Positive Implicative, Implicative, and Commutative SB-Neutrosophic Ideals in BCK/BCI-Algebras

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Abstract—**This article introduces the notions of positive implicative SB-neutrosophic ideal, implicative SB-neutrosophic ideal, and commutative SB-neutrosophic ideal in the context of BCK/BCI-algebras. We establish conditions for an SB-neutrosophic set to be a positive implicative SB-neutrosophic ideal, an implicative SB-neutrosophic ideal, and a commutative SB-neutrosophic ideal. The relationships between SB-neutrosophic ideal and positive implicative SB-neutrosophic ideal, SB-neutrosophic ideal and implicative SB-neutrosophic ideal, as well as SB-neutrosophic ideal and commutative SB-neutrosophic ideal, are explored. Characterizations of the positive implicative SB-neutrosophic ideal, the implicative SB-neutrosophic ideal, and the commutative SB-neutrosophic ideal are provided. Moreover, the extension property for both positive implicative and implicative SB-neutrosophic ideals is examined.**

Index Terms—**SB-Neutrosophic Set (SB-NSS), SB- neutrosophic ideal (SB-NSI), Positive Implicative SB-Neutrosophic Ideal (PISB-NSI), Implicative SB-Neutrosophic Ideal (ISB-NSI), Commutative SB-Neutrosophic Ideal (CSB-NSI).**

Abbreviations

We provide a list of abbreviations used in this article.
SB stands for the initial letters of the author's names, Satyanarayana and Baji.
SB-NSS: SB-Neutrosophic Set
SB-NSI: SB-Neutrosophic Ideal
ISB-NSI: Implicative SB-Neutrosophic Ideal
PISB-NSI: Positive Implicative SB-Neutrosophic Ideal
CSB-NSI: Commutative SB-Neutrosophic Ideal

I. Introduction

ZADEH. L. A. [1], a professor of computer science at the University of California, introduced the concept of fuzzy set (FS) in 1965. The concept of FSs entails assessing the membership strength of each element within a set. In 1986, Atanassove [2] extended the concept of a fuzzy set to an intuitionistic fuzzy set (IFS) by incorporating an additional function known as a non-membership function. The term "neutrosophic," introduced by F. Smarandache, derives from the concept of "neutrosophy" as its etymological foundation. Neutrosophy refers to the knowledge of neutral thinking, and this concept of neutrality distinguishes neutrosophic sets from both fuzzy sets and intuitionistic fuzzy sets. In 1995, F. Smarandache introduced the concept of indeterminacy, which was denoted by 'I', as an independent component (see [3], [4]). He formulated the neutrosophic set using three components (T, I, F) = (True, Indeterminate, False). More information is available on the website: http://fs.gallup.unm.edu/FlorentinSmarandache.htm. Neutrosophic logic finds applications in dealing with transportation problems (see [5]) and in resolving multi-attribute decision-making problems (see [6]). In 1966, BCK/BCI-algebras were introduced by Y. Imai and K. Iseki. These are two categories of non-classical logic algebras (see [7], [8]). Subsequently, research has extended to the neutrosophic fuzzification of ideals in BCK/BCI-algebras.

Several variations have been introduced into the world of neutrosophic structures to help us better understand indeterminacy and uncertainty. In 2005, Wang, H., extended the idea with the introduction of interval-valued neutrosophic sets (see [9]). Following this, Y. B. Jun et al. interval-valued neutrosophic sets were then applied to the ideals in BCI/BCK algebras (see [10]). M. M. Takalo et al. introduced the MBJ-neutrosophic structures as a generalisation of the neutrosophic sets, in which the indeterminacy function is represented by interval-valued fuzzy sets (see [11]). F. Smarandache et al. introduced neutrosophic N-structures in which the truth, uncertainty, and falsity membership functions are negatively valued functions (see [12]). Later, neutrosophic N-structures are applied to positive implicative ideals (see [13]) and commutative ideals (see [14]) in BCK-algebras. In developing neutrosophic structures, Y. B. Jun and others presented the concept of (Φ,Ψ)-neutrosophic ideals and (Φ,Ψ)-neutrosophic subalgebras in BCI/BCK-algebras (see [15], [16]). Additionally, Song, S. Z., and others introduced the concept of generalised NSS (see [17]). Subsequently, R. A. Borzooei and others proposed the idea of commutative generalised neutrosophic ideals in BCI/BCK-algebras (see [18]). Satyanarayana et al. introduced the idea of BS-NSS in the context of BCI/BCK algebras, where false membership is represented by an interval-valued fuzzy set (see [19]).

Recently, B. Satyanarayana et al. introduced the concept of SB-neutrosophic sets as a slightly extended version of neutrosophic sets, and it has been applied to BCK/BCI-algebras (see [20]). When extending the NSS concept, an interval-valued fuzzy set (IVFS) is used as the truth membership function due to its broader generalization of the fuzzy set. SB-neutrosophic structures are helpful when there's a high level of uncertainty in the data, especially regarding the truth membership function. Besides that, in cases where there is a low level of uncertainty in the indeterminate membership function and false membership function, SB-Neutrosophic structures are also valuable. In Figure 1, we've created a visual guide that highlights the different types of generalisations of neutrosophic structures. This makes it easier for readers to follow and understand the detailed concepts discussed.

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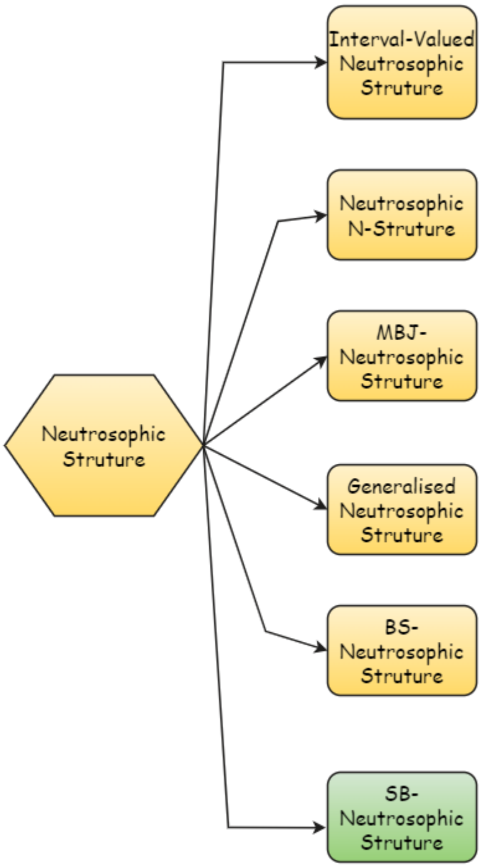

Fig. 1. Generalisations of Neutrosophic Structures

In this article, we present the fundamental definitions required for our discussion in Section II. These include definitions of BCK/BCI-algebras, subalgebra, ideal, positive implicative ideal, commutative ideal, implicative ideal, fuzzy set, complement of a fuzzy set, NSS, SB-NSS, level sets, SB-NSSA, and SB-NSI. In Section III, we introduce PI-SBNSI with illustrative examples. We discuss the conditions for SB-NSS to be a PI-SBNSI, as well as the conditions for SB-NSI to be a PI-SBNSI. Furthermore, we develop the PI-SBNSI extension property. In Section IV, we present I-SBNSI along with illustrative examples. We discuss the characteristics of I-SBNSI, conditions under which SB-NSI can qualify as I-SBNSI, and conditions under which SB-NSSA can qualify as I-SBNSI. Additionally, we develop the I-SBNSI extension property. In Section V, we present C-SBNSI with illustrative examples. We also demonstrate that the inverse image of a C-SBNSI under homomorphism remains a C-SBNSI and that the image of a C-SBNSI under an onto homomorphism is also a C-SBNSI. Thus, a comprehensive study of PI-SBNSI, I-SBNSI, and C-SBNSI in Sections III, IV, and V contributes to a deeper understanding of their relationships and characteristics, paving the way for further research and practical application.

II. Preliminaries

Definition 2.1. *([21], [22], [23]) Let $\mathcal{K}(\neq \phi)$ be a set with a binary operation denoted by "$\cdot$" and "0" is a constant. $\mathcal{K}$ is called a BCI-algebra if the following axioms satisfied by the $\mathcal{K}$ for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$.*

$$((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{p}_1 \cdot \mathfrak{u}_1)) \cdot (\mathfrak{u}_1 \cdot \mathfrak{r}_1) = 0 \quad (1)$$

$$(\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \cdot \mathfrak{r}_1 = 0 \quad (2)$$

$$\mathfrak{p}_1 \cdot \mathfrak{p}_1 = 0 \quad (3)$$

$$\mathfrak{p}_1 \cdot \mathfrak{r}_1 = 0, \mathfrak{r}_1 \cdot \mathfrak{p}_1 = 0 \Rightarrow \mathfrak{p}_1 = \mathfrak{r}_1 \quad (4)$$

If the following axiom is satisfied by a BCI-algebra

$$0 \cdot \mathfrak{p}_1 = 0 \quad (5)$$

for all $\mathfrak{p}_1 \in \mathcal{K}$, then it is called a BCK-algebra.

In any BCK/BCI-algebra, the following properties hold ([23])

$$\mathfrak{p}_1 \cdot 0 = \mathfrak{p}_1 \quad (6)$$

$$\mathfrak{p}_1 \leq \mathfrak{r}_1 \Rightarrow \mathfrak{p}_1 \cdot \mathfrak{u}_1 \leq \mathfrak{r}_1 \cdot \mathfrak{u}_1, \mathfrak{u}_1 \cdot \mathfrak{r}_1 \leq \mathfrak{u}_1 \cdot \mathfrak{p}_1 \quad (7)$$

$$(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1 = (\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1 \quad (8)$$

$$(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1) \leq \mathfrak{p}_1 \cdot \mathfrak{r}_1 \quad (9)$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$, where $\mathfrak{p}_1 \leq \mathfrak{r}_1$ if and only if $\mathfrak{p}_1 \cdot \mathfrak{r}_1 = 0$. In any BCI-algebra $\mathcal{K}$, the following conditions hold [23]

$$\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)) = \mathfrak{p}_1 \cdot \mathfrak{r}_1 \quad (10)$$

$$0 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1) = (0 \cdot \mathfrak{p}_1) \cdot (0 \cdot \mathfrak{r}_1) \quad (11)$$

Definition 2.2. *If the following condition is satisfied in a BCK-algebra $\mathcal{K}$*

$$\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1) = \mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1) \quad (12)$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, then it is called a commutative BCK-algebra.

Definition 2.3. *[22] If the following condition holds in a BCK-algebra $\mathcal{K}$*

$$\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1) = \mathfrak{p}_1 \quad (13)$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, then it is called an implicative.

Definition 2.4. *[22] If the following condition holds in a BCK-algebra $\mathcal{K}$*

$$(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1) = (\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1 \quad (14)$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$, then it is called a positive implicative.

Definition 2.5. *[23] If the following condition holds in a BCI-algebra $\mathcal{K}$*

$$(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1 = (\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1 \quad (15)$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$, then it is called an associative.

Definition 2.6. *Let $\mathcal{K}$ be a BCK/BCI-algebra. A subset $\mathcal{H}(\neq \emptyset)$ of $\mathcal{K}$ is said to be*

- *a subalgebra of $\mathcal{K}$, if for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{H}$, $\mathfrak{p}_1 \cdot \mathfrak{r}_1 \in \mathcal{H}$.*
- *an ideal of $\mathcal{K}$, if for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, $0 \in \mathcal{H}$ and $\mathfrak{r}_1, \mathfrak{p}_1 \cdot \mathfrak{r}_1 \in \mathcal{H} \Rightarrow \mathfrak{p}_1 \in \mathcal{H}$.*
- *a positive implicative ideal of $\mathcal{K}$, if for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$, $0 \in \mathcal{H}$ and $(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1, \mathfrak{r}_1 \cdot \mathfrak{u}_1 \in \mathcal{H} \Rightarrow \mathfrak{p}_1 \cdot \mathfrak{u}_1 \in \mathcal{H}$.*
- *a commutative ideal of $\mathcal{K}$, if for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$, $0 \in \mathcal{H}$ and $(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1, \mathfrak{u}_1 \in \mathcal{H} \Rightarrow \mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \in \mathcal{H}$.*

Proposition 2.7. *Let $\mathfrak{I}$ and $\mathfrak{J}$ be ideals of $\mathcal{K}$ with $\mathfrak{I} \subseteq \mathfrak{J}$. If $\mathfrak{I}$ is an implicative ideal, then $\mathfrak{J}$ is also an implicative ideal.*

Let $\mathfrak{I}$ and $\mathfrak{J}$ be ideals of $\mathcal{K}$ with $\mathfrak{I} \subseteq \mathfrak{J}$. If $\mathfrak{I}$ is an implicative ideal, then $\mathfrak{J}$ is also an implicative ideal.

Definition 2.8. *[1] Let $\mathcal{K}$ be a non-empty set. A fuzzy set in $\mathcal{K}$ is a mapping $\mathfrak{B}_T : \mathcal{K} \to [0,1]$.*

Definition 2.9. *[24] A fuzzy set $\mathfrak{B}_T : \mathcal{K} \to [0,1]$ is called a fuzzy sub-algebra of $\mathcal{K}$, if*

$$\mathfrak{B}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \geq min\{\mathfrak{B}_T(\mathfrak{p}_1), \mathfrak{B}_T(\mathfrak{r}_1)\} \text{ for all } \mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}.$$

An interval number, denoted as $\widetilde{h} = [h^-, h^+]$, represents a closed subinterval of $[I]$, where $0 \leq h^- \leq h^+ \leq 1$. Here, $[I]$ refers to the set of all interval numbers. The interval $[h, h]$ is indicated by the number $h \in [0,1]$ for whatever follows. Let us define the refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in $[I]$. We also define the symbols '$\preccurlyeq$', '$\succcurlyeq$', and '$=$' in the case of two elements in $[I]$. Consider two interval numbers $\widetilde{h}_1 = [{h_1}^-, {h_1}^+]$ and $\widetilde{h}_2 = [{h_2}^-, {h_2}^+]$. Then

- $rmin\{\widetilde{h}_1, \widetilde{h}_2\} = \left[min\{{h_1}^-, {h_2}^-\}, min\{{h_1}^+, {h_2}^+\}\right]$
- $rmax\{\widetilde{h}_1, \widetilde{h}_2\} = \left[max\{{h_1}^-, {h_2}^-\}, max\{{h_1}^+, {h_2}^+\}\right]$
- $\widetilde{h}_1 \succcurlyeq \widetilde{h}_2 \Leftrightarrow {h_1}^- \geq {h_2}^-,\ {h_1}^+ \geq {h_2}^+$
- $\widetilde{h}_1 \preccurlyeq \widetilde{h}_2 \Leftrightarrow {h_1}^- \leq {h_2}^-,\ {h_1}^+ \leq {h_2}^+$
- $\widetilde{h}_1 = \widetilde{h}_2 \Leftrightarrow {h_1}^- = {h_2}^-,\ {h_1}^+ = {h_2}^+$

Let $\widetilde{h}_I \in [I]$ where $i \in \sqcap$. We define

- $\underset{i\in\sqcap}{rinf}\widetilde{h}_I = \left[\underset{i\in\sqcap}{inf}{h_I}^-, \underset{i\in\sqcap}{inf}{h_I}^+\right]$
- $\underset{i\in\sqcap}{rsup}\widetilde{h}_I = \left[\underset{i\in\sqcap}{sup}{h_I}^-, \underset{i\in\sqcap}{sup}{h_I}^+\right]$

Definition 2.10. *[25] An interval-valued fuzzy set in $\mathcal{K}(\neq \emptyset)$ is a function $\widetilde{\mathfrak{B}} : \mathcal{K} \to [I]$. The set of all IVFS in $\mathcal{K}$ is denoted by $[I]^{\mathcal{K}}$. For every IVFS $\widetilde{\mathfrak{B}} \in [I]^{\mathcal{K}}$ and $\mathfrak{p}_1 \in \mathcal{K}$, $\widetilde{\mathfrak{B}}(\mathfrak{p}_1) = [\mathfrak{B}^-(\mathfrak{p}_1), \mathfrak{B}^+(\mathfrak{p}_1)]$ is called the degree of belonging of an element $\mathfrak{p}_1 \in \widetilde{\mathfrak{B}}$, where $\mathfrak{B}^- : \mathcal{K} \to [I]$ is called a lower fuzzy set and $\mathfrak{B}^+ : \mathcal{K} \to [I]$ is called an upper fuzzy set in $\mathcal{K}$.*

Definition 2.11. *[4] Let $\mathcal{K}$ be a non-empty set. A neutrosophic set (NSS) in $\mathcal{K}$ is a structure of the form $\mathfrak{B} = \{\langle \mathfrak{p}_1; \mathfrak{B}_T(\mathfrak{p}_1), \mathfrak{B}_I(\mathfrak{p}_1), \mathfrak{B}_F(\mathfrak{p}_1)\rangle : \mathfrak{p}_1 \in \mathcal{K}\}$ where $\mathfrak{B}_T : \mathcal{K} \to [0,1]$ is a degree of belonging, $\mathfrak{B}_I : \mathcal{K} \to [0,1]$ is a degree of indeterminacy of belonging, and $\mathfrak{B}_F : \mathcal{K} \to [0,1]$ is a degree of non-belonging.*

Definition 2.12. *[20] Let $\mathcal{K}$ be a non-empty set. An SB-NSS in $\mathcal{K}$ is a structure of the form*

$$\mathfrak{B} = \{\langle \mathfrak{p}_1; \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1), \mathfrak{B}_I(\mathfrak{p}_1), \mathfrak{B}_F(\mathfrak{p}_1)\rangle : \mathfrak{p}_1 \in \mathcal{K}\}$$

Here, $\mathfrak{B}_I$ and $\mathfrak{B}_F$ are fuzzy sets in $\mathcal{K}$, representing the degree of indeterminacy of belonging and the degree of non-belonging, respectively. $\widetilde{\mathfrak{B}}_T$ is an interval-valued fuzzy set in $\mathcal{K}$, representing an interval-valued degree of belonging of an element to $\mathfrak{B}$.

For simplicity, the SB-neutrosophic set will be denoted by the symbol $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$

Definition 2.13. *[20] Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$. We define SB-level sets as*

$$\mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2]) = \{\mathfrak{p}_1 \in \mathcal{K} : \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq [l_1, l_2]\}$$
$$\mathcal{U}(\mathfrak{B}_I; m) = \{\mathfrak{p}_1 \in \mathcal{K} : \mathfrak{B}_I(\mathfrak{p}_1) \geq m\}$$
$$\mathcal{L}(\mathfrak{B}_F; n) = \{\mathfrak{p}_1 \in \mathcal{K} : \mathfrak{B}_F(\mathfrak{p}_1) \leq n\}$$

where $m, n \in [0,1]$ and $[l_1, l_2] \in [I]$.

Definition 2.14. *[20] Let $\mathcal{K}$ be a BCK/BCI-algebra. An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ is called an SB-neutrosophic subalgebra of $\mathcal{K}$ if it satisfies*
(SB-NSSA 1) $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1)\}$
(SB-NSSA 2) $\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \geq min\{\mathfrak{B}_I(\mathfrak{p}_1), \mathfrak{B}_I(\mathfrak{r}_1)\}$
(SB-NSSA 3) $\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \leq max\{\mathfrak{B}_F(\mathfrak{p}_1), \mathfrak{B}_F(\mathfrak{r}_1)\}$
for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.

Definition 2.15. *[20] Let $\mathcal{K}$ be a BCK/BCI-algebra. An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ is called an SB-NSI of $\mathcal{K}$ if it satisfies*

$$\widetilde{\mathfrak{B}}_T(0) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1),\ \mathfrak{B}_I(0) \geq \mathfrak{B}_I(\mathfrak{p}_1), \mathfrak{B}_F(0) \leq \mathfrak{B}_F(\mathfrak{p}_1) \tag{16}$$

(SB-NSI 1) $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1)\}$
(SB-NSI 2) $\mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1), \mathfrak{B}_I(\mathfrak{r}_1)\}$
(SB-NSI 3) $\mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1), \mathfrak{B}_F(\mathfrak{r}_1)\}$
for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.

III. Positive Implicative SB-neutrosophic ideal (PISB-NSI)

Definition 3.1. *Let $\mathcal{K}$ be a BCK-algebra. An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ is called a PISB-NSI of $\mathcal{K}$ if it satisfies Condition (16) and*

$$\left(\begin{array}{l} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\} \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\} \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\} \end{array}\right) \tag{17}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$.

Example 3.2. *Consider a BCK-algebra $\mathcal{K} = \{0, a, b, c, d\}$, as defined in Table I.*

TABLE I
BCK-ALGEBRA

·	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

Let us define an SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ as shown in Table II.

TABLE II
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	$[0.6, 1]$	0.9	0.1
a	$[0.5, 0.8]$	0.6	0.3
b	$[0.4, 0.6]$	0.3	0.5
c	$[0.2, 0.4]$	0.3	0.5
d	$[0.2, 0.3]$	0.1	0.8

It is normal to check that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$.

Example 3.3. *Consider a BCK-algebra $\mathcal{K} = \{0, 1, 2, 3\}$, as defined in Table III.*

TABLE III
BCK-ALGEBRA

$\cdot$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

TABLE IV
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	[0.53, 0.95]	0.88	0.31
1	[0.43, 0.83]	0.57	0.45
2	[0.11, 0.35]	0.39	0.79
3	[0.11, 0.35]	0.39	0.79

Let us define an SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *in* $\mathcal{K}$ *as shown in Table IV.*

By using standard computation, it is clear that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *is a PISB-NSI of* $\mathcal{K}$.

Theorem 3.4. *In a BCK-algebra* $\mathcal{K}$*, every PISB-NSI is an SB-NSI of* $\mathcal{K}$.

Proof: Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be a PISB-NSI of $\mathcal{K}$. If we take $\mathfrak{u}_1 = 0$ in Condition (17) and use Equation (6), then we have

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot 0) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot 0), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1 \cdot 0)\}\\
&\Rightarrow \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1)\},\\
&\mathfrak{B}_I(\mathfrak{p}_1 \cdot 0) \geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot 0), \mathfrak{B}_I(\mathfrak{r}_1 \cdot 0)\}\\
&\Rightarrow \mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1), \mathfrak{B}_I(\mathfrak{r}_1)\},\\
&\mathfrak{B}_F(\mathfrak{p}_1 \cdot 0) \leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot 0), \mathfrak{B}_F(\mathfrak{r}_1 \cdot 0)\}\\
&\Rightarrow \mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1), \mathfrak{B}_F(\mathfrak{r}_1)\},
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. Thus, every PISB-NSI is an SB-NSI of $\mathcal{K}$. ■

The following example confirms that the converse of Theorem 3.4 is not true.

Example 3.5. *Consider a BCK-algebra* $\mathcal{K} = \{0, 1, 2, 3\}$*, as shown in Table V.*

TABLE V
BCK-ALGEBRA

$\cdot$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Let us define an SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *in* $\mathcal{K}$ *as shown in Table VI. It is normal to check that* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *is an SB-NSI of* $\mathcal{K}$*. Since*

$$\widetilde{\mathfrak{B}}_T(2 \cdot 1) \prec rmin\{\widetilde{\mathfrak{B}}_T((2 \cdot 1) \cdot 1), \widetilde{\mathfrak{B}}_T(1 \cdot 1)\},$$

$\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *is not a PISB-NSI of* $\mathcal{K}$.

TABLE VI
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	$[0.5, 1]$	1	0.3
1	$[0.4, 0.7]$	0.3	0.7
2	$[0.4, 0.7]$	0.3	0.7
3	$[0.2, 0.4]$	0.1	1

Corollary 3.6. *Let* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *be a PISB-NSI of* $\mathcal{K}$*. If* $\mathfrak{p}_1 \leq \mathfrak{r}_1$ *in* $\mathcal{K}$*, then* $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1)$, $\mathfrak{B}_I(\mathfrak{p}_1) \geq \mathfrak{B}_I(\mathfrak{r}_1)$, *and* $\mathfrak{B}_F(\mathfrak{p}_1) \leq \mathfrak{B}_F(\mathfrak{r}_1)$*. i.e.,* $\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I$ *are order-reversing and* $\mathfrak{B}_F$ *is order-preserving.*

Lemma 3.7. *[20] Let* $\mathcal{K}$ *be a BCK/BCI-algebra. Then, every SB-NSI* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *of* $\mathcal{K}$ *satisfies the following assertion*

$$\mathfrak{p}_1 \cdot \mathfrak{r}_1 \leq \mathfrak{u}_1 \Rightarrow \begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\ \mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I(\mathfrak{r}_1), \mathfrak{B}_I(\mathfrak{u}_1)\} \\ \mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F(\mathfrak{r}_1), \mathfrak{B}_F(\mathfrak{u}_1)\} \end{pmatrix} \tag{18}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$.

Lemma 3.8. *[20] If an SB-NSS* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *in a BCK- algebra* $\mathcal{K}$ *satisfies the Condition (18), then* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *is an SB-NSI of* $\mathcal{K}$.

Theorem 3.9. *An SB-NSS* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *in a BCK-algebra* $\mathcal{K}$ *is a PISB-NSI of* $\mathcal{K}$ *if and only if it is an SB-NSI of* $\mathcal{K}$*, and the following conditions hold*

$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \end{pmatrix} \tag{19}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.

Proof: Assume that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. If $\mathfrak{u}_1$ is replaced by $\mathfrak{r}_1$ in Condition (17), we get

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1 \cdot \mathfrak{r}_1)\}\\
&= rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(0)\}\\
&= \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1),\\
\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_I(\mathfrak{r}_1 \cdot \mathfrak{r}_1)\}\\
&= min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_I(0)\}\\
&= \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1),\\
\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_F(\mathfrak{r}_1 \cdot \mathfrak{r}_1)\}\\
&= max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_F(0)\}\\
&= \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1),
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.

Conversely, let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSI of $\mathcal{K}$ satisfying the Condition (19). Since

$$\begin{aligned}
((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1) &\leq ((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1)\\
&= ((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$, it follows from Corollary 3.6 that

$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\ \mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\ \mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \end{pmatrix} \tag{20}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Now, using Conditions (19), (17), and (20), we have

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \\
&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\} \\
&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\}, \\
&\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \\
&\geq min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)), \mathfrak{B}_I(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\} \\
&\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\}, \\
&\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{u}_1) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \\
&\leq max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)), \mathfrak{B}_F(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\} \\
&\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{r}_1 \cdot \mathfrak{u}_1)\},
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Thus, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. ∎

Theorem 3.10. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSI of $\mathcal{K}$. Then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ if and only if the following conditions hold for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$*

$$\left(\begin{array}{c}
\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\
\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\
\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)
\end{array}\right) \quad (21)$$

Proof: Suppose that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$ by Theorem 3.4, and it satisfies the Condition (19) by Theorem 3.9. Since

$$\begin{aligned}
((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1 &= ((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1 \\
&\leq ((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. It follows from Corollary 3.6 that

$$\left(\begin{array}{c}
\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\
\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\
\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)
\end{array}\right) \quad (22)$$

Now, utilizing the Equation (8), Conditions (19) and (22), we obtain

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) = \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \\
&\succcurlyeq \widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \\
&\succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \\
&\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) = \mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \\
&\geq \mathfrak{B}_I(((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \\
&\geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \\
&\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) = \mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \\
&\leq \mathfrak{B}_F(((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1) \cdot \mathfrak{u}_1) \\
&\leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1),
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Hence, the Condition (21) holds.

Conversely, let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSI of $\mathcal{K}$ that satisfies the Condition (21). By substituting $\mathfrak{u}_1 = \mathfrak{r}_1$ into Condition (21) and utilising Equations (3) and (6), we obtain

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \\
&\qquad \Rightarrow \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \\
&\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \\
&\qquad \Rightarrow \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \\
&\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \\
&\qquad \Rightarrow \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1),
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Therefore, by Theorem 3.9, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ∎

Theorem 3.11. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS of $\mathcal{K}$. Then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ if and only if it satisfies the Condition (16) and*

$$\left(\begin{array}{c}
\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \\
\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\
\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \geq min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \\
\mathfrak{B}_I(\mathfrak{u}_1)\} \\
\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \leq max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \\
\mathfrak{B}_F(\mathfrak{u}_1)\}
\end{array}\right) \quad (23)$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$.

Proof: Suppose that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Consequently, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$ by Theorem 3.4. Thus, Condition (16) is valid. Utilising (SB NSI 2), (SB NSI 3), (SB NSI 4), (3), (6), (8), and Condition (21), we obtain

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\
&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}, \\
&\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\} \\
&= min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)), \mathfrak{B}_I(\mathfrak{u}_1)\} \\
&\geq min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_I(\mathfrak{u}_1)\} \\
&= min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\}, \\
&\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\} \\
&= max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)), \mathfrak{B}_F(\mathfrak{u}_1)\} \\
&\leq max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_F(\mathfrak{u}_1)\} \\
&= max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\},
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$.

Conversely, let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$ that satisfies Conditions (16) and (23). Then,

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &= \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot 0) \\
&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot 0) \cdot 0) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}, \\
\mathfrak{B}_I(\mathfrak{p}_1) &= \mathfrak{B}_I(\mathfrak{p}_1 \cdot 0) \\
&\geq min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot 0) \cdot 0) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\} \\
&= min\{\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{u}_1)\}, \\
\mathfrak{B}_F(\mathfrak{p}_1) &= \mathfrak{B}_F(\mathfrak{p}_1 \cdot 0) \\
&\leq max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot 0) \cdot 0) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\} \\
&= max\{\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\},
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{u}_1 \in \mathcal{K}$. Thus, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. Taking $\mathfrak{u}_1 = 0$ in Condition (23) and utilising Equations (6) and (16), we obtain

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot 0), \widetilde{\mathfrak{B}}_T(0)\} \\ &= rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(0)\} \\ &= \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\geq min\{\mathfrak{B}_I(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot 0), \mathfrak{B}_I(0)\} \\ &= min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_I(0)\} \\ &= \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\leq max\{\mathfrak{B}_F(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot 0), \mathfrak{B}_F(0)\} \\ &= max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_F(0)\} \\ &= \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1),\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. It follows from Theorem 3.9 that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

Theorem 3.12. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS of $\mathcal{K}$. Then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ if and only if the following conditions hold for all $\mathfrak{p}_1, \mathfrak{r}_1, a, b \in \mathcal{K}$ $(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot a) \cdot b = 0$*

$$\Rightarrow \begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a), \widetilde{\mathfrak{B}}_T(b)\} \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \geq min\{\mathfrak{B}_I(a), \mathfrak{B}_I(b)\} \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \leq max\{\mathfrak{B}_F(a), \mathfrak{B}_F(b)\} \end{pmatrix} \quad (24)$$

Proof: Assume $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Consequently, by Theorem 3.4, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$.

Let $\mathfrak{p}_1, \mathfrak{r}_1, a, b \in \mathcal{K}$ be such that $(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot a) \cdot b = 0$. Utilising Theorem 3.9 and Lemma 3.7, we have

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a), \widetilde{\mathfrak{B}}_T(b)\}, \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \geq min\{\mathfrak{B}_I(a), \mathfrak{B}_I(b)\}, \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \leq max\{\mathfrak{B}_F(a), \mathfrak{B}_F(b)\}\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. Hence, Condition (24) holds in $\mathcal{K}$.

Conversely, let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$ that satisfies the Condition (24). Suppose $\mathfrak{p}_1, a, b \in \mathcal{K}$ satisfy the relation $(((\mathfrak{p}_1 \cdot 0) \cdot 0) \cdot a) \cdot b = 0$ and so

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &= \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot 0) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a), \widetilde{\mathfrak{B}}_T(b)\}, \\ \mathfrak{B}_I(\mathfrak{p}_1) &= \mathfrak{B}_I(\mathfrak{p}_1 \cdot 0) \geq min\{\mathfrak{B}_I(a), \mathfrak{B}_I(b)\}, \\ \mathfrak{B}_F(\mathfrak{p}_1) &= \mathfrak{B}_F(\mathfrak{p}_1 \cdot 0) \leq max\{\mathfrak{B}_F(a), \mathfrak{B}_F(b)\}.\end{aligned}$$

Thus, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$ by Lemma 3.8. Since $(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot ((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1)) \cdot 0 = 0$ for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, we have

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \widetilde{\mathfrak{B}}_T(0)\} \\ &= \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_I(0)\} \\ &= \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1), \mathfrak{B}_F(0)\} \\ &= \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1).\end{aligned}$$

It follows from Theorem 3.9 that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

Theorem 3.13. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS of $\mathcal{K}$. Then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ if and only if the following condition holds for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1, a, b \in \mathcal{K}$ such that $(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \cdot a) \cdot b = 0$*

$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a), \widetilde{\mathfrak{B}}_T(b)\} \\ \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \geq min\{\mathfrak{B}_I(a), \mathfrak{B}_I(b)\} \\ \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \leq max\{\mathfrak{B}_F(a), \mathfrak{B}_F(b)\} \end{pmatrix} \quad (25)$$

Proof: Assume that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Consequently, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$, by Theorem 3.4. Let $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1, a, b \in \mathcal{K}$ be such that $(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \cdot a) \cdot b = 0$. Utilising Theorem 3.10 and Lemma 3.7, we have

$$\begin{aligned}\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) &\succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a), \widetilde{\mathfrak{B}}_T(b)\}, \\ \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) &\geq \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\ &\geq min\{\mathfrak{B}_I(a), \mathfrak{B}_I(b)\}, \\ \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) &\leq \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \\ &\leq max\{\mathfrak{B}_F(a), \mathfrak{B}_F(b)\}.\end{aligned}$$

Conversely, let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$ that satisfies the Condition (25). Let $\mathfrak{p}_1, \mathfrak{r}_1, a, b \in \mathcal{K}$ be such that $(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot a) \cdot b = 0$. Then,

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &= \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)) \\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a), \widetilde{\mathfrak{B}}_T(b)\}, \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &= \mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)) \\ &\geq min\{\mathfrak{B}_I(a), \mathfrak{B}_I(b)\}, \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &= \mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{r}_1)) \\ &\leq max\{\mathfrak{B}_F(a), \mathfrak{B}_F(b)\}.\end{aligned}$$

It follows from Theorem 3.12 that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

Theorem 3.14. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS of $\mathcal{K}$. Then, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ if and only if the following condition holds*

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a_1), \widetilde{\mathfrak{B}}_T(a_2), \ldots, \widetilde{\mathfrak{B}}_T(a_n)\}, \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\geq min\{\mathfrak{B}_I(a_1), \mathfrak{B}_I(a_2), \ldots, \mathfrak{B}_I(a_n)\}, \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) &\leq max\{\mathfrak{B}_F(a_1), \mathfrak{B}_F(a_2), \ldots, \mathfrak{B}_F(a_n)\},\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, a_1, a_2, \ldots, a_n \in \mathcal{K}$, with
$(\ldots(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{r}_1) \cdot a_1) \cdot a_2 \ldots) \cdot a_n = 0.$

Theorem 3.15. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS of $\mathcal{K}$. Then, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ if and only if the following condition holds*

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \\ &\qquad \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(a_1), \widetilde{\mathfrak{B}}_T(a_2), \ldots, \widetilde{\mathfrak{B}}_T(a_n)\}, \\ &\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \\ &\qquad \geq min\{\mathfrak{B}_I(a_1), \mathfrak{B}_I(a_2), \ldots, \mathfrak{B}_I(a_n)\}, \\ &\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{u}_1) \cdot (\mathfrak{r}_1 \cdot \mathfrak{u}_1)) \\ &\qquad \leq max\{\mathfrak{B}_F(a_1), \mathfrak{B}_F(a_2), \ldots, \mathfrak{B}_F(a_n)\},\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, a_1, a_2 \ldots, a_n \in \mathcal{K}$, with
$(\ldots(((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) \cdot a_1) \cdot a_2 \ldots) \cdot a_n = 0.$

Theorem 3.16. *An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ is a PISB-NSI of $\mathcal{K}$ if and only if the non-empty sets*

$\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$, $\mathcal{U}(\mathfrak{B}_I;m)$, *and* $\mathcal{L}(\mathfrak{B}_F;n)$ *are positive implicative ideals of* $\mathcal{K}$ *for all* $m,n\in[0,1]$ *and* $[l_1,l_2]\in[I]$.

Proof: Suppose that $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Let $m,n\in[0,1]$ and $[l_1,l_2]\in[I]$ be such that $\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$, $\mathcal{U}(\mathfrak{B}_I;m)$, and $\mathcal{L}(\mathfrak{B}_F;n)$ are non-empty. Obviously,
$0\in\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$, $0\in\mathcal{U}(\mathfrak{B}_I;m)$, and $0\in\mathcal{L}(\mathfrak{B}_F;n)$.

For any $a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2,c_3\in\mathcal{K}$ if $(a_1\cdot a_2)\cdot a_3$, $a_2\cdot a_3\in\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$, $(b_1\cdot b_2)\cdot b_3$, $b_2\cdot b_3\in\mathcal{U}(\mathfrak{B}_I;m)$, and $(c_1\cdot c_2)\cdot c_3$, $c_2\cdot c_3\in\mathcal{L}(\mathfrak{B}_F;n)$ then

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(a_1\cdot a_3)&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((a_1\cdot a_2)\cdot a_3),\widetilde{\mathfrak{B}}_T(a_2\cdot a_3)\}\\&\succcurlyeq rmin\{[l_1,l_2],[l_1,l_2]\}=[l_1,l_2],\\ \mathfrak{B}_I(b_1\cdot b_3)&\geq min\{\mathfrak{B}_I((b_1\cdot b_2)\cdot b_3),\mathfrak{B}_I(b_2\cdot b_3)\}\\&\geq min\{m,m\}=m,\\ \mathfrak{B}_F(c_1\cdot c_3)&\leq max\{\mathfrak{B}_F((c_1\cdot c_2)\cdot c_3),\mathfrak{B}_F(c_2\cdot c_3)\}\\&\leq max\{n,n\}=n.\end{aligned}$$

Therefore, $a_1\cdot a_3\in\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$, $b_1\cdot b_3\in\mathcal{U}(\mathfrak{B}_I;m)$, and $c_1\cdot c_3\in\mathcal{L}(\mathfrak{B}_F;n)$. Hence, $\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$, $\mathcal{U}(\mathfrak{B}_I;m)$, and $\mathcal{L}(\mathfrak{B}_F;n)$ are positive implicative ideals of $\mathcal{K}$.

Conversely, assume that the non-empty sets $\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2]),\mathcal{U}(\mathfrak{B}_I;m)$, and $\mathcal{L}(\mathfrak{B}_F;n)$ are positive implicative ideals of $\mathcal{K}$ for all $m,n\in[0,1]$ and $[l_1,l_2]\in[I]$.
Suppose that

$$\widetilde{\mathfrak{B}}_T(0)\prec\widetilde{\mathfrak{B}}_T(a),\mathfrak{B}_I(0)<\mathfrak{B}_I(a),\text{ and }\mathfrak{B}_F(0)>\mathfrak{B}_F(a)$$

for some $a\in\mathcal{K}$. Then,

$$0\notin\mathcal{U}(\widetilde{\mathfrak{B}}_T;\widetilde{\mathfrak{B}}_T(a))\cap\mathcal{U}(\mathfrak{B}_I;\mathfrak{B}_I(a))\cap\mathcal{L}(\mathfrak{B}_F;\mathfrak{B}_F(a))$$

which is a contradiction. Hence,

$$\widetilde{\mathfrak{B}}_T(0)\succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1),\mathfrak{B}_I(0)\geq\mathfrak{B}_I(\mathfrak{p}_1),\mathfrak{B}_F(0)\leq\mathfrak{B}_F(\mathfrak{p}_1)$$

for all $\mathfrak{p}_1\in\mathcal{K}$. Suppose that

$$\widetilde{\mathfrak{B}}_T(a_0\cdot c_0)\prec rmin\{\widetilde{\mathfrak{B}}_T((a_0\cdot b_0)\cdot c_0),\widetilde{\mathfrak{B}}_T(b_0\cdot c_0)\}$$

for some $a_0,b_0,c_0\in\mathcal{K}$. Let $\widetilde{\mathfrak{B}}_T((a_0\cdot b_0)\cdot c_0)=[\delta_1,\delta_2]$, $\widetilde{\mathfrak{B}}_T(b_0\cdot c_0)=[\delta_3,\delta_4]$, and $\widetilde{\mathfrak{B}}_T(a_0\cdot c_0)=[l_1,l_2]$. Then

$$\begin{aligned}[l_1,l_2]&\prec rmin\{[\delta_1,\delta_2],[\delta_3,\delta_4]\\&=[min\{\delta_1,\delta_3\},min\{\delta_2,\delta_4\}]\text{ and so}\end{aligned}$$

$l_1\prec min\{\delta_1,\delta_3\}$ and $l_2\prec min\{\delta_2,\delta_4\}$. Taking

$$\begin{aligned}[\eta_1,\eta_2]&=\frac{1}{2}[\widetilde{\mathfrak{B}}_T(a_0\cdot c_0)\\&\quad+rmin\{\widetilde{\mathfrak{B}}_T((a_0\cdot b_0)\cdot c_0),\widetilde{\mathfrak{B}}_T(b_0\cdot c_0)\}]\\&=\frac{1}{2}[[l_1,l_2]+[min\{\delta_1,\delta_3\},min\{\delta_2,\delta_4\}]]\\&=[\frac{1}{2}(l_1+min\{\delta_1,\delta_3\}),\frac{1}{2}(l_2+min\{\delta_2,\delta_4\})].\end{aligned}$$

It follows that

$$\begin{aligned}l_1<\eta_1&=\frac{1}{2}(l_1+min\{\delta_1,\delta_3\})<min\{\delta_1,\delta_3\},\\ l_2<\eta_2&=\frac{1}{2}(l_2+min\{\delta_2,\delta_4\})<min\{\delta_2,\delta_4\}.\end{aligned}$$

Hence,
$[min\{\delta_1,\delta_3\},min\{\delta_2,\delta_4\}]\succ[\eta_1,\eta_2]\succ[l_1,l_2]=\widetilde{\mathfrak{B}}_T(a_0)$.
Therefore $a_0\cdot c_0\notin\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$. On the other hand,

$$\begin{aligned}\widetilde{\mathfrak{B}}_T((a_0\cdot b_0)\cdot c_0)=[\delta_1,\delta_2]&\succcurlyeq[min\{\delta_1,\delta_3\},min\{\delta_2,\delta_4\}]\\&\succ[\eta_1,\eta_2],\\ \widetilde{\mathfrak{B}}_T(b_0\cdot c_0)=[\delta_3,\delta_4]&\succcurlyeq[min\{\delta_1,\delta_3\},min\{\delta_2,\delta_4\}]\\&\succ[\eta_1,\eta_2].\end{aligned}$$

i.e., $((a_0\cdot b_0)\cdot c_0)$, $(b_0\cdot c_0)\in\mathcal{U}(\widetilde{\mathfrak{B}}_T;[l_1,l_2])$. This is a contradiction, and therefore $\widetilde{\mathfrak{B}}_T(a_0\cdot c_0)\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((a_0\cdot b_0)\cdot c_0),\widetilde{\mathfrak{B}}_T(b_0\cdot c_0)\}$ for all $a_0,b_0,c_0\in\mathcal{K}$. Also, if

$$\mathfrak{B}_I(a_0\cdot c_0)<min\{\mathfrak{B}_I((a_0\cdot b_0)\cdot c_0),\mathfrak{B}_I(b_0\cdot c_0)\}$$

for some $a_0,b_0,c_0\in\mathcal{K}$, then $(a_0\cdot b_0)\cdot c_0$, $(b_0\cdot c_0)\in\mathcal{U}(\mathfrak{B}_I;m_0)$, but $a_0\cdot c_0\notin\mathcal{U}(\mathfrak{B}_I;m_0)$ for $m_0=min\{\mathfrak{B}_I((a_0\cdot b_0)\cdot c_0),\mathfrak{B}_I(b_0\cdot c_0)\}$.
This is a contradiction, and thus

$$\mathfrak{B}_I(a_0\cdot c_0)\geq min\{\mathfrak{B}_I((a_0\cdot b_0)\cdot c_0),\mathfrak{B}_I(b_0\cdot c_0)\}$$

for all $a_0,b_0,c_0\in\mathcal{K}$. Similarly, we can show that

$$\mathfrak{B}_F(a_0\cdot c_0)\leq max\{\mathfrak{B}_F((a_0\cdot b_0)\cdot c_0),\mathfrak{B}_F(b_0\cdot c_0)\}$$

for all $\mathfrak{p}_1,\mathfrak{r}_1\in\mathcal{K}$. Consequently, $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

Theorem 3.17 (Extension property for PISB-NSI). *Let* $\mathcal{M}=(\widetilde{\mathcal{M}}_T,\mathcal{M}_I,\mathcal{M}_F)$ *and* $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ *be two SB-NSIs of* $\mathcal{K}$ *with the condition that* $\mathcal{M}(0)=\mathfrak{B}(0)$ *and* $\mathcal{M}\subseteq\mathfrak{B}$. *i.e.,* $\widetilde{\mathcal{M}}_T(0)=\widetilde{\mathfrak{B}}_T(0)$, $\mathcal{M}_I(0)=\mathfrak{B}_I(0)$, $\mathcal{M}_F(0)=\mathfrak{B}_F(0)$, *and* $\widetilde{\mathcal{M}}_T(\mathfrak{p}_1)\preccurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathcal{M}_I(\mathfrak{p}_1)\leq\mathfrak{B}_I(\mathfrak{p}_1)$, $\mathcal{M}_F(\mathfrak{p}_1)\geq\mathfrak{B}_F(\mathfrak{p}_1)$ *for all* $\mathfrak{p}_1\in\mathcal{K}$. *If* $\mathcal{M}=(\widetilde{\mathcal{M}}_T,\mathcal{M}_I,\mathcal{M}_F)$ *is a PISB-NSI of* $\mathcal{K}$, *then so is* $\mathfrak{B}$.

Proof: Suppose that $\mathcal{M}=(\widetilde{\mathcal{M}}_T,\mathcal{M}_I,\mathcal{M}_F)$ is a PISB-NSI of $\mathcal{K}$. Utilising Equation (8) and Equation (3), we obtain

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\\&=\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&=\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\succcurlyeq\widetilde{\mathcal{M}}_T(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\succcurlyeq\widetilde{\mathcal{M}}_T(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\\&=\widetilde{\mathcal{M}}_T(((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\\&=\widetilde{\mathcal{M}}_T(((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\\&=\widetilde{\mathcal{M}}_T(0)=\widetilde{\mathfrak{B}}_T(0).\end{aligned}$$

It follows from Definition 3.1 that

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\succcurlyeq min\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)),\\&\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\}\\&\succcurlyeq min\{\widetilde{\mathfrak{B}}_T(0),\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\}=\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1).\\&\Rightarrow\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\succcurlyeq\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\end{aligned}$$

for any $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$.

$$\begin{aligned}&\mathfrak{B}_I(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\\&\quad=\mathfrak{B}_I(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\quad=\mathfrak{B}_I(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\quad\geq\mathcal{M}_I(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\quad\geq\mathcal{M}_I(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\\&\quad=\mathcal{M}_I(((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\\&\quad=\mathcal{M}_I(((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\\&\quad=\mathcal{M}_I(0)=\mathfrak{B}_I(0).\end{aligned}$$

It follows from Definition 3.1 that

$$\begin{aligned}&\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\geq min\{\mathfrak{B}_I(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)),\\&\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\}\\&\geq min\{\mathfrak{B}_I(0),\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\}=\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1).\\&\Rightarrow\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\geq\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\\&\text{for any }\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}.\end{aligned}$$

$$\begin{aligned}&\mathfrak{B}_F(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\\&\quad=\mathfrak{B}_F(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\quad=\mathfrak{B}_F(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\quad\leq\mathcal{M}_F(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\quad\leq\mathcal{M}_F(((\mathfrak{p}_1\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\\&\quad=\mathcal{M}_F(((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\cdot\mathfrak{u}_1)\\&\quad=\mathcal{M}_F(((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1))\\&\quad=\mathcal{M}_F(0)=\mathfrak{B}_F(0).\end{aligned}$$

It follows from Definition 3.1 that

$$\begin{aligned}&\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\\&\leq max\{\mathfrak{B}_F(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\cdot((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)),\\&\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\}\\&\leq max\{\mathfrak{B}_F(0),\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\}=\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1).\\&\Rightarrow\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\leq\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\\&\text{for any }\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}.\end{aligned}$$

Hence, $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

Corollary 3.18. *In $\mathcal{K}$, every PISB-NSI of $\mathcal{K}$ is an SB-NSSA.*

Theorem 3.19. *If $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI in $\mathcal{K}$, and $\mathcal{J}(0)=\{\mathfrak{p}_1\in\mathcal{K}|\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)=\widetilde{\mathfrak{B}}_T(0),\mathfrak{B}_I(\mathfrak{p}_1)=\mathfrak{B}_I(0),\mathfrak{B}_F(\mathfrak{p}_1)=\mathfrak{B}_F(0)\}$, then $\mathcal{J}(0)$ is a positive implicative ideal of $\mathcal{K}$.*

Proof: Let $\mathfrak{p}_1,\mathfrak{r}_1\in\mathcal{K}$ such that $(\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1,\mathfrak{r}_1\cdot\mathfrak{u}_1\in\mathcal{J}(0)$. Since $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$, we have

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&=min\{\widetilde{\mathfrak{B}}_T(0),\widetilde{\mathfrak{B}}_T(0)\}=\widetilde{\mathfrak{B}}_T(0),\\\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&=min\{\mathfrak{B}_I(0),\mathfrak{B}_I(0)\}=\mathfrak{B}_I(0),\\\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&=max\{\mathfrak{B}_F(0),\mathfrak{B}_F(0)\}=\mathfrak{B}_F(0).\end{aligned}$$

On the other hand, we know from Condition (16), $\widetilde{\mathfrak{B}}_T(0)\succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0)\geq\mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(0)\leq\mathfrak{B}_F(\mathfrak{p}_1)$ for all $\mathfrak{p}_1\in\mathcal{K}$. Thus, $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1)=\widetilde{\mathfrak{B}}_T(0)$, $\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1)=\mathfrak{B}_I(0)$, and $\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)=\mathfrak{B}_F(0)$. This implies $\mathfrak{p}_1\cdot\mathfrak{u}_1\in\mathcal{J}(0)$. Obviously, $0\in\mathcal{J}(0)$. Therefore, $\mathcal{J}(0)$ is a positive implicative ideal of $\mathcal{K}$. ■

Theorem 3.20. *In a positive implicative BCK-algebra $\mathcal{K}$, every SB-NSI of $\mathcal{K}$ is a PISB-NSI of $\mathcal{K}$.*

Proof: Let $\mathcal{K}$ be a positive implicative BCK-algebra, and $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. If we replace $\mathfrak{p}_1$ with $\mathfrak{p}_1\cdot\mathfrak{u}_1$ and $\mathfrak{r}_1$ with $\mathfrak{r}_1\cdot\mathfrak{u}_1$ in (SB-NSI 2), (SB-NSI 3), (SB-NSI 4), then we obtain

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)),\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&=rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{u}_1)\},\\\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)),\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&=min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{u}_1)\},\\\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)),\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&=max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\},\end{aligned}$$

for all $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$. Obviously,

$$\widetilde{\mathfrak{B}}_T(0)\succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1),\mathfrak{B}_I(0)\geq\mathfrak{B}_I(\mathfrak{p}_1),\mathfrak{B}_F(0)\leq\mathfrak{B}_F(\mathfrak{p}_1)$$

for all $\mathfrak{p}_1\in\mathcal{K}$. Therefore, $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

Theorem 3.21. *If $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$, then $\widetilde{\mathfrak{B}}_T$ is an interval-valued fuzzy positive implicative ideal, while $\mathfrak{B}_I$ and $\overline{\mathfrak{B}}_F$ are fuzzy positive implicative ideal of $\mathcal{K}$.*

Proof: Suppose $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Then, for all $\mathfrak{p}_1\in\mathcal{K}$, $\widetilde{\mathfrak{B}}_T(0)\succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0)\geq\mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(0)\leq\mathfrak{B}_F(\mathfrak{p}_1)\Rightarrow-\mathfrak{B}_F(0)\geq-\mathfrak{B}_F(\mathfrak{p}_1)\Rightarrow 1-\mathfrak{B}_F(0)\geq 1-\mathfrak{B}_F(\mathfrak{p}_1)\Rightarrow\overline{\mathfrak{B}}_F(0)\geq\overline{\mathfrak{B}}_F(\mathfrak{p}_1)$. Also,

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)&\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&\Rightarrow-\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)\geq-max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\\&\qquad\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&\Rightarrow 1-\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)\geq 1-max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\\&\qquad\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&\Rightarrow\overline{\mathfrak{B}}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)\geq min\{1-\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\\&\qquad 1-\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\&\Rightarrow\overline{\mathfrak{B}}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1)\geq min\{\overline{\mathfrak{B}}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\\&\qquad\overline{\mathfrak{B}}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\end{aligned}$$

Therefore, $\widetilde{\mathfrak{B}}_T$ is an interval-valued fuzzy positive implicative ideal, while $\mathfrak{B}_I$ and $\overline{\mathfrak{B}}_F$ are fuzzy positive implicative ideal of $\mathcal{K}$. ■

Theorem 3.22. *Let $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ be an SB-NSI of $\mathcal{K}$. Then $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$ iff it satisfies the condition*

$$\begin{pmatrix}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{r}_1)=\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\\\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{r}_1)=\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\\\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{r}_1)=\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\end{pmatrix}\ \textit{for all }\mathfrak{p}_1,\mathfrak{r}_1\in\mathcal{K}.\tag{26}$$

Proof: Assume that $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. Put $\mathfrak{u}_1=\mathfrak{r}_1$ in the Definition 3.1, then we get

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{r}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{r}_1)\}\\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\widetilde{\mathfrak{B}}_T(0)\}\\ &\succcurlyeq \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\\ \mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{r}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{r}_1)\}\\ &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\mathfrak{B}_I(0)\}\\ &\geq \mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\\ \mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{r}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{r}_1)\}\\ &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1),\mathfrak{B}_F(0)\}\\ &\leq \mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1).\end{aligned}$$

Therefore,

$$\begin{pmatrix}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{r}_1)\preccurlyeq\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\\ \mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{r}_1)\leq\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\\ \mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{r}_1)\geq\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\end{pmatrix}\text{ for all }\mathfrak{p}_1,\mathfrak{r}_1\in\mathcal{K}. \tag{27}$$

Since $(\mathfrak{p}_1\cdot\mathfrak{r}_1)\leq\mathfrak{u}_1\Rightarrow(\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1\leq\mathfrak{p}_1\cdot\mathfrak{r}_1$, by Corollary 3.6,

$$\begin{pmatrix}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{r}_1)\succcurlyeq\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\\ \mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{r}_1)\geq\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\\ \mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{r}_1)\leq\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{r}_1)\end{pmatrix}\text{ for all }\mathfrak{p}_1,\mathfrak{r}_1\in\mathcal{K}. \tag{28}$$

From Conditions (27) and (28), we conclude that Condition (26) holds in $\mathcal{K}$.

Conversely, let $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ be an SB-NSI of $\mathcal{K}$ satisfying the Condition (26). Since $\mathcal{K}$ is a BCK-algebra, we have $((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)\leq(\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1$ for all $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$. By Corollary 3.6, we have

$$\begin{pmatrix}\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\succcurlyeq\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\\ \mathfrak{B}_I(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\geq\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\\ \mathfrak{B}_F(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1))\leq\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1)\end{pmatrix} \tag{29}$$

Now, by using Condition (26), Definition 2.15, and Condition (29), we obtain

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1) &= \widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)),\\ &\quad\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1) &= \mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\\ &\geq min\{\mathfrak{B}_I(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)),\\ &\quad\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\ &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{r}_1\cdot\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1) &= \mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\\ &\leq max\{\mathfrak{B}_F(((\mathfrak{p}_1\cdot\mathfrak{u}_1)\cdot\mathfrak{u}_1)\cdot(\mathfrak{r}_1\cdot\mathfrak{u}_1)),\\ &\quad\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}\\ &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot\mathfrak{r}_1)\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{r}_1\cdot\mathfrak{u}_1)\}.\end{aligned}$$

Thus, $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is a PISB-NSI of $\mathcal{K}$. ■

IV. Implicative SB-neutrosophic ideal (ISB-NSI)

Definition 4.1. *Let $\mathcal{K}$ be a BCK-algebra. An SB-NSS $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ of $\mathcal{K}$ is called an ISB-NSI of $\mathcal{K}$ if it satisfies (16) and*

$$\begin{pmatrix}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}\\ \mathfrak{B}_I(\mathfrak{p}_1)\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\}\\ \mathfrak{B}_F(\mathfrak{p}_1)\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\}\end{pmatrix} \tag{30}$$

for all $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$.

Example 4.2. *Consider a BCK-algebra $\mathcal{K}=\{0,1,2,3,4\}$ as given in Table VII.*

TABLE VII
BCK-ALGEBRA

·	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Let us define an SB-NSS $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ in $\mathcal{K}$ as shown in the Table VIII. By routine calculations we can show

TABLE VIII
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	[0.57, 0.73]	0.93	0.41
1	[0.57, 0.73]	0.93	0.41
2	[0.57, 0.73]	0.93	0.41
3	[0.35, 0.61]	0.77	0.79
4	[0.35, 0.61]	0.77	0.79

that $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$.

Theorem 4.3. *Every ISB-NSI of $\mathcal{K}$ is an SB-NSI of $\mathcal{K}$.*

Proof: Let $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. If we put $\mathfrak{r}_1=0$ in (30), then we obtain

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(0\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}\\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(0\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\}\\ &\geq min\{\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(0\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\}\\ &\leq max\{\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\},\end{aligned}$$

for all $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$. This shows that $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. ■

The following example shows that the converse of Theorem 4.3 is not true in general.

Example 4.4. *Consider a BCK-algebra $\mathcal{K}=\{0,1,2,3\}$ as given in Table IX.*

Let us define an SB-NSS $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ in $\mathcal{K}$ as shown in the Table X.

TABLE IX
BCK-ALGEBRA

$\cdot$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

TABLE X
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	[0.35, 0.87]	0.49	0.77
1	[0.19, 0.53]	0.11	0.95
2	[0.55, 0.87]	0.49	0.77
3	[0.19, 0.53]	0.11	0.95

It is easy to check that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$, but it is not an ISB-NSI of $\mathcal{K}$ because of the following reasons

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(1) &= [0.19, 0.53] \prec [0.55, 0.87] = \widetilde{\mathfrak{B}}_T(2)\\ &= rmin\{\widetilde{\mathfrak{B}}_T((1\cdot(3\cdot 1))\cdot 2), \widetilde{\mathfrak{B}}_T(2)\}\\ \mathfrak{B}_I(1) &= 0.11 < 0.49 = \mathfrak{B}_I(2)\\ &= min\{\mathfrak{B}_I((1\cdot(3\cdot 1))\cdot 2), \mathfrak{B}_I(2)\},\\ \mathfrak{B}_F(1) &= 0.95 > 0.77 = \mathfrak{B}_F(2)\\ &= max\{\mathfrak{B}_F((1\cdot(3\cdot 1))\cdot 2), \mathfrak{B}_F(2)\}.\end{aligned}$$

Theorem 4.5. *If $\mathcal{K}$ is an implicative BCK-algebra, then every SB-NSI of $\mathcal{K}$ is an ISB-NSI.*

Proof: Assume $\mathcal{K}$ is an implicative BCK-algebra and $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. Utilising Definition 4.1 and Equation (13), we obtain

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot\mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}\\ \Rightarrow\ &\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ &\mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I(\mathfrak{p}_1\cdot\mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\}\\ \Rightarrow\ &\mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\ &\mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F(\mathfrak{p}_1\cdot\mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}\\ \Rightarrow\ &\mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

It follows that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. ■

Corollary 4.6. *In $\mathcal{K}$, every ISB-NSI is an SB-NSSA.*

Corollary 4.7. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an ISB-NSI of $\mathcal{K}$. If $\mathfrak{p}_1 \leq \mathfrak{r}_1$ in $\mathcal{K}$, then $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1)$, $\mathfrak{B}_I(\mathfrak{p}_1) \geq \mathfrak{B}_I(\mathfrak{r}_1)$, and $\mathfrak{B}_F(\mathfrak{p}_1) \leq \mathfrak{B}_F(\mathfrak{r}_1)$ i.e., $\widetilde{\mathfrak{B}}_T$ and $\mathfrak{B}_I$ are order-reversing and $\mathfrak{B}_F$ are order-preserving.*

Theorem 4.8. *If $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$ and*

$$\begin{aligned}\mathfrak{G}(0) = \{\mathfrak{p}_1 \in \mathcal{K} \mid\ &\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) = \widetilde{\mathfrak{B}}_T(0), \mathfrak{B}_I(\mathfrak{p}_1) = \mathfrak{B}_I(0),\\ &\mathfrak{B}_F(\mathfrak{p}_1) = \mathfrak{B}_F(0)\}.\end{aligned}$$

Then $\mathfrak{G}(0)$ is an implicative ideal of $\mathcal{K}$.

Proof: Let $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$ such that $(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1, \mathfrak{u}_1 \in \mathfrak{G}(0)$. Since $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$, we have

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}\\ &= rmin\{\widetilde{\mathfrak{B}}_T(0), \widetilde{\mathfrak{B}}_T(0)\} = \widetilde{\mathfrak{B}}_T(0),\\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\}\\ &= min\{\mathfrak{B}_I(0), \mathfrak{B}_I(0)\} = \mathfrak{B}_I(0),\\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}\\ &= max\{\mathfrak{B}_F(0), \mathfrak{B}_F(0)\} = \mathfrak{B}_F(0).\end{aligned}$$

On the other hand, we know from Equation (16) that $\widetilde{\mathfrak{B}}_T(0) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0) \geq \mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(0) \leq \mathfrak{B}_F(\mathfrak{p}_1)$ for all $\mathfrak{p}_1 \in \mathcal{K}$. Thus, $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) = \widetilde{\mathfrak{B}}_T(0)$, $\mathfrak{B}_I(\mathfrak{p}_1) = \mathfrak{B}_I(0)$, and $\mathfrak{B}_F(\mathfrak{p}_1) = \mathfrak{B}_F(0)$. This implies $\mathfrak{p}_1 \in \mathfrak{G}(0)$. Obviously, $0 \in \mathfrak{G}(0)$. Therefore, $\mathfrak{G}(0)$ is an implicative ideal of $\mathcal{K}$. ■

Theorem 4.9. *An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ of $\mathcal{K}$ is an implicative SB-NSI of $\mathcal{K}$ if and only if the non-empty sets $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])$, $\mathcal{U}(\mathfrak{B}_I; m)$, and $\mathcal{L}(\mathfrak{B}_F; n)$ are implicative ideals of $\mathcal{K}$ for all $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$.*

Proof: The proof of the theorem follows a similar approach to the proof provided in Theorem 3.16. ■

Theorem 4.10 (Extension property for ISB-NSI)**.** *Let $\mathcal{M} = (\widetilde{\mathcal{M}}_T, \mathcal{M}_I, \mathcal{M}_F)$ and $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be two SB-NSIs of $\mathcal{K}$ with the condition that $\mathcal{M}(0) = \mathfrak{B}(0)$ and $\mathcal{M} \subseteq \mathfrak{B}$. i.e., $\widetilde{\mathcal{M}}_T(0) = \widetilde{\mathfrak{B}}_T(0)$, $\mathcal{M}_I(0) = \mathfrak{B}_I(0)$, $\mathcal{M}_F(0) = \mathfrak{B}_F(0)$, and $\widetilde{\mathcal{M}}_T(\mathfrak{p}_1) \preccurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathcal{M}_I(\mathfrak{p}_1) \leq \mathfrak{B}_I(\mathfrak{p}_1)$, $\mathcal{M}_F(\mathfrak{p}_1) \geq \mathfrak{B}_F(\mathfrak{p}_1)$ for all $\mathfrak{p}_1 \in \mathcal{K}$.*

If $\mathcal{M} = (\widetilde{\mathcal{M}}_T, \mathcal{M}_I, \mathcal{M}_F)$ is an ISB-NSI of $\mathcal{K}$, then so is $\mathfrak{B}$.

Proof: To prove that $\mathfrak{B}$ is an ISB-NSI of $\mathcal{K}$, it is sufficient to show that for any $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$ the level subsets $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])$, $\mathcal{U}(\mathfrak{B}_I; m)$, and $\mathcal{L}(\mathfrak{B}_F; n)$ are either empty or an implicative ideals of $\mathcal{K}$. If the level subset $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])$, $\mathcal{U}(\mathfrak{B}_I; m)$, and $\mathcal{L}(\mathfrak{B}_F; n)$ are non-empty then

$$\mathcal{U}(\widetilde{\mathcal{M}}_T; [l_1, l_2]) \neq \phi,\ \mathcal{U}(\mathcal{M}_I; m) \neq \phi,\ \mathcal{L}(\mathcal{M}_F; n) \neq \phi,$$

and $\begin{pmatrix}\mathcal{U}(\widetilde{\mathcal{M}}_T; [l_1, l_2]) \subseteq \mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])\\ \mathcal{U}(\mathcal{M}_I; m) \subseteq \mathcal{U}(\mathfrak{B}_I; m)\\ \mathcal{L}(\mathcal{M}_F; n) \subseteq \mathcal{L}(\mathfrak{B}_F; n)\end{pmatrix}$. In fact, if

$$\begin{pmatrix}\mathfrak{p}_1 \in \mathcal{U}(\widetilde{\mathcal{M}}_T; [l_1, l_2])\\ \Rightarrow \widetilde{\mathcal{M}}_T(\mathfrak{p}_1) \succcurlyeq [l_1, l_2]\\ \Rightarrow \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq [l_1, l_2]\\ \Rightarrow \mathfrak{p}_1 \in \mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])\end{pmatrix}, \quad \begin{pmatrix}\mathfrak{p}_1 \in \mathcal{U}(\mathcal{M}_I; m)\\ \Rightarrow \mathcal{M}_I(\mathfrak{p}_1) \geq m\\ \Rightarrow \mathfrak{B}_I(\mathfrak{p}_1) \geq m\\ \Rightarrow \mathfrak{p}_1 \in \mathcal{U}(\mathfrak{B}_I; m)\end{pmatrix}$$

and $\begin{pmatrix}\mathfrak{p}_1 \in \mathcal{U}(\mathcal{M}_F; n)\\ \Rightarrow \mathcal{M}_F(\mathfrak{p}_1) \leq n\\ \Rightarrow \mathfrak{B}_F(\mathfrak{p}_1) \leq n\\ \Rightarrow \mathfrak{p}_1 \in \mathcal{L}(\mathfrak{B}_F; n)\end{pmatrix}$.

By hypothesis $\mathcal{M} = (\widetilde{\mathcal{M}}_T, \mathcal{M}_I, \mathcal{M}_F)$ is an ISB-NSI of $\mathcal{K}$. By the Theorem 4.9, the non-empty level subsets $\mathcal{U}(\widetilde{\mathcal{M}}_T; [l_1, l_2])$, $\mathcal{U}(\mathcal{M}_I; m)$, and $\mathcal{L}(\mathcal{M}_F; n)$ are implicative ideals of $\mathcal{K}$, for any $m, n \in [0, 1]$ and $[l_1, l_2] \in [I]$. By Proposition 2.7, $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])$, $\mathcal{U}(\mathfrak{B}_I; m)$, and $\mathcal{L}(\mathfrak{B}_F; n)$ are implicative ideals of $\mathcal{K}$. It follows again from Theorem 4.9, that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. ■

Theorem 4.11. *An SB-NSI* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *is an ISB-NSI of* $\mathcal{K}$ *iff the following conditions hold*

$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_I(\mathfrak{p}_1) \geq \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_F(\mathfrak{p}_1) \leq \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \end{pmatrix} \text{ for all } \mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}. \tag{31}$$

Proof: Suppose that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. Put $\mathfrak{u}_1 = 0$ in Equation (30). Then, we obtain

$$\begin{aligned} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot 0), \widetilde{\mathfrak{B}}_T(0)\} \\ &= \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)), \\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot 0), \mathfrak{B}_I(0)\} \\ &= \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)), \\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot 0), \mathfrak{B}_F(0)\} \\ &= \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)), \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.
Conversely, assume that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ satisfies Condition (31), for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. Since $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$,

$$\begin{aligned} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}, \\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\}, \\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}, \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. ■

Theorem 4.12. *An SB-NSSA* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *of* $\mathcal{K}$ *is an ISB-NSI of* $\mathcal{K}$ *iff the following condition holds*

$$(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1 \leq \mathfrak{w} \Rightarrow$$
$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{w})\} \\ \mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I(\mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{w})\} \\ \mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F(\mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{w})\} \end{pmatrix} \tag{32}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1, \mathfrak{w} \in \mathcal{K}$.

Proof: Assume that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. Let $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1, \mathfrak{w} \in \mathcal{K}$ be such that $(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1 \leq \mathfrak{w}$. By Theorem 4.3, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. It follows from Theorem 4.11 and Lemma 3.7, that we have

$$\begin{aligned} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{w})\} \\ \mathfrak{B}_I(\mathfrak{p}_1) \geq \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \geq min\{\mathfrak{B}_I(\mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{w})\} \\ \mathfrak{B}_F(\mathfrak{p}_1) \leq \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \leq max\{\mathfrak{B}_F(\mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{w})\} \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1, \mathfrak{w} \in \mathcal{K}$.

Conversely, assume that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSSA of $\mathcal{K}$ and satisfies the Condition (32).

If we take $\mathfrak{p}_1 = 0$ and $\mathfrak{u}_1 = \mathfrak{w} = \mathfrak{p}_1$ in (32), then we can obtain $\widetilde{\mathfrak{B}}_T(0) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0) \geq \mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(0) \leq \mathfrak{B}_F(\mathfrak{p}_1)$ for all $\mathfrak{p}_1 \in \mathcal{K}$.

Since $(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot ((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1) \leq \mathfrak{u}_1$, it follows from the hypothesis

$$\begin{aligned} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\} \\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\} \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. ■

Theorem 4.13. *Let* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *is an SB-NSI of* $\mathcal{K}$. *Then the following are equivalent*

(a) $\mathfrak{B}$ *is an ISB-NSI.*

(b) $$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_I(\mathfrak{p}_1) \geq \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_F(\mathfrak{p}_1) \leq \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \end{pmatrix} \text{ for all } \mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}.$$

(c) $$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) = \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_I(\mathfrak{p}_1) = \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_F(\mathfrak{p}_1) = \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \end{pmatrix} \text{ for all } \mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}.$$

Proof: $(a) \Rightarrow (b)$
Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. Put $\mathfrak{u}_1 = 0$ in Equation (30), we obtain

$$\begin{aligned} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot 0), \widetilde{\mathfrak{B}}_T(0)\} \\ &= \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)), \\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot 0), \mathfrak{B}_I(0)\} \\ &= \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)), \\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot 0), \mathfrak{B}_F(0)\} \\ &= \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)), \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. Hence, the Condition (b) holds.

$(b) \Rightarrow (c)$
Observe that for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, $\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1) \leq \mathfrak{p}_1$. Using Corollary 4.7, we have $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \geq \mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \leq \mathfrak{B}_F(\mathfrak{p}_1)$.

It follows from (b) that $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) = \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))$, $\mathfrak{B}_I(\mathfrak{p}_1) = \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))$, and $\mathfrak{B}_F(\mathfrak{p}_1) = \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))$. Hence, the Condition (c) holds.

$(c) \Rightarrow (a)$
Since $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$, we have

$$\begin{aligned} &\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ &\qquad \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}, \\ &\mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ &\qquad \geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\}, \\ &\mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ &\qquad \leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}, \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Using (c) we obtain

$$\begin{aligned} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\} \\ \mathfrak{B}_I(\mathfrak{p}_1) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\} \\ \mathfrak{B}_F(\mathfrak{p}_1) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\} \end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Obviously, $\mathfrak{B}$ satisfies $\widetilde{\mathfrak{B}}_T(0) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0) \geq \mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(0) \leq \mathfrak{B}_F(\mathfrak{p}_1)$ for all $\mathfrak{p}_1 \in \mathcal{K}$.

Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. ■

Theorem 4.14. *Every ISB-NSI* $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ *of* $\mathcal{K}$ *satisfies*

$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \geq \mathfrak{B}_I(\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \leq \mathfrak{B}_F(\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) \end{pmatrix} \tag{33}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.

Proof: In $\mathcal{K}$, by utilising Equations (2), (7), (1), and (8), we obtain

$$\begin{aligned}&\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1)\le\mathfrak{p}_1\\&\Rightarrow\mathfrak{r}_1\cdot\mathfrak{p}_1\ge\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\\&\Rightarrow\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1)))\le\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)\\&\Rightarrow(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1)\le\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)\\&\Rightarrow(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1)))\le\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)\end{aligned}$$

It follows from Corollary 4.7,

$$\begin{pmatrix}\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\\ \succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\\ \mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\\ \ge\mathfrak{B}_I(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\\ \mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\\ \le\mathfrak{B}_F(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\end{pmatrix}\quad(34)$$

Since $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI, we have
$\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\cdot 0),\widetilde{\mathfrak{B}}_T(0)\}$
$\Rightarrow\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\succcurlyeq\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))$,
$\mathfrak{B}_I(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\ge min\{\mathfrak{B}_I(((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\cdot 0),\mathfrak{B}_I(0)\}$
$\Rightarrow\mathfrak{B}_I(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\ge\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\ge\mathfrak{B}_I(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))$,
$\mathfrak{B}_F(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\le max\{\mathfrak{B}_F(((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\cdot 0),\mathfrak{B}_F(0)\}$
$\Rightarrow\mathfrak{B}_F(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\le\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))\cdot(\mathfrak{r}_1\cdot(\mathfrak{p}_1\cdot(\mathfrak{p}_1\cdot\mathfrak{r}_1))))\le\mathfrak{B}_F(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))$.
Hence, the result is proved. ■

Theorem 4.15. *Let $\mathcal{J}\subseteq\mathcal{K}$ and $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$ defined as*

$$\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)=\begin{cases}[\eta_1,\eta_2], & \text{if } \mathfrak{p}_1\in\mathcal{J}\\ [0,0], & \text{otherwise}\end{cases}$$

$$\mathfrak{B}_I(\mathfrak{p}_1)=\begin{cases}m, & \text{if } \mathfrak{p}_1\in\mathcal{J}\\ 0, & \text{otherwise}\end{cases}$$

$$\mathfrak{B}_F(\mathfrak{p}_1)=\begin{cases}n, & \text{if } \mathfrak{p}_1\in\mathcal{J}\\ 1, & \text{otherwise}\end{cases}$$

Where $\eta_1,\eta_2,m\in(0,1]$ with $\eta_1<\eta_2$ and $n\in[0,1)$. Then the following are equivalent

(i) *$\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$.*
(ii) *$\mathcal{J}$ is an implicative ideal of $\mathcal{K}$.*

Proof: $(i)\Rightarrow(ii)$
Suppose $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. Let $\mathfrak{p}_1\in\mathcal{J}$, $\widetilde{\mathfrak{B}}_T(0)\ge\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)=[\eta_1,\eta_2]\Rightarrow\widetilde{\mathfrak{B}}_T(0)\ge[\eta_1,\eta_2]\Rightarrow 0\in\mathcal{J}$. Let $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$ such that $(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1\in\mathcal{J}$ and $\mathfrak{u}_1\in\mathcal{J}$. We have $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}=rmin\{[\eta_1,\eta_2],[\eta_1,\eta_2]\}=[\eta_1,\eta_2]$ and so $\mathfrak{p}_1\in\mathcal{J}$. Hence, $\mathcal{J}$ is an implicative ideal of $\mathcal{K}$.
$(ii)\Rightarrow(i)$
Let $\mathfrak{p}_1\in\mathcal{K}$. If $\mathfrak{p}_1\in\mathcal{J}$, then $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)=[\eta_1,\eta_2]$, $\mathfrak{B}_I(\mathfrak{p}_1)=m$, and $\mathfrak{B}_F(\mathfrak{p}_1)=n$. Since $0\in\mathcal{J}$, we have $\widetilde{\mathfrak{B}}_T(0)=[\eta_1,\eta_2]$, $\mathfrak{B}_I(0)=m$, and $\mathfrak{B}_F(0)=n$. Therefore, $\widetilde{\mathfrak{B}}_T(0)=\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0)=\mathfrak{B}_I(\mathfrak{p}_1)$, and $\mathfrak{B}_F(0)=\mathfrak{B}_F(\mathfrak{p}_1)$. If $\mathfrak{p}_1\notin\mathcal{J}$, then $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)=[0,0]$, $\mathfrak{B}_I(\mathfrak{p}_1)=0$, and $\mathfrak{B}_F(\mathfrak{p}_1)=1$. Therefore,

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T(0)=[\eta_1,\eta_2]\succ[0,0]=\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)\Rightarrow\widetilde{\mathfrak{B}}_T(0)\succ\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)\\&\mathfrak{B}_I(0)=m>0=\mathfrak{B}_I(\mathfrak{p}_1)\Rightarrow\mathfrak{B}_I(0)>\mathfrak{B}_I(\mathfrak{p}_1)\\&\mathfrak{B}_F(0)=n<0=\mathfrak{B}_F(\mathfrak{p}_1)\Rightarrow\mathfrak{B}_F(0)<\mathfrak{B}_F(\mathfrak{p}_1).\end{aligned}$$

Therefore,
$\widetilde{\mathfrak{B}}_T(0)\succcurlyeq\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)$, $\mathfrak{B}_I(0)\ge\mathfrak{B}_I(\mathfrak{p}_1)$, $\mathfrak{B}_F(0)\le\mathfrak{B}_F(\mathfrak{p}_1)$ for all $\mathfrak{p}_1\in\mathcal{K}$.

Let $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$. If $(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1\in\mathcal{J}$ and $\mathfrak{u}_1\in\mathcal{J}$, then $\mathfrak{p}_1\in\mathcal{J}$ and so

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)&=[\eta_1,\eta_2]=rmin\{[\eta_1,\eta_2],[\eta_1,\eta_2]\}\\&=rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1)&=m=min\{m,m\}\\&=min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1)&=n=max\{n,n\}\\&=max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

If any one of $(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1$ and $\mathfrak{u}_1$ is contained in $\mathcal{J}$ say $(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1\in\mathcal{J}$, then

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)&\succcurlyeq 0=rmin\{[\eta_1,\eta_2],[0,0]\}\\&=rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1)&\ge 0=min\{m,0\}\\&=min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1)&\le 1=max\{n,1\}\\&=max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

If $(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1\notin\mathcal{J}$ and $\mathfrak{u}_1\notin\mathcal{J}$, then

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)&\succcurlyeq 0=rmin\{[0,0],[0,0]\}\\&=rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1)&\ge 0=min\{0,0\}\\&=min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1)&\le 1=max\{1,1\}\\&=max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}\\&\mathfrak{B}_I(\mathfrak{p}_1)\ge min\{\mathfrak{B}_I((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_I(\mathfrak{u}_1)\}\\&\mathfrak{B}_F(\mathfrak{p}_1)\le max\{\mathfrak{B}_F((\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))\cdot\mathfrak{u}_1),\mathfrak{B}_F(\mathfrak{u}_1)\}\end{aligned}$$

for all $\mathfrak{p}_1,\mathfrak{r}_1,\mathfrak{u}_1\in\mathcal{K}$.

Hence, $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$. ■

Corollary 4.16. *Let $\mathcal{J}\subseteq\mathcal{K}$ and $\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$ defined as*

$$\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1)=\begin{cases}[1,1], & \text{if } \mathfrak{p}_1\in\mathcal{J}\\ [0,0], & \text{otherwise}\end{cases}$$

$$\mathfrak{B}_I(\mathfrak{p}_1)=\begin{cases}1, & \text{if } \mathfrak{p}_1\in\mathcal{J}\\ 0, & \text{otherwise}\end{cases}$$

$$\mathfrak{B}_F(\mathfrak{p}_1)=\begin{cases}0, & \text{if } \mathfrak{p}_1\in\mathcal{J}\\ 1, & \text{otherwise}\end{cases}$$

Then the following are equivalent

(i) *$\mathfrak{B}=(\widetilde{\mathfrak{B}}_T,\mathfrak{B}_I,\mathfrak{B}_F)$ is an ISB-NSI of $\mathcal{K}$.*
(ii) *$\mathcal{J}$ is an implicative ideal of $\mathcal{K}$.*

V. Commutative SB-neutrosophic ideal (CSB-NSI)

Definition 5.1. *Let $\mathcal{K}$ be a BCK-algebra. An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ is called a CSB-NSI of $\mathcal{K}$ if it satisfies (16) and*

(CSB-NSI 1) $\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},$

(CSB-NSI 2) $\mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},$

(CSB-NSI 3) $\mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\},$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$.

Example 5.2. *Consider a BCK-algebra $\mathcal{K} = \{0, a, b, c, d\}$ as given in Table XI.*

TABLE XI
BCK-ALGEBRA

·	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Let us define an SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ as shown in the Table XII. It is normal to check that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$.

TABLE XII
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	$[0.8, 1]$	1	0.2
a	$[0.5, 0.7]$	0.8	0.4
b	$[0.4, 0.6]$	0.6	0.6
c	$[0.3, 0.5]$	0.4	0.8
d	$[0.1, 0.3]$	0.2	1

Theorem 5.3. *In a BCK-algebra $\mathcal{K}$, every CSB-NSI is an SB-NSI of $\mathcal{K}$.*

Proof: Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$. If we take $\mathfrak{r}_1 = 0$ in (CSB-NSI 1), (CSB-NSI 2), (CSB-NSI 3) and utilising Equations (5) and (6), then

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (0 \cdot (0 \cdot \mathfrak{p}_1))) &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot 0) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\}\\
&\Rightarrow \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\
\mathfrak{B}_I(\mathfrak{p}_1 \cdot (0 \cdot (0 \cdot \mathfrak{p}_1))) &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot 0) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\}\\
&\Rightarrow \mathfrak{B}_I(\mathfrak{p}_1) \geq min\{\mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\
\mathfrak{B}_F(\mathfrak{p}_1 \cdot (0 \cdot (0 \cdot \mathfrak{p}_1))) &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot 0) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}\\
&\Rightarrow \mathfrak{B}_F(\mathfrak{p}_1) \leq max\{\mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\},
\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. ■

The following example illustrates that the converse of the Theorem 5.3 may not be true.

Example 5.4. *Consider a BCK-algebra $\mathcal{K} = \{0, a, b, c, d\}$ as defined in Table XIII.*

TABLE XIII
BCK-ALGEBRA

·	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS in $\mathcal{K}$ as defined in Table XIV. It is normal to check that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$. Since

TABLE XIV
SB-NEUTROSOPHIC SET

$\mathcal{K}$	Interval-valued grade of membership	Grade of indeterminacy	Grade of non-membership
0	$[0.7, 1]$	1	0.1
a	$[0.5, 0.8]$	0.5	0.2
b	$[0.2, 0.5]$	0.2	0.3
c	$[0.2, 0.5]$	0.1	0.3
d	$[0.2, 0.5]$	0.1	0.3

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T(b \cdot (d \cdot (d \cdot b))) &= \widetilde{\mathfrak{B}}_T(b)\\
&= [0.2, 0.5] \prec [0.7, 1] = \widetilde{\mathfrak{B}}_T(0)\\
&= rmin\{\widetilde{\mathfrak{B}}_T((b \cdot d) \cdot 0), \widetilde{\mathfrak{B}}_T(0)\},\\
\mathfrak{B}_I(b \cdot (d \cdot (d \cdot b))) &= \mathfrak{B}_I(b)\\
&= 0.2 < 1 = \mathfrak{B}_I(0)\\
&= min\{\mathfrak{B}_I((b \cdot d) \cdot 0), \mathfrak{B}_I(0)\},\\
\mathfrak{B}_F(b \cdot (d \cdot (d \cdot b))) &= \mathfrak{B}_F(b)\\
&= 0.3 > 0.1 = \mathfrak{B}_F(0)\\
&= max\{\mathfrak{B}_F((b \cdot d) \cdot 0), \mathfrak{B}_F(0)\},
\end{aligned}$$

$\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is not a CSB-NSI of $\mathcal{K}$.

Theorem 5.5. *Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSS of $\mathcal{K}$. Then, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$ if and only if it is an SB-NSI and the following condition holds*

$$\begin{pmatrix} \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \geq \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \leq \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \end{pmatrix} \tag{35}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$.

Proof: Assume that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$. If we set $\mathfrak{u}_1 = 0$ in equations (CSB-NSI 1), (CSB-NSI 2), and (CSB-NSI 3), then we obtain the Condition stated in (35).

Conversely, suppose that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{K}$ that satisfies the Condition (35). Then

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot \mathfrak{r}_1)\\ &\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\geq \mathfrak{B}_I(\mathfrak{p}_1 \cdot \mathfrak{r}_1)\\ &\geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\leq \mathfrak{B}_F(\mathfrak{p}_1 \cdot \mathfrak{r}_1)\\ &\leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\},\end{aligned}$$

for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$. ■

Theorem 5.6. *In a commutative BCK-algebra, every SB-NSI is a CSB-NSI.*

Proof: Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of a commutative BCK-algebra $\mathcal{K}$. Using Equations (8), (9), (12), and (15), we derive

$$\begin{aligned}&((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \cdot ((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)) \cdot \mathfrak{u}_1\\ &\quad = ((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \cdot \mathfrak{u}_1) \cdot ((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)\\ &\quad \leq (\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)\\ &\quad = (\mathfrak{p}_1 \cdot (\mathfrak{p}_1 \cdot \mathfrak{r}_1)) \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)) = 0.\end{aligned}$$

That is, $((\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \cdot ((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1)) \leq \mathfrak{u}_1$ for all $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. According to Lemma 3.7, we have

$$\begin{aligned}&\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)))\\ &\quad \succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ &\mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)))\\ &\quad \geq min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\ &\mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)))\\ &\quad \leq max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$. ■

Theorem 5.7. *An SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{K}$ is a CSB-NSI of $\mathcal{K}$ if and only if, for any $m, n \in [0,1]$ and $[l_1, l_2] \in [I]$, the non-empty sets $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [l_1, l_2])$, $\mathcal{U}(\mathfrak{B}_I; m)$, and $\mathcal{L}(\mathfrak{B}_F; n)$ are commutative ideals of $\mathcal{K}$.*

Proof: The proof of the theorem follows a similar approach to the proof provided in Theorem 3.16. ■

Theorem 5.8. *Given $\mathcal{J}$ is a commutative ideal of a BCK/BCI-algebra $\mathcal{K}$. Let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ be an SB-NSI of $\mathcal{K}$ defined by*

$$\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1) = \begin{cases}[\eta_1, \eta_2], & \text{if } \mathfrak{p}_1 \in \mathcal{J}\\ [0,0], & \text{otherwise}\end{cases}$$

$$\mathfrak{B}_I(\mathfrak{p}_1) = \begin{cases}m, & \text{if } \mathfrak{p}_1 \in \mathcal{J}\\ 0, & \text{otherwise}\end{cases}$$

$$\mathfrak{B}_F(\mathfrak{p}_1) = \begin{cases}n, & \text{if } \mathfrak{p}_1 \in \mathcal{J}\\ 1, & \text{otherwise}\end{cases}$$

Where $\eta_1, \eta_2, m \in (0,1]$ with $\eta_1 < \eta_2$ and $n \in [0,1)$. Then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$ such that $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [\eta_1, \eta_2]) = \mathcal{J}$, $\mathcal{U}(\mathfrak{B}_I; m) = \mathcal{J}$, and $\mathcal{L}(\mathfrak{B}_F; n) = \mathcal{J}$.

Proof: Let $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$. If $(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1 \in \mathcal{J}$ and $\mathfrak{u}_1 \in \mathcal{J}$, then $(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) \in \mathcal{J}$, and so

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &= [\eta_1, \eta_2]\\ &= rmin\{[\eta_1, \eta_2], [\eta_1, \eta_2]\}\\ &= rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &= m = min\{m, m\}\\ &= min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &= n = max\{n, n\}\\ &= max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

If any one of $(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1$ and $\mathfrak{u}_1$ is contained in $\mathcal{J}$, say $(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1 \in \mathcal{J}$, then $\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) = [\eta_1, \eta_2]$, $\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) = m$, $\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) = n$, $\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1) = [0,0]$, $\mathfrak{B}_I(\mathfrak{u}_1) = 0$, and $\mathfrak{B}_F(\mathfrak{u}_1) = 1$. Hence,

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\succcurlyeq [0,0]\\ &= rmin\{[\eta_1, \eta_2], [0,0]\}\\ &= rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\geq 0 = min\{m, 0\}\\ &= min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\leq 1 = max\{n, 1\}\\ &= max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

If $(\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1 \notin \mathcal{J}$ and $\mathfrak{u}_1 \notin \mathcal{J}$, then $\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) = [0,0]$, $\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) = 0$, $\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1) = 1$, $\widetilde{\mathfrak{B}}_T(\mathfrak{u}_1) = [0,0]$, $\mathfrak{B}_I(\mathfrak{u}_1) = 0$, and $\mathfrak{B}_F(\mathfrak{u}_1) = 1$. It follows that

$$\begin{aligned}\widetilde{\mathfrak{B}}_T(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\succcurlyeq [0,0]\\ &= rmin\{[0,0], [0,0]\}\\ &= rmin\{\widetilde{\mathfrak{B}}_T((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \widetilde{\mathfrak{B}}_T(\mathfrak{u}_1)\},\\ \mathfrak{B}_I(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\geq 0 = min\{0, 0\}\\ &= min\{\mathfrak{B}_I((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_I(\mathfrak{u}_1)\},\\ \mathfrak{B}_F(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))) &\leq 1 = max\{1, 1\}\\ &= max\{\mathfrak{B}_F((\mathfrak{p}_1 \cdot \mathfrak{r}_1) \cdot \mathfrak{u}_1), \mathfrak{B}_F(\mathfrak{u}_1)\}.\end{aligned}$$

It is obvious that

$$\widetilde{\mathfrak{B}}_T(0) \succcurlyeq \widetilde{\mathfrak{B}}_T(\mathfrak{p}_1), \mathfrak{B}_I(0) \geq \mathfrak{B}_I(\mathfrak{p}_1), \mathfrak{B}_F(0) \leq \mathfrak{B}_F(\mathfrak{p}_1)$$

for all $\mathfrak{p}_1 \in \mathcal{K}$. Consequently, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{K}$. Clearly, we also observe that $\mathcal{U}(\widetilde{\mathfrak{B}}_T; [\eta_1, \eta_2]) = \mathcal{J}$, $\mathcal{U}(\mathfrak{B}_I; m) = \mathcal{J}$, and $\mathcal{L}(\mathfrak{B}_F; n) = \mathcal{J}$. ■

A mapping $\lambda : \mathcal{K} \to \mathcal{Y}$ of BCK/BCI-algebras is called a homomorphism if $\lambda(\mathfrak{p}_1 \cdot \mathfrak{r}_1) = \lambda(\mathfrak{p}_1) \cdot \lambda(\mathfrak{r}_1)$ for all $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. Note that if $\lambda : \mathcal{K} \to \mathcal{Y}$ is a homomorphism, then $\lambda(0) = 0$.

Consider a homomorphism $\lambda : \mathcal{K} \to \mathcal{Y}$ in a BCK/BCI-algebra.

For any SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{Y}$, we introduce a new SB-NSS $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$. This induced SB-NSS defined by $\widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1) = \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1))$, $\mathfrak{B}_I^\lambda(\mathfrak{p}_1) = \mathfrak{B}_I(\lambda(\mathfrak{p}_1))$, and $\mathfrak{B}_F^\lambda(\mathfrak{p}_1) = \mathfrak{B}_F(\lambda(\mathfrak{p}_1))$ for all $\mathfrak{p}_1 \in \mathcal{K}$.

Lemma 5.9. *Let $\lambda : \mathcal{K} \to \mathcal{Y}$ be a homomorphism of a BCK /BCI-algebras. If an SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{Y}$ is an SB-NSI of $\mathcal{Y}$, then the corresponding induced SB-NSS $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, {\mathfrak{B}_I}^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$ is also an SB-NSI of $\mathcal{K}$.*

Proof: For any $\mathfrak{p}_1 \in \mathcal{K}$, we have

$$\widetilde{\mathfrak{B}}_T^\lambda(0) = \widetilde{\mathfrak{B}}_T(\lambda(0)) = \widetilde{\mathfrak{B}}_T(0) \succcurlyeq \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)) = \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1),$$
$$\mathfrak{B}_I^\lambda(0) = \mathfrak{B}_I(\lambda(0)) = \mathfrak{B}_I(0) \geq \mathfrak{B}_I(\lambda(\mathfrak{p}_1)) = \mathfrak{B}_I^\lambda(\mathfrak{p}_1),$$
$$\mathfrak{B}_F^\lambda(0) = \mathfrak{B}_F(\lambda(0)) = \mathfrak{B}_F(0) \leq \mathfrak{B}_F(\lambda(\mathfrak{p}_1)) = \mathfrak{B}_F^\lambda(\mathfrak{p}_1).$$

Let $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, then

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1) &= \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)) \\
&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)), \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{r}_1))\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)), \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{r}_1))\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{r}_1)\}, \\
\mathfrak{B}_I^\lambda(\mathfrak{p}_1) &= \mathfrak{B}_I(\lambda(\mathfrak{p}_1)) \\
&\geq min\{\mathfrak{B}_I(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)), \mathfrak{B}_I(\lambda(\mathfrak{r}_1))\} \\
&= min\{\mathfrak{B}_I(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)), \mathfrak{B}_I(\lambda(\mathfrak{r}_1))\} \\
&= min\{\mathfrak{B}_I^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \mathfrak{B}_I^\lambda(\mathfrak{r}_1)\}, \\
\mathfrak{B}_F^\lambda(\mathfrak{p}_1) &= \mathfrak{B}_F(\lambda(\mathfrak{p}_1)) \\
&\leq max\{\mathfrak{B}_F(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)), \mathfrak{B}_F(\lambda(\mathfrak{r}_1))\} \\
&= max\{\mathfrak{B}_F(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)), \mathfrak{B}_F(\lambda(\mathfrak{r}_1))\} \\
&= max\{\mathfrak{B}_F^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \mathfrak{B}_F^\lambda(\mathfrak{r}_1)\}.
\end{aligned}$$

Therefore, $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ is an SB-NSI of $\mathcal{K}$. ■

Theorem 5.10. *Let $\lambda : \mathcal{K} \to \mathcal{Y}$ be a homomorphism of a BCK/BCI-algebras. If an SB-NSS $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ in $\mathcal{Y}$ is a CSB-NSI of $\mathcal{Y}$, then the induced SB-NSS $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$ is a CSB-NSI of $\mathcal{K}$.*

Proof: Assume $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{Y}$. According to Theorem 5.3, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{Y}$. Consequently, by applying Lemma 5.9, we assert that $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ is an SB-NSI of $\mathcal{Y}$. For any $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$, we have

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))) = \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)))) \\
&= \widetilde{\mathfrak{B}}_T((\lambda(\mathfrak{p}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot\lambda(\mathfrak{p}_1))))) \\
&\succcurlyeq \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)) \\
&= \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)) = \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \\
&\mathfrak{B}_I^\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))) = \mathfrak{B}_I(\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)))) \\
&= \mathfrak{B}_I((\lambda(\mathfrak{p}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot\lambda(\mathfrak{p}_1))))) \\
&\geq \mathfrak{B}_I(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)) \\
&= \mathfrak{B}_I(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)) = \mathfrak{B}_I^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \\
&\mathfrak{B}_F^\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))) = \mathfrak{B}_F(\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)))) \\
&= \mathfrak{B}_F((\lambda(\mathfrak{p}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot\lambda(\mathfrak{p}_1))))) \\
&\leq \mathfrak{B}_F(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)) \\
&= \mathfrak{B}_F(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)) = \mathfrak{B}_F^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1).
\end{aligned}$$

Hence, according to Theorem 5.5, $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ is a CSB-NSI of $\mathcal{K}$. ■

Lemma 5.11. *Let $\lambda : \mathcal{K} \to \mathcal{Y}$ be an onto homomorphism of BCK/BCI-algebras, and let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ represent an SB-NSS in $\mathcal{Y}$. If the induced SB-NSS $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$ is an SB-NSI of $\mathcal{K}$, then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is also an SB-NSI of $\mathcal{Y}$.*

Proof: Suppose that the induced SB-NSS $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$ is an SB-NSI of $\mathcal{K}$. For any element a in $\mathcal{Y}$, there exists $\mathfrak{p}_1 \in \mathcal{K}$ such that $\lambda(\mathfrak{p}_1) = a$. Thus,

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T(0) &= \widetilde{\mathfrak{B}}_T(\lambda(0)) = \widetilde{\mathfrak{B}}_T^\lambda(0) \\
&\qquad \succcurlyeq \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1) = \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)) = \widetilde{\mathfrak{B}}_T(a), \\
\mathfrak{B}_I(0) &= \mathfrak{B}_I(\lambda(0)) = \mathfrak{B}_I^\lambda(0) \\
&\qquad \geq \mathfrak{B}_I^\lambda(\mathfrak{p}_1) = \mathfrak{B}_I(\lambda(\mathfrak{p}_1)) = \mathfrak{B}_I(a), \\
\mathfrak{B}_F(0) &= \mathfrak{B}_F(\lambda(0)) = \mathfrak{B}_F^\lambda(0) \\
&\qquad \leq \mathfrak{B}_F^\lambda(\mathfrak{p}_1) = \mathfrak{B}_F(\lambda(\mathfrak{p}_1)) = \mathfrak{B}_F(a).
\end{aligned}$$

Let $a, b \in \mathcal{Y}$. Then $\lambda(\mathfrak{p}_1) = a$ and $\lambda(\mathfrak{r}_1) = b$ for some $\mathfrak{p}_1, \mathfrak{r}_1 \in \mathcal{K}$. Therefore,

$$\begin{aligned}
\widetilde{\mathfrak{B}}_T(a) &= \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)) = \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1) \\
&\succcurlyeq rmin\{\widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{r}_1)\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)), \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{r}_1))\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)), \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{r}_1))\} \\
&= rmin\{\widetilde{\mathfrak{B}}_T(a\cdot b), \widetilde{\mathfrak{B}}_T(b)\}, \\
\mathfrak{B}_I(a) &= \mathfrak{B}_I(\lambda(\mathfrak{p}_1)) = \mathfrak{B}_I^\lambda(\mathfrak{p}_1) \\
&\geq min\{\mathfrak{B}_I^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \mathfrak{B}_I^\lambda(\mathfrak{r}_1)\} \\
&= min\{\mathfrak{B}_I(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)), \mathfrak{B}_I(\lambda(\mathfrak{r}_1))\} \\
&= min\{\mathfrak{B}_I(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)), \mathfrak{B}_I(\lambda(\mathfrak{r}_1))\} \\
&= min\{\mathfrak{B}_I(a\cdot b), \mathfrak{B}_I(b)\}, \\
\mathfrak{B}_F(a) &= \mathfrak{B}_F(\lambda(\mathfrak{p}_1)) = \mathfrak{B}_F^\lambda(\mathfrak{p}_1) \\
&\leq max\{\mathfrak{B}_F^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1), \mathfrak{B}_F^\lambda(\mathfrak{r}_1)\} \\
&= max\{\mathfrak{B}_F(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)), \mathfrak{B}_F(\lambda(\mathfrak{r}_1))\} \\
&= max\{\mathfrak{B}_F(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)), \mathfrak{B}_F(\lambda(\mathfrak{r}_1))\} \\
&= max\{\mathfrak{B}_F(a\cdot b), \mathfrak{B}_F(b)\}.
\end{aligned}$$

Therefore, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{Y}$. ■

Theorem 5.12. *Let $\lambda : \mathcal{K} \to \mathcal{Y}$ be an onto homomorphism of BCK/BCI-algebras, and let $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ represent an SB-NSS in $\mathcal{Y}$. If the induced SB-NSS $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$ is a CSB-NSI of $\mathcal{K}$, then $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is also a CSB-NSI of $\mathcal{Y}$.*

Proof: Assume that the induced SB-NSS denoted by $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ in $\mathcal{K}$ is a CSB-NSI of $\mathcal{K}$. According to Theorem 5.3, $\mathfrak{B}^\lambda = (\widetilde{\mathfrak{B}}_T^\lambda, \mathfrak{B}_I^\lambda, \mathfrak{B}_F^\lambda)$ is an SB-NSI of $\mathcal{Y}$. Consequently, by applying Lemma 5.11, we assert that $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is an SB-NSI of $\mathcal{Y}$.

For any $a, b, c \in \mathcal{Y}$, there exists $\mathfrak{p}_1, \mathfrak{r}_1, \mathfrak{u}_1 \in \mathcal{K}$ such that $\lambda(\mathfrak{p}_1) = a$, $\lambda(\mathfrak{r}_1) = b$, and $\lambda(\mathfrak{u}_1) = c$. It follows that,

$$\begin{aligned}
&\widetilde{\mathfrak{B}}_T(a\cdot(b\cdot(b\cdot a))) \\
&= \widetilde{\mathfrak{B}}_T((\lambda(\mathfrak{p}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot\lambda(\mathfrak{p}_1))))) \\
&= \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)))) \\
&= \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1))) \\
&\succcurlyeq \widetilde{\mathfrak{B}}_T^\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1) = \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1\cdot\mathfrak{r}_1)) \\
&= \widetilde{\mathfrak{B}}_T(\lambda(\mathfrak{p}_1)\cdot\lambda(\mathfrak{r}_1)) = \widetilde{\mathfrak{B}}_T(a\cdot b), \\
&\mathfrak{B}_I(a\cdot(b\cdot(b\cdot a))) \\
&= \mathfrak{B}_I((\lambda(\mathfrak{p}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot(\lambda(\mathfrak{r}_1)\cdot\lambda(\mathfrak{p}_1))))) \\
&= \mathfrak{B}_I(\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)))) \\
&= \mathfrak{B}_I^\lambda(\mathfrak{p}_1\cdot(\mathfrak{r}_1\cdot(\mathfrak{r}_1\cdot\mathfrak{p}_1)))
\end{aligned}$$

$$\geq \mathfrak{B}_I^\lambda(\mathfrak{p}_1 \cdot \mathfrak{r}_1) = \mathfrak{B}_I(\lambda(\mathfrak{p}_1 \cdot \mathfrak{r}_1))$$
$$= \mathfrak{B}_I(\lambda(\mathfrak{p}_1) \cdot \lambda(\mathfrak{r}_1)) = \mathfrak{B}_I(a \cdot b),$$
$$\mathfrak{B}_F(a \cdot (b \cdot (b \cdot a)))$$
$$= \mathfrak{B}_F((\lambda(\mathfrak{p}_1) \cdot (\lambda(\mathfrak{r}_1) \cdot (\lambda(\mathfrak{r}_1) \cdot \lambda(\mathfrak{p}_1)))))$$
$$= \mathfrak{B}_F(\lambda(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1))))$$
$$= \mathfrak{B}_F^\lambda(\mathfrak{p}_1 \cdot (\mathfrak{r}_1 \cdot (\mathfrak{r}_1 \cdot \mathfrak{p}_1)))$$
$$\leq \mathfrak{B}_F^\lambda(\mathfrak{p}_1 \cdot \mathfrak{r}_1) = \mathfrak{B}_F(\lambda(\mathfrak{p}_1 \cdot \mathfrak{r}_1))$$
$$= \mathfrak{B}_F(\lambda(\mathfrak{p}_1) \cdot \lambda(\mathfrak{r}_1)) = \mathfrak{B}_F(a \cdot b).$$

Hence, according to Theorem 5.5, $\mathfrak{B} = (\widetilde{\mathfrak{B}}_T, \mathfrak{B}_I, \mathfrak{B}_F)$ is a CSB-NSI of $\mathcal{Y}$. ■

VI. Conclusion

In this study, we discussed in detail the concepts of commutative SB-neutrosophic ideals, positive implicative SB-neutrosophic ideals, and implicative SB-neutrosophic ideals, exploring their properties and characteristics. Through our research, we have provided valuable insights into these theories to highlight their essential characteristics and functions in the larger field of neutrosophic theory. Our results contribute to a deeper understanding of these algebraic structures, which can provide a platform for new research and applications in different fields.

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