\sigma,\sigma,\varrho)$.

(ii) $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)=0$ then $\varrho=\sigma$.

Proof. (i) (a) $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)$

$\preceq_{i_2} \mathcal{S}_{bps}(\varrho,\varrho,\varrho)+\mathcal{S}_{bps}(\varrho,\varrho,\varrho)+\mathcal{S}_{bps}(\sigma,\sigma,\varrho)$
$\quad -2\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$
$= \mathcal{S}_{bps}(\sigma,\sigma,\varrho)$.

(b) $\mathcal{S}_{bps}(\sigma,\sigma,\varrho)$
$\preceq_{i_2} \mathcal{S}_{bps}(\sigma,\sigma,\sigma)+\mathcal{S}_{bps}(\sigma,\sigma,\sigma)+\mathcal{S}_{bps}(\varrho,\varrho,\sigma)$
$\quad -2\mathcal{S}_{bps}(\sigma,\sigma,\sigma)$
$= \mathcal{S}_{bps}(\varrho,\varrho,\sigma)$.

From (a) and (b), we have $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)=\mathcal{S}_{bps}(\sigma,\sigma,\varrho)$.

(ii) By the condition (2) of Definition I.2, we have

$$\mathcal{S}_{bps}(\varrho,\varrho,\varrho) \preceq_{i_2} \mathcal{S}_{bps}(\varrho,\varrho,\sigma)=0. \tag{1}$$

$$\mathcal{S}_{bps}(\sigma,\sigma,\sigma) \preceq_{i_2} \mathcal{S}_{bps}(\sigma,\sigma,\varrho)=0. \tag{2}$$

From (1) and (2), we get $\varrho=\sigma$.

Definition I.11: In a BCPSMS $(\mathcal{D},\mathcal{S}_{bps})$, a sequence $\{\varrho_n\}\subseteq\mathcal{D}$ is referred as a Cauchy's sequence in $(\mathcal{D},\mathcal{S}_{bps})$ if for each $\varepsilon>0$, then there exists $\wp\in\mathcal{C}_2^+$ and $n_0\in\mathcal{N}$ such that $||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)-\wp||<\epsilon$, for all $n,m\in\mathcal{N}$ and $n,m\geq n_0$.

Definition I.12: A BCPSMS $(\mathcal{D},\mathcal{S}_{bps})$ is complete if for every Cauchy's sequence in $\mathcal{D}$ is converges in $\mathcal{D}$.

Lemma I.13: Let $(\mathcal{D},\mathcal{S}_{bps})$ be a BCPSMS and $\{\varrho_n\}$ be a sequence in $\mathcal{D}$. Then $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{D}$ if and only if $\lim\limits_{n\to\infty}\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)=\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$.

Proof. Let $\{\varrho_n\}$ be a sequence in $\mathcal{D}$. Let $\varepsilon>0$ then there exists a real number $\mathcal{K}=\frac{\epsilon}{2}+i_1\frac{\epsilon}{2}+i_2\frac{\epsilon}{2}+i_1i_2\frac{\epsilon}{2}$ then $0\preceq_{i_2}\mathcal{K}\in\mathcal{C}_2^+$ and for this radius $\mathcal{K}$ there exists $n_0\in\mathcal{N}$ such that $\varrho_n\in B_{\mathcal{S}_{bps}}(\varrho_m,\mathcal{K})$ for all $m,n\geq n_0$.

i.e., $\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)\preceq_{i_2}\mathcal{K}+\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$

$\Rightarrow ||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)||<\epsilon$, for all $m,n\geq n_0$.

Therefore $\lim\limits_{n\to\infty}\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)=\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$.

Conversely, suppose that $\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)\to\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$ as $m,n\to\infty$.

For each $0\preceq_{i_2}\mathcal{K}\in\mathcal{C}_2^+$, then there exists $\varepsilon>o$ such that for all $\zeta\in\mathcal{C}_2^+$, $||\zeta||<\varepsilon\Rightarrow\zeta\preceq_{i_2}\mathcal{K}$.

For $\varepsilon>o$, then there exists $n_0\in\mathcal{N}$ such that $||\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)-\mathcal{S}_{bps}(\varrho,\varrho,\varrho)||<\epsilon$, for all $m,n\geq n_0$.

Therefore $\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_m)\preceq_{i_2}\mathcal{K}+\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$, for all $m,n\geq n_0$.

Lemma I.14: Let $(\mathcal{D},\mathcal{S}_{bps})$ be a BCPSMS. A sequence $\{\varrho_n\}$ in $\mathcal{D}$ converges to ϱ then ϱ is unique.

Proof. Let a sequence $\{\varrho_n\}$ in $\mathcal{D}$ converges to ϱ and σ. Based on the condition (i) of Lemma 1.10, we have

$\mathcal{S}_{bps}(\varrho,\varrho,\sigma)$
$\preceq_{i_2} 2\mathcal{S}_{bps}(\varrho,\varrho,\varrho_n)+\mathcal{S}_{bps}(\sigma,\sigma,\varrho_n)-2\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n)$
$\preceq_{i_2} 2(\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho)-\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_n))+$
$\quad \mathcal{S}_{bps}(\varrho_n,\varrho_n,\sigma)-\mathcal{S}_{bps}(\sigma,\sigma,\sigma)+\mathcal{S}_{bps}(\sigma,\sigma,\sigma)$.

Taking limit $n\to\infty$, we have

$\quad \mathcal{S}_{bps}(\varrho,\varrho,\sigma)\preceq_{i_2}\mathcal{S}_{bps}(\sigma,\sigma,\sigma)$.

Hence, $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)=\mathcal{S}_{bps}(\sigma,\sigma,\sigma)$.

Similarly, we can show that $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)=\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$.

Therefore $\mathcal{S}_{bps}(\varrho,\varrho,\sigma)$ = $\mathcal{S}_{bps}(\sigma,\sigma,\sigma)$= $\mathcal{S}_{bps}(\varrho,\varrho,\varrho)$.

Hence $\varrho=\sigma$.

Theorem I.15: ([8]) Let a POSET $(\mathcal{D},\preceq)$ and assume that $(\mathcal{D},\mathcal{S}^*)$ is a complete PSMS with a partial $\mathcal{S}$-metric $\mathcal{S}^*$ on $\mathcal{D}$. Suppose $\Theta:\mathcal{D}\to\mathcal{D}$ is a nondecreasing and continuous mapping such that

$$\mathcal{S}^*(\Theta\sigma,\Theta\varsigma,\Theta\psi)\leq\kappa.max\{\mathcal{S}^*(\sigma,\varsigma,\psi),\mathcal{S}^*(\sigma,\sigma,\Theta\sigma), \mathcal{S}^*(\varsigma,\varsigma,\Theta\varsigma),\mathcal{S}^*(\psi,\psi,\Theta\psi), \tfrac{1}{2}[\mathcal{S}^*(\sigma,\sigma,\Theta\varsigma)+\mathcal{S}^*(\sigma,\sigma,\Theta\psi)]\},$$

for all $\sigma,\varsigma,\psi\in\mathcal{D}$ with $\psi\preceq\varsigma\preceq\sigma$ where $0<\kappa<1$.

If there exists an $\sigma_0\in\mathcal{D}$ with $\sigma_0\preceq\Theta\sigma_0$, then there exists $\sigma\in\mathcal{D}$ such that $\sigma=\Theta\sigma$. Moreover, $\mathcal{S}^*(\sigma,\sigma,\sigma)=0$.

Theorem I.16: ([4]) Let a complete BPMS $(\mathcal{D},\varrho_{bcb})$ and two continuous mappings $\Psi,\Omega:\mathcal{D}\to\mathcal{D}$ such that

$$\varrho_{bcb}(\Psi\phi,\Omega\varrho)\preceq_{i_2}\varrho.max\{\varrho_{bcb}(\phi,\varrho),\varrho_{bcb}(\phi,\Psi\phi), \varrho_{bcb}(\varrho,\Omega\varrho),\tfrac{1}{2}[\varrho_{bcb}(\phi,\Omega\varrho)+\varrho_{bcb}(\varrho,\Psi\phi)]\},$$

for all $\phi,\varrho\in\mathcal{D}$, where $0\leq\varrho<1$. Then, (Ψ,Ω) has a unique common fixed point and $\varrho_{bcb}(\phi^*,\phi^*)=0$.

By the motivation of the Theorem I.15 and Theorem I.16, in this paper we extend the notion of bicomplex partial metric space [4] to BCPSMS and obtain fixed points for certain contractions. To verify the importance and effectiveness of our main result, examples are given. As a consequence of our result, we study the existence solutions of a two point boundary value problem.

II. Main Results

Theorem II.1: *Let $(\mathcal{D},\mathcal{S}_{bps})$ be a complete BCPSMS and a function $\mathcal{E}:\mathcal{D}\to\mathcal{D}$ be a continuous mapping such that*

$$\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma) \preceq_{i_2}\alpha\ max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho), \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho), \frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}, \tag{3}$$

for all $\varrho,\sigma,\varsigma\in\mathcal{D}$, where $0\leq\alpha<1$. Then $\mathcal{E}$ has a unique fixed point in $\mathcal{D}$.

Proof: Let $\varrho_0\in\mathcal{D}$, define a sequence $\{\varrho_n\}$ as:

$$\varrho_{n+1}=\mathcal{E}\varrho_n,\forall n\in\mathcal{N}. \tag{4}$$

If $\varrho_n=\varrho_{n+1}$, then $\{\varrho_n\}$ is a fixed point of $\mathcal{E}$ in $\mathcal{D}$.

$\varrho_n\neq\varrho_{n+1}$, for all $n\in\mathcal{N}$.

Consider $\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})$

$$= \mathcal{S}_{bps}(\mathcal{E}\varrho_{n-1}, \mathcal{E}\varrho_{n-1}, \mathcal{E}\varrho_n)$$

$$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varrho_{n-1}), \mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \varrho_n), \tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \varrho_{n-1}), \frac{\mathcal{S}_{bps}(\varrho_n, \varrho_n, \mathcal{E}\varrho_n)(1+\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \mathcal{E}\varrho_{n-1}))}{1+\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)}\}$$

$$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_n), \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_n), \tfrac{1}{3}\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n-1})\}, \frac{\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})(1+\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n))}{1+\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)}\}$$

$$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_n), \tfrac{1}{3}\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n-1}), \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})\}$$

$$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) - 2\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+1}), \tfrac{1}{3}\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_n) + \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_n) + \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n) - 2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_n)), \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})\}$$

$$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}), \frac{1}{3}(2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n))\}. \quad (5)$$

Case(i). If $max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}), \frac{1}{3}(2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})+ \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n))\} = \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})$,

then from (5), we have

$$\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}),$$

which is contradiction, since $\alpha < 1$.

Case(ii). If $max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}), \frac{1}{3}(2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n))\} = \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)$,

then from (5), we have

$$\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n). \quad (6)$$

Case(iii). If $max\{\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n), \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}), \frac{1}{3}(2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})+ \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n))\} = \frac{1}{3}(2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})+ \mathcal{S}_{bps}(\varrho_{n-1}+\varrho_{n-1}, \varrho_n))$,

then from (5) we have

$$\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \alpha.\tfrac{1}{3}(2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1})+ \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n))$$

$$\Rightarrow 3\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \alpha.2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)$$

$$\Rightarrow \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \frac{\alpha}{(3-2\alpha)}\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n). \quad (7)$$

Since $\alpha < 1$, we have $\frac{\alpha}{(3-2\alpha)} < 1$.

Let $\lambda = Max\{\alpha, \frac{\alpha}{(3-2\alpha)}\}$.

Therefore from (6) and (7), we have

$$\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \lambda.\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n). \quad (8)$$

Similarly,

$$\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+2}) \preceq_{i_2} \lambda.\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}). \quad (9)$$

Then from (8) and (9), we can conclude that

$\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \preceq_{i_2} \mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)$,

for all $n \in \mathcal{N}$.

Hence, for all $n = 0, 1, 2, ...$, we get

$$\begin{aligned}\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+2}) &\preceq_{i_2} \lambda.\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) \\ &\preceq_{i_2} \lambda.(\lambda.\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)) \\ &\preceq_{i_2} \lambda^2\mathcal{S}_{bps}(\varrho_{n-1}, \varrho_{n-1}, \varrho_n)) \\ &\;\;\vdots \\ &\preceq_{i_2} \lambda^{n+1}\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1)). \quad (10)\end{aligned}$$

For $m, n \in \mathcal{N}, m > n$, we have

$$\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_m)$$

$$\preceq_{i_2}\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_m, \varrho_m, \varrho_{n+1}) - 2\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+1})$$

$$\preceq_{i_2} 2\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_m)$$

$$\preceq_{i_2} 2\lambda^n\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1) + \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+2}) + \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+2}) + \mathcal{S}_{bps}(\varrho_m, \varrho_m, \varrho_{n+2}) - 2\mathcal{S}_{bps}(\varrho_{n+2}, \varrho_{n+2}, \varrho_{n+2})$$

$$\preceq_{i_2} 2\lambda^n\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1) + 2\lambda^{n+1}\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1) + \mathcal{S}_{bps}(\varrho_{n+2}, \varrho_{n+2}, \varrho_m)$$

$$\vdots$$

$$\preceq_{i_2} 2[\lambda^n + \lambda^{n+1} + ... + \lambda^{m-1}]\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1)$$

$$\preceq_{i_2} 2.\tfrac{\lambda^n}{1-\lambda}\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1).$$

Hence $||\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_m)|| \leq 2.\frac{\lambda^n}{1-\lambda}||\mathcal{S}_{bps}(\varrho_0, \varrho_0, \varrho_1)|| \to 0$ as $n \to \infty$.

Therefore $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{D}$.

Since $(\mathcal{D}, \mathcal{S}_{bps})$ is complete, then there exists $\varsigma \in \mathcal{D}$ such that $\varrho_n \to \varsigma$ as $n \to \infty$. And

$$\mathcal{S}_{bps}(\varsigma, \varsigma, \varsigma) = \lim_{n\to\infty}\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varsigma) = \lim_{n\to\infty}\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varrho_n) = 0.$$

In the view of continuous of $\mathcal{E}$, it follows that

$$\varrho_{n+1} = \mathcal{E}\varrho_n \to \mathcal{E}\varsigma \text{ as } n \to \infty$$

i.e., $\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varsigma) = \lim_{n\to\infty}\mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varsigma) = \lim_{n\to\infty}\mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varrho_n) = 0.$

Let us consider

$||\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma)||$

$\leq ||\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varrho_n) + \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varrho_n)$
$+\mathcal{S}_{bps}(\varsigma, \varsigma, \mathcal{E}\varrho_n) - 2\mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varrho_n)||$
$\leq 2||\mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varsigma)|| + ||\mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1})||$
$\leq 2||\mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varsigma)|| + ||\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varsigma)||$,

as $n \to \infty$, we obtain $||\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma)|| \leq 0$

Hence $\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma) = 0$

$\Rightarrow \mathcal{E}\varsigma = \varsigma$.

Therefore ς follows as a fixed point of $\mathcal{E}$ in $\mathcal{D}$.

Uniqueness: Let μ, ν be two fixed points of $\mathcal{E}$ in $\mathcal{D}$, then $\mathcal{E}\mu = \mu$ and $\mathcal{E}\nu = \nu$. Consider

$\mathcal{S}_{bps}(\mathcal{E}\mu, \mathcal{E}\mu, \mathcal{E}\nu)$
$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\mu, \mu, \nu), \mathcal{S}_{bps}(\mathcal{E}\nu, \mathcal{E}\nu, \mathcal{E}\mu),$
$\mathcal{S}_{bps}(\mathcal{E}\nu, \mathcal{E}\nu, \nu), \frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\nu, \mathcal{E}\nu, \mu),$
$\frac{\mathcal{S}_{bps}(\nu,\nu,\mathcal{E}\nu)(1+\mathcal{S}_{bps}(\mu,\mu,\mathcal{E}\mu)}{(1+\mathcal{S}_{bps}(\mu,\mu,\nu))}\}$
$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\mu, \mu, \nu), \mathcal{S}_{bps}(\nu, \nu, \mu),$
$\frac{1}{3}\mathcal{S}_{bps}(\nu, \nu, \mu)\}$
$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\mu, \mu, \nu), \mathcal{S}_{bps}(\nu, \nu, \mu)\}$
$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\mu, \mu, \nu), \mathcal{S}_{bps}(\mu, \mu, \nu)\}$
$\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\mu, \mu, \nu)$,

which is contradiction. Hence $\mu = \nu$.

Therefore $\mathcal{E}$ has a unique fixed point in $\mathcal{D}$.

In the lack of the continuity criterion for mapping $\mathcal{E}$, we have the following theorem.

Theorem II.2: *Let $(\mathcal{D}, \mathcal{S}_{bps})$ be a complete BCPSMS and function $\mathcal{E}$: $\mathcal{D} \to \mathcal{D}$ a mapping such that*

$\mathcal{S}_{bps}(\mathcal{E}\varrho, \mathcal{E}\sigma, \mathcal{E}\varsigma)$
$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho, \sigma, \varsigma), \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varrho),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma), \frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho),$
$\frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}$,

for all $\varrho, \sigma, \varsigma \in \mathcal{D}$, where $0 \leq \alpha < 1$. Then $\mathcal{E}$ has a unique fixed point in $\mathcal{D}$.

Proof. Following from the Theorem II.1, $\{\varrho_n\}$ is a Cauchy's sequence in $\mathcal{D}$.

Since $\mathcal{D}$ is complete, there exists $\varsigma \in \mathcal{D}$ such that $\varrho_n \to \varsigma$ as $n \to \infty$.

Since $\mathcal{E}$ is not continuous, we have $\mathcal{S}_{bps}(\varsigma, \varsigma, \mathcal{E}\varsigma) = \mathcal{L} > 0$

$\mathcal{L} = \mathcal{S}_{bps}(\varsigma, \varsigma, \mathcal{E}\varsigma)$
$\preceq_{i_2} \mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1}) + \mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1})+$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho_{n+1}) - 2\mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \varrho_{n+1})$
$\preceq_{i_2} 2\mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1}) + \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho_{n+1})$
$\preceq_{i_2} 2\mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1}) + \mathcal{S}_{bps}(\varrho_{n+1}, \varrho_{n+1}, \mathcal{E}\varsigma)$
$\preceq_{i_2} 2\mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1}) + \mathcal{S}_{bps}(\mathcal{E}\varrho_n, \mathcal{E}\varrho_n, \mathcal{E}\varsigma)$
$\preceq_{i_2} 2\mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1}) + \alpha.max\{\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varsigma),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varrho_n), \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma),$
$\frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho_n), \frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\varrho_n,\varrho_n,\mathcal{E}\varrho_n))}{1+s_{bps}(\varrho_n,\varrho_n,\varsigma)}\}$
$\preceq_{i_2} 2\mathcal{S}_{bps}(\varsigma, \varsigma, \varrho_{n+1}) + \alpha.max\{\mathcal{S}_{bps}(\varrho_n, \varrho_n, \varsigma),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho_{n+1}), \mathcal{S}_{bps}(\varsigma, \varsigma, \mathcal{E}\varsigma),$
$\frac{1}{3}\mathcal{S}_{bps}(\varrho_n, \varrho_n, \mathcal{E}\varsigma), \frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\varrho_n,\varrho_n,\varrho_{n+1}))}{1+s_{bps}(\varrho_n,\varrho_n,\varsigma)}\}$.

As $n \to \infty$, we have $\mathcal{L} \preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varsigma, \varsigma, f\varsigma)$.

Therefore

$$||\mathcal{L}|| \leq \alpha.||\mathcal{L}||,$$

which is contradiction, since $\alpha < 1$.

Then $\mathcal{S}_{bps}(\varsigma, \varsigma, \mathcal{E}\varsigma) = 0$

$$\Rightarrow \mathcal{E}\varsigma = \varsigma$$

Hence ς is the fixed point of $\mathcal{E}$ in $\mathcal{D}$.

The uniqueness of the fixed point follows from above Theorem 2.1.

Corollary II.3: Theorem II.1 continues to be true if (3) is replaced by

$$\mathcal{S}_{bps}(\mathcal{E}\varrho, \mathcal{E}\sigma, \mathcal{E}\varsigma) \preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho, \sigma, \varsigma),$$

for all $\varrho, \sigma, \varsigma \in \mathcal{D}$ and $0 \leq \alpha < 1$. Then $\mathcal{E}$ has a unique fixed point in $\mathcal{D}$.

Corollary II.4: Theorem II.1 continues to be true if (3) is replaced by

$\mathcal{S}_{bps}(\mathcal{E}\varrho, \mathcal{E}\sigma, \mathcal{E}\varsigma)$
$\preceq_{i_2} \alpha\, max\{\mathcal{S}_{bps}(\varrho, \sigma, \varsigma), \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varrho), \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma),$
$\frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho)\}$,

for all $\varrho, \sigma, \varsigma \in \mathcal{D}$ and $0 \leq \alpha < 1$. Then $\mathcal{E}$ has a unique fixed point in $\mathcal{D}$.

III. Examples

Example III.1: Let $\mathcal{D} \in [0, \infty)$ and we define $\mathcal{S}$: $\mathcal{D}^3 \to \mathcal{C}_2^+$ by $\mathcal{S}_{bps}(\varrho, \sigma, \varsigma) = (1 + i_2)(|\varrho - \varsigma| + |\sigma - \varsigma|)$, for each ϱ, σ, and $\varsigma \in \mathcal{D}$.

Clearly $\mathcal{S}_{bps}$ is a complete BCPSMS.

We define a mapping $\mathcal{E}$: $\mathcal{D} \to \mathcal{D}$ by $\mathcal{E}\varrho = \frac{\varrho+1}{4}$.

We now verify the inequality (3) with $\alpha = \frac{1}{4}$.

$\mathcal{S}_{bps}(\mathcal{E}\varrho, \mathcal{E}\sigma, \mathcal{E}\varsigma)$
$= \mathcal{S}_{bps}(\frac{\varrho+1}{4}, \frac{\sigma+1}{4}, \frac{\varsigma+1}{4})$
$= (1 + i_2)[|\frac{\varrho+1}{4} - \frac{\varsigma+1}{4}| + |\frac{\sigma+1}{4} - \frac{\varsigma+1}{4}|]$
$= (1 + i_2)[|\frac{\varrho-\varsigma}{4}| + |\frac{\sigma-\varsigma}{4}|]$
$= \frac{1}{4}(1 + i_2)[|\varrho - \varsigma| + |\sigma - \varsigma|]$
$\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho, \sigma, \varsigma)$
$\preceq_{i_2} \alpha max\{\mathcal{S}_{bps}(\varrho, \sigma, \varsigma), \mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \mathcal{E}\varrho),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varsigma), \frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma, \mathcal{E}\varsigma, \varrho),$
$\frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}$

Hence $\mathcal{E}$ satisfies all the conditions of Theorem II.1, '$\frac{1}{3}$' follows as a unique fixed point of $\mathcal{E}$.

Example III.2: Let $\mathcal{D} = [0,1]$ and we define $\mathcal{S}$: $\mathcal{D}^3 \to [1,\infty)$ by $\mathcal{S}_{bps}(\varrho,\sigma,\varsigma) = (1+i_2)max\{\varrho,\sigma,\varsigma\}$, for each ϱ,σ, and $\varsigma \in \mathcal{D}$.

Clearly $\mathcal{S}_{bps}$ is a complete BCPSMS.

We define a mapping $\mathcal{E}$: $\mathcal{D} \to \mathcal{D}$ by

$$\mathcal{E}\varrho = \begin{cases} \frac{\varrho^2}{2} & if \ \varrho \in [0,\frac{1}{2}) \\ \frac{1}{4} & if \ \varrho \in (\frac{1}{2},1] \end{cases}$$

We now verify the inequality (3) with $\alpha = \frac{1}{2}$.

Case (i): If $\varrho,\sigma,\varsigma \in [0,\frac{1}{2}]$ and $\varrho \geq \sigma \geq \varsigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{\varrho^2}{2},\tfrac{\sigma^2}{2},\tfrac{\varsigma^2}{2})\\
&= (1+i_2)max\{\tfrac{\varrho^2}{2},\tfrac{\sigma^2}{2},\tfrac{\varsigma^2}{2}\}\\
&= (1+i_2).\tfrac{\varrho^2}{2}\\
&= \tfrac{1}{2}(1+i_2).\varrho\\
&\preceq_{i_2} \tfrac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}\\
&\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)\\
&\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),\\
&\qquad \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),\\
&\qquad \tfrac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.
\end{aligned}$$

Similarly, when $\varrho < \sigma < \varsigma$ the inequality (3) holds.

Case (ii): If $\varrho,\sigma,\varsigma \in (\frac{1}{2},1]$ and $\varrho \geq \sigma \geq \varsigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})\\
&= (1+i_2)max\{\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}\} = (1+i_2).\tfrac{1}{4}\\
&\preceq_{i_2} \tfrac{1}{2}(1+i_2).\varrho\\
&\preceq_{i_2} \tfrac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}\\
&\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)\\
&\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),\\
&\qquad \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),\\
&\qquad \tfrac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.
\end{aligned}$$

Similarly, when $\varrho < \sigma < \varsigma$ the inequality (3) holds.

Case (iii): If $\varrho \in [0,\frac{1}{2}],\sigma,\varsigma \in (\frac{1}{2},1]$ and $\sigma \geq \varsigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{\varrho^2}{2},\tfrac{1}{4},\tfrac{1}{4})\\
&= (1+i_2)max\{\tfrac{\varrho^2}{2},\tfrac{1}{4},\tfrac{1}{4}\} = (1+i_2).\tfrac{1}{4}\\
&\preceq_{i_2} \tfrac{1}{2}(1+i_2).\sigma\\
&\preceq_{i_2} \tfrac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}\\
&\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)\\
&\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),\\
&\qquad \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),\\
&\qquad \tfrac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.
\end{aligned}$$

Similarly, $\sigma < \varsigma$ the inequality (3) holds.

Case (iv): If $\varrho,\sigma \in [0,\frac{1}{2}],\varsigma \in (\frac{1}{2},1]$ and $\varrho \geq \sigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{\varrho^2}{2},\tfrac{\sigma^2}{2},\tfrac{1}{4})\\
&= (1+i_2)max\{\tfrac{\varrho^2}{2},\tfrac{\sigma^2}{2},\tfrac{1}{4}\} = (1+i_2).\tfrac{1}{4}\\
&\preceq_{i_2} \tfrac{1}{2}(1+i_2).\varsigma\\
&\preceq_{i_2} \tfrac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}\\
&\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)\\
&\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),\\
&\qquad \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),\\
&\qquad \tfrac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.
\end{aligned}$$

Similarly, when $\varrho < \sigma$, the inequality (3) holds.

Case (v): If $\sigma,\varsigma \in [0,\frac{1}{2}],\varrho \in (\frac{1}{2},1]$ and $\sigma \geq \varsigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{1}{4},\tfrac{\sigma^2}{2},\tfrac{\varsigma^2}{2})\\
&= (1+i_2)max\{\tfrac{1}{4},\tfrac{\sigma^2}{2},\tfrac{\varsigma^2}{2}\} = (1+i_2).\tfrac{1}{4}\\
&\preceq_{i_2} \tfrac{1}{2}(1+i_2).\varrho\\
&\preceq_{i_2} \tfrac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}\\
&\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)\\
&\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),\\
&\qquad \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),\\
&\qquad \tfrac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.
\end{aligned}$$

Similarly, when $\sigma < \varsigma$ the inequality (3) holds.

Case (vi): If $\varsigma \in [0,\frac{1}{2}],\varrho,\sigma \in (\frac{1}{2},1]$ and $\varrho \geq \sigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{1}{4},\tfrac{1}{4},\tfrac{\varsigma^2}{2})\\
&= (1+i_2)max\{\tfrac{1}{4},\tfrac{1}{4},\tfrac{\varsigma^2}{2}\} = (1+i_2).\tfrac{1}{4}\\
&\preceq_{i_2} \tfrac{1}{2}(1+i_2).\varrho\\
&\preceq_{i_2} \tfrac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}\\
&\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)\\
&\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),\\
&\qquad \mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\tfrac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),\\
&\qquad \tfrac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.
\end{aligned}$$

Similarly, when $\varrho < \sigma$ the inequality (3) holds.

Case (vii): If $\sigma \in [0,\frac{1}{2}],\varrho,\varsigma \in (\frac{1}{2},1]$ and $\varrho \geq \varsigma$, then

$$\begin{aligned}
&\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)\\
&= \mathcal{S}_{bps}(\tfrac{1}{4},\tfrac{\sigma^2}{2},\tfrac{1}{4})\\
&= (1+i_2)max\{\tfrac{1}{4},\tfrac{\sigma^2}{2},\tfrac{1}{4}\} = (1+i_2).\tfrac{1}{4}
\end{aligned}$$

$\preceq_{i_2} \frac{1}{2}(1+i_2).\varrho$

$\preceq_{i_2} \frac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}$

$\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)$

$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),$
$\frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.$

Similarly, when $\varrho<\varsigma$ the inequality (3) holds.

Case (viii): If $\varrho,\varsigma\in[0,\frac{1}{2}],\sigma\in(\frac{1}{2},1]$ and $\varrho\geq\varsigma$, then

$\mathcal{S}_{bps}(\mathcal{E}\varrho,\mathcal{E}\sigma,\mathcal{E}\varsigma)$

$=\mathcal{S}_{bps}(\frac{\varrho^2}{2},\frac{1}{4},\frac{\varsigma^2}{2})$

$=(1+i_2)max\{\frac{\varrho^2}{2},\frac{1}{4},\frac{\varsigma^2}{2}\}=(1+i_2).\frac{1}{4}$

$\preceq_{i_2} \frac{1}{2}(1+i_2).\sigma$

$\preceq_{i_2} \frac{1}{2}.(1+i_2)max\{\varrho,\sigma,\varsigma\}$

$\preceq_{i_2} \alpha.\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)$

$\preceq_{i_2} \alpha.max\{\mathcal{S}_{bps}(\varrho,\sigma,\varsigma),\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\mathcal{E}\varrho),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varsigma),\frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma,\mathcal{E}\varsigma,\varrho),$
$\frac{\mathcal{S}_{bps}(\varsigma,\varsigma,\mathcal{E}\varsigma)(1+\mathcal{S}_{bps}(\sigma,\sigma,\mathcal{E}\sigma))}{1+\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)}\}.$

Similarly, when $\varrho<\varsigma$ the inequality (3) holds.

Hence $\mathcal{E}$ satisfies all the conditions of Theorem II.1, '0' follows as a unique fixed point of $\mathcal{E}$.

IV. Applications

In this section, we obtain solution of the following two point boundary value problem

$$\frac{d^2\varrho}{d\omega^2}=-\mathcal{F}(\omega,\varrho(\omega)), \qquad (11)$$

for each $\omega\in[0,1]$ and the initial conditions are $\varrho(0)=\varrho(1)=0$.

The Green's function corresponding to given differential equation is

$$\mathcal{G}(\omega,\kappa)=\begin{cases}\omega(1-\kappa) & if\ 0\leq\omega\leq\kappa\leq 1\\ \kappa(1-\omega) & if\ \ 0\leq\kappa\leq\omega\leq 1\end{cases} \qquad (12)$$

The solution of (11) is the same as finding the solution $\varrho(\omega)$ of the given integral equation

$\varrho(\omega)=\int_0^1\mathcal{G}(\omega,\kappa)\mathcal{F}(\kappa,\varrho(\kappa))d\kappa$, for each $\omega\in[0,1]$.

Let $\mathcal{U}=\mathcal{C}([0,1],\mathcal{R})$ be the class of all real valued continuous functions on $[0,1]$. We define $\preceq_{i_2}$ in $\mathcal{C}_2^+$ by $\varrho\preceq_{i_2}\sigma$ if and only if $\varrho\leq\sigma$.

Define $\mathcal{S}_{bps}:\mathcal{U}\times\mathcal{U}\to\mathcal{C}_2^+$ defined by $\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)=(1+i_2)[|\varrho-\varsigma|+|\sigma-\varsigma|+2]$, for each ϱ,σ and $\varsigma\in\mathcal{U}$.

We define a operator $\mathcal{E}:\mathcal{U}\to\mathcal{U}$ by

$$\mathcal{E}(\varrho)=\int_0^1\mathcal{G}(\omega,\kappa)\mathcal{F}(\kappa,\varrho(\kappa))d\kappa, \qquad (13)$$

for each $\omega\in[0,1]$.

Clearly the solution of (11) is a fixed point of $\mathcal{E}$.

Theorem IV.1: *Consider the differential equation* (11). *Suppose that:*

$|\mathcal{F}(\kappa,\varrho(\kappa))-\mathcal{F}(\kappa,\varsigma(\kappa))|+|\mathcal{F}(\kappa,\sigma(\kappa))-\mathcal{F}(\kappa,\varsigma(\kappa))|$
$\preceq_{i_2} max\{|\varrho(\kappa)-\varsigma(\kappa)|+|\sigma(\kappa)-\varsigma(\kappa)|\}.$

Then the integral operator defined as in (13) *has a unique solution.*

Proof. We define a BCPSMS on $\mathcal{U}$ by

$\mathcal{S}_{bps}(\varrho,\sigma,\varsigma)=(1+i_2)[|\varrho-\varsigma|+|\sigma-\varsigma|+2]$,

for each ϱ,σ and $\varsigma\in\mathcal{U}$.

Clearly $\mathcal{S}_{bps}$ is a complete BCPSMS.

$\mathcal{S}_{bps}(\mathcal{E}\varrho(\kappa),\mathcal{E}\sigma(\kappa),\mathcal{E}\varsigma(\kappa))$

$=(1+i_2)[|\mathcal{E}\varrho(\kappa)-\mathcal{E}\varsigma(\kappa)|+|\mathcal{E}\sigma(\kappa)-\mathcal{E}\varsigma(\kappa)|+2]$

$=(1+i_2)[|\int_0^1\mathcal{G}(\omega,\kappa)(\mathcal{F}(\kappa,\varrho(\kappa))-\mathcal{F}(\kappa,\varsigma(\kappa)))d\kappa|+$
$|\int_0^1\mathcal{G}(\omega,\kappa)(\mathcal{F}(\kappa,\sigma(\kappa))-\mathcal{F}(\kappa,\varsigma(\kappa)))d\kappa|+2]$

$=(1+i_2)[\int_0^1\mathcal{G}(\omega,\kappa)d\kappa(|\mathcal{F}(\kappa,\varrho(\kappa))-\mathcal{F}(\kappa,\varsigma(\kappa))|+$
$|\mathcal{F}(\kappa,\sigma(\kappa))-\mathcal{F}(\kappa,\varsigma(\kappa))|+2)]$

$\preceq_{i_2}(1+i_2)[\int_0^1\mathcal{G}(\omega,\kappa)d\kappa.(max\{|\varrho(\kappa)-\varsigma(\kappa)|$
$+|\sigma(\kappa)-\varsigma(\kappa)|+2)]$

$\preceq_{i_2}(1+i_2)[(\int_0^\omega\mathcal{G}(\omega,\kappa)d\kappa+\int_\omega^1\mathcal{G}(\omega,\kappa)d\kappa).$
$max\{|\varrho(\kappa)-\varsigma(\kappa)|+|\sigma(\kappa)-\varsigma(\kappa)|+2]$

$\preceq_{i_2}(1+i_2)[(\int_0^\omega\kappa(1-\omega)d\kappa+\int_\omega^1\omega(1-\kappa)d\kappa)$
$max\{|\varrho(\kappa)-\varsigma(\kappa)|+|\sigma(\kappa)-\varsigma(\kappa)|+2]$

$\preceq_{i_2}(1+i_2)[(\frac{\omega}{2}-\frac{\omega^2}{2})max\{|\varrho(\kappa)-\varsigma(\kappa)|+|\sigma(\kappa)-\varsigma(\kappa)|+2]$

$\preceq_{i_2}\frac{1}{8}.max\{(1+i_2)[|\varrho(\kappa)-\varsigma(\kappa)|+|\sigma(\kappa)-\varsigma(\kappa)|+2]\}$

$\preceq_{i_2}\frac{1}{8}max\{\mathcal{S}_{bps}(\varrho(\kappa),\sigma(\kappa),\varsigma(\kappa)\}$

$\preceq_{i_2}\frac{1}{8}max\{\mathcal{S}_{bps}(\varrho(\kappa),\sigma(\kappa),\varsigma(\kappa)),$
$\mathcal{S}_{bps}(\mathcal{E}\varsigma(\kappa),\mathcal{E}\varsigma(\kappa),\mathcal{E}\varrho(\kappa)),\mathcal{S}_{bps}(\mathcal{E}\varsigma(\kappa),\mathcal{E}\varsigma(\kappa),\varsigma(\kappa)),$
$\frac{1}{3}\mathcal{S}_{bps}(\mathcal{E}\varsigma(\kappa),\mathcal{E}\varsigma(\kappa),\varrho(\kappa)),$
$\frac{\mathcal{S}_{bps}((\varsigma(\kappa),\varsigma(\kappa),\mathcal{E}\varsigma(\kappa))[1+\mathcal{S}_{bps}(\sigma(\kappa),\sigma(\kappa),\mathcal{E}\sigma(\kappa))}{1+\mathcal{S}_{bps}(\varrho(\kappa),\sigma(\kappa),\varsigma(\kappa))}\},$

we note that $sup_{w\in[0,1]}\int_0^1\mathcal{G}(\omega,\kappa)d\kappa=\frac{1}{8}$.

Hence, $\mathcal{E}$ satisfies all the conditions of Theorem II.1, then the function $\mathcal{E}$ has a unique fixed point. As a result, the integral equation (11) has a solution in $\mathcal{U}$, ensuring the existence of a solution to the integral equation (11).

V. Conclusions

We extend the concept of a bicomplex partial S-metric space and establish the existence of fixed points for certain generalized contraction mappings. The bicomplex partial S-metric space is particularly significant, as it does not necessarily arise from any standard metric space, making it a compact and unique framework. Through illustrative examples, we demonstrated that these extensions, improvements, and generalizations are valid and meaningful. The paper concludes by addressing a boundary value problem, and the results offer a concrete approach for further exploration in this emerging area of bicomplex partial S-metric theory.

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M Madhuri was born in Andhra Pradesh, India. She graduated in Master of Science (2011) from M R P G college at Vizianagaram, under Andhra University in Andhra Pradesh. She is a Research scholar in the Department of Mathematics, School of Science, GITAM Deemed to be University, Visakhapatnam, India. She is currently working as an Assistant Professor in the Department of Mathematics, Lendi Institute of Engineering and Technology, Vizianagaram, Andhra Pradesh, India. Her research interests lie in fixed-point theory.

M V R Kameswari was born in Andhra Pradesh, India. She is an Associate Professor in the Department of Mathematics, School of Science, GITAM Deemed to be University, Rushikonda, Visakhapatnam, Andhra Pradesh, India. She received Ph.D Degree at Andhra University, Visakhapatnam, India. The results of her research have been published in several National Journals, International Proceedings, and International Journals Indexed by Scopus. Her research interest includes fixed points and approximation of fixed points.