# Simpson Type Inequalities for Twice-differentiable Functions Arising from Tempered Fractional Integral Operators

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***Abstract*—Simpson inequalities for first-order differentiable convex functions and various fractional integrals have been studied extensively. However, Simpson type inequalities for twice-differentiable functions are researched slightly. Therefore, in the present paper, we endeavor to study fractional inequalities of Simpson type for twice-differentiable convex functions. To achieve this goal, we establish a new twice-differentiable Simpson's identity by using tempered fractional integral operators. Based upon it, we prove several fractional Simpson type inequalities whose second derivatives in absolute value are convex. Finally, we give some examples to illustrate the correctness of the obtained results.**

***Index Terms*—Convex functions; Simpson type inequalities; Riemann–Liouville fractional integrals; tempered fractional integrals**

## I. Introduction and preliminaries

THE Simpson type inequality is one of the classic inequalities in analysis, which has intuitive geometric meanings and a wide range of application values in the field of engineering mathematics. Let us state it as the following theorem.

**Theorem 1.1:** Let $F : [a,b] \to \mathbb{R}$ denotes a four times continuously differentiable mapping on $(a,b)$, and let $\left\|F^{(4)}\right\|_\infty = \sup\limits_{x\in(a,b)} \left|F^{(4)}(x)\right| < \infty$. Then, the following inequality holds:

$$\begin{aligned}\Bigg|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]\\ -\frac{1}{b-a}\int_a^b F(x)\mathrm{d}x\Bigg| \le \frac{1}{2880}\left\|F^{(4)}\right\|_\infty (b-a)^4.\end{aligned} \tag{1}$$

Due to the fact that the convex theory is an available way to solve many problems in various branches of mathematics, a lot of authors have investigated the Simpson type inequalities via different differentiable convex functions. Hereby, we enumerate several existing results concerning with different classes of functions, such as $\varphi$-convex functions [1], geometrically relative convex functions [2], strongly $s$-convex functions [3], $h$-convex functions [4], harmonically-preinvex functions [5] and so on. More recent results with respect to (1) and other related outcomes, we refer the reader to Refs. [6], [7], [8], [9], [10], [11], [12] and the references cited therein.

In order to meet the need of the later exploration, let us look back some mathematical preliminaries about fractional calculus theory as below.

**Definition 1.1:** Let $[a,b]$ be a real interval and $\alpha > 0$. Then, for a function $F \in L^1([a,b])$, the left-sided and right-sided Riemann–Liouville fractional integrals are respectively defined by

$$\mathcal{I}_{a^+}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} F(t)\mathrm{d}t, \quad x > a, \tag{2}$$

and

$$\mathcal{I}_{b^-}^{\alpha} F(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} F(t)\mathrm{d}t, \quad x < b. \tag{3}$$

Using the Riemann–Liouville fractional integrals above, Sarikaya et al. extended the classical Hermite–Hadamard's integral inequality to the form of fractional integrals as below.

**Theorem 1.2:** [13] Suppose that $F : [a,b] \to \mathbb{R}$ is a positive function with $0 \le a < b$ as well as $F \in L^1([a,b])$. If the function $F$ is convex defined on $[a,b]$ and $\alpha > 0$, then one acquires the undermentioned fractional integral inequalities

$$\begin{aligned}F\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\mathcal{I}_{a^+}^{\alpha} F(x) + \mathcal{I}_{b^-}^{\alpha} F(x)\right]\\ &\le \frac{F(a)+F(b)}{2}.\end{aligned} \tag{4}$$

It is undeniable that the fractional integral operators has a great influence on realizing the differentiation and integration of real order and complex order. Moreover, it emerged rapidly due to it is widely used in mathematical modeling of engineering problems, such as dynamic modeling for dealing with complex systems, decision modeling for structural engineering, as well as stochastic modeling and so on. For example, Li et al. in Ref. [14] studied the fractional order Chebyshev cardinal function and presented solutions for two different forms of fractional order delay differential equations. In Ref. [15], the author considered the oscillation of a class fractional differential equations and established some oscillatory criteria for the equations. Moreover, Alomari and Massoun [16] investigated time-fractional coupled Korteweg-de Vries differential equations and proposed an efficient solution method based on Caputo definition. For more applications with regard to this issue, see the published articles [17], [18], [19], [20] and the references cited in them.

Besides, there are many results of the integral inequalities, especially Simpson-type inequalities considering different

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types of fractional integrals. For example, some authors developed Riemann–Liouville fractional integrals to develop Simpson-type inequalities for various of differentiable functions, as for convex functions [21], $h$-convex functions [22] and $(s, m)$-convex functions [23]. Furthermore, Kermausuor [24] utilized the Katugampola fractional integrals, as a extension of the Riemann–Liouville and Hadamard fractional integrals, to investigate Simpson-type inequalities for $s$-convex functions in the second sense. The authors in Ref. [25] deduced the parameterized Simpson-type inequalities in accordance with differentiable convex functions via generalized fractional integrals. And Şanlı [26] offered a couple of Simpson-type integral inequalities taking advantage of the conformable fractional integrals and exhibited several applications relating to special means. For more Simpson-type inequalities acquired by virtue of fractional integrals involving first-order differentiable functions, one can refer to the literatures [27], [28], [29], [30], [31], [32] and their bibliographies.

Fractional versions of Simpson inequalities for first-order differentiable convex functions are extensively researched. However, Simpson type inequalities for twice-differentiable functions are investigated slightly. In the literature, several papers were focused on Simpson-type inequalities for twice-differentiable convex functions. For example, Hezenci et al. in [33] gave certain error bounds of Simpson-type for Riemann–Liouville fractional integrals, in which the absolute value of the second derivatives of the functions belongs to convex functions. Furthermore, the authors in Ref. [34] established some generalized Simpson-type integral inequalities involving the convexity of twice-differentiable function, and they mentioned several special cases of the obtained integral inequalities. Recent interesting studies for twice-differentiable functions, considering with fractional Simpson-type integral inequalities and others, can be found in the articles [35], [36], [37], [38], [39], [40] and the references cited therein.

Recently, Sabzikar et al. introduced the notion of the tempered fractional integrals.

**Definition 1.2:** [41] Let $[a, b]$ be a real interval and $\lambda \geq 0$, $\alpha > 0$. Then for a function $F \in L^1([a, b])$, the left-sided and right-sided tempered fractional integrals are respectively defined by

$$\begin{aligned} &\mathcal{I}_{a^+}^{\alpha,\lambda} F(x) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} F(t) \mathrm{d}t, \quad x > a, \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\mathcal{I}_{b^-}^{\alpha,\lambda} F(x) \\ &\quad = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} F(t) \mathrm{d}t, \quad x < b. \end{aligned} \tag{6}$$

The most significant feature of tempered fractional integrals is that they generalize some types of fractional integrals such as Riemann–Liouville fractional integrals, Riemann integrals etc. These important special cases of the integral operators (5) and (6) are mentioned as follows:

(i) If we let $\lambda = 0$, then the operators (5) and (6) reduce to Riemann–Liouville fractional integrals.

(ii) If we choose $\lambda = 0$ and $\alpha = 1$, then the operators (5) and (6) reduce to Riemann integrals.

For related development pertaining to the tempered fractional integrals, see the published articles [42], [43], [44], [45], [46], [47] and the references therein.

Enlightened by the above-referenced works, particularly the results displayed in the papers [38] and [33], the current paper focuses on investigating Simpson-type inequalities in relation with the discovered tempered fractional integral identity. The obtained results here can be transferred to the Riemann–Liouville fractional integral inequalities for $\lambda = 0$, and the Riemann integral inequalities for $\alpha = 1$ together with $\lambda = 0$.

The general structure of the paper consists of four sections including an introduction. In Sec. II, after giving a general literature survey and definition of some fractional integral operators, we present a Simpson equality for twice-differentiable functions using tempered fractional integrals. In Sec. III, by taking advantage of the equality, and considering the functions whose second derivatives are convex, we establish several Simpson type inequalities. In Sec. IV, we give some special examples to illustrate the results we obtained in Sec. III.

## II. Some lemmas

In this section, we firstly give an identity on twice-differentiable functions for establishing the main results.

**Lemma 2.1:** Suppose that $F : [a, b] \to \mathbb{R}$ is an absolutely continuous mapping on $(a, b)$ satisfying that $F'' \in L_1([a, b])$. Then, for $\alpha > 0$ and $\lambda \geq 0$, we the have the coming equality

$$\begin{aligned} &\frac{1}{6}\left[F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right] \\ &\quad - \frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^\alpha \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1)} \\ &\quad \times \left[\mathcal{I}_{a^+}^{\alpha,\lambda} F\left(\frac{a+b}{2}\right) + \mathcal{I}_{b^-}^{\alpha,\lambda} F\left(\frac{a+b}{2}\right)\right] \\ &= \frac{(b-a)^2}{8\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1)} \\ &\quad \times \int_0^1 \left(h(1) - h(t) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1)(1-t)\right) \\ &\quad \times \left[F''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right. \\ &\quad\quad \left. + F''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right] \mathrm{d}t, \end{aligned} \tag{7}$$

where

$$h(t) = \int_0^t \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, s) \mathrm{d}s, \tag{8}$$

and

$$\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, s) = \int_0^s u^{\alpha-1} e^{-\lambda \frac{b-a}{2} u} \mathrm{d}u. \tag{9}$$

*Proof:* By using integration by parts, we obtain that

$$K_1 = \int_0^1 \left( h(1) - h(t) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1)(1-t) \right) \times F''\left( \frac{1-t}{2}a + \frac{1+t}{2}b \right) \mathrm{d}t$$
$$= -\frac{2\left[ h(1) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) \right]}{b-a} F'\left( \frac{a+b}{2} \right) - \frac{2}{b-a}\int_0^1 \left( \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, t) \right) \times F'\left( \frac{1-t}{2}a + \frac{1+t}{2}b \right) \mathrm{d}t$$
$$= -\frac{2\left[ h(1) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) \right]}{b-a} F'\left( \frac{a+b}{2} \right) + \frac{4F(b)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) + \frac{8F\left(\frac{a+b}{2}\right)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \frac{4}{(b-a)^2}\int_0^1 t^{\alpha-1} e^{-\lambda\frac{b-a}{2}t} F\left( \frac{1-t}{2}a + \frac{1+t}{2}b \right) \mathrm{d}t.$$

Using the change of the variable $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ for $t \in [0,1]$, it can be rewritten as follows

$$K_1 = -\frac{2\left[ h(1) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) \right]}{b-a} F'\left( \frac{a+b}{2} \right) + \frac{4F(b)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) + \frac{8F\left(\frac{a+b}{2}\right)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \left( \frac{2}{b-a} \right)^{\alpha+2} \Gamma(\alpha) \mathcal{I}_{a^+}^{\alpha,\lambda} F\left( \frac{a+b}{2} \right). \tag{10}$$

Similarly, we get that

$$K_2 = \int_0^1 \left( h(1) - h(t) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1)(1-t) \right) \times F''\left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \mathrm{d}t$$
$$= \frac{2\left[ h(1) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) \right]}{b-a} F'\left( \frac{a+b}{2} \right) - \frac{2}{b-a}\int_0^1 \left( \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, t) \right) \times F'\left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \mathrm{d}t$$
$$= \frac{2\left[ h(1) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) \right]}{b-a} F'\left( \frac{a+b}{2} \right) + \frac{4F(a)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) + \frac{8F\left(\frac{a+b}{2}\right)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \frac{4}{(b-a)^2}\int_0^1 t^{\alpha-1} e^{-\lambda\frac{b-a}{2}t} F\left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \mathrm{d}t$$
$$= \frac{2\left[ h(1) - \frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) \right]}{b-a} F'\left( \frac{a+b}{2} \right) + \frac{4F(a)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) + \frac{8F\left(\frac{a+b}{2}\right)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \left( \frac{2}{b-a} \right)^{\alpha+2} \Gamma(\alpha) \mathcal{I}_{b^-}^{\alpha,\lambda} F\left( \frac{a+b}{2} \right). \tag{11}$$

From Eqs. (10) and (11), we have that

$$K_1 + K_2 = \frac{4(F(a)+F(b))}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) + \frac{16F\left(\frac{a+b}{2}\right)}{3(b-a)^2} \cdot \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \left( \frac{2}{b-a} \right)^{\alpha+2} \Gamma(\alpha) \times \left[ \mathcal{I}_{a^+}^{\alpha,\lambda} F\left( \frac{a+b}{2} \right) + \mathcal{I}_{b^-}^{\alpha,\lambda} F\left( \frac{a+b}{2} \right) \right]. \tag{12}$$

Multiplying the both sides of (12) by $\frac{(b-a)^2}{8\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)}$, we obtain the desired identity. This ends the proof. ■

**Remark 2.1:** If we consider taking $\lambda = 0$, then we have Lemma 1 established by Hezenci et al. in [33]

Let us mention the definition of $\lambda$-incomplete gamma function.

**Definition 2.1:** [44] The $\lambda$-incomplete gamma function is defined as follows:

$$\gamma_\lambda(\alpha, s) = \int_0^s t^{\alpha-1} e^{-\lambda t} \mathrm{d}t, \quad \alpha > 0, \; s, \lambda \geq 0.$$

If we consider taking $\lambda = 1$, then the $\lambda$-incomplete gamma function reduces to the incomplete gamma function [48].

$$\gamma(\alpha, s) = \int_0^s t^{\alpha-1} e^{-t} \mathrm{d}t, \quad \alpha > 0, \; s > 0.$$

The following facts will be required in establishing the following lemma.

**Remark 2.2:** For the real numbers $\alpha > 0$ and $s, \lambda \geq 0$, the following identities hold:

$$(i) \quad \int_0^1 \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, s) \mathrm{d}s = \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) - \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha+1, 1), \tag{13}$$

$$(ii) \quad \int_0^t \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, s) \mathrm{d}s = t\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, t) - \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha+1, t). \tag{14}$$

*Proof:* $(i)$ From the Definition of $\lambda$-incomplete gamma function, we have that

$$\int_0^1 \gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, s) \mathrm{d}s = \int_0^1 \int_0^s u^{\alpha-1} e^{-\lambda\left(\frac{b-a}{2}\right)u} \mathrm{d}u \mathrm{d}s.$$

By changing the order of the integration, we get that

$$\begin{aligned}
&\int_0^1 \gamma_{\lambda(\frac{b-a}{2})}(\alpha, s)\mathrm{d}s \\
&= \int_0^1 \int_u^1 u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}s\mathrm{d}u \\
&= \int_0^1 (1-u) u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u \\
&= \int_0^1 u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u - \int_0^1 u^{\alpha} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u \\
&= \gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1) - \gamma_{\lambda(\frac{b-a}{2})}(\alpha+1, 1).
\end{aligned}$$

We obtain the result (13).

$(ii)$ From the definition of $\lambda$-incomplete gamma function, we have that

$$\begin{aligned}
&\int_0^t \gamma_{\lambda(\frac{b-a}{2})}(\alpha, s)\mathrm{d}s \\
&= \int_0^t \int_0^s u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u\mathrm{d}s.
\end{aligned}$$

By changing the order of the integration, we get that

$$\begin{aligned}
&\int_0^t \gamma_{\lambda(\frac{b-a}{2})}(\alpha, s)\mathrm{d}s \\
&= \int_0^t \int_u^t u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}s\mathrm{d}u \\
&= \int_0^t (t-u) u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u \\
&= t\int_0^t u^{\alpha-1} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u - \int_0^t u^{\alpha} e^{-\lambda(\frac{b-a}{2})u} \mathrm{d}u \\
&= t\gamma_{\lambda(\frac{b-a}{2})}(\alpha, t) - \gamma_{\lambda(\frac{b-a}{2})}(\alpha+1, t).
\end{aligned}$$

We obtain the result (14). This ends the proof. ■

**Lemma 2.2:** Let us consider the function $\Delta : [0,1] \to \mathbb{R}$ by $\Delta(\lambda, \alpha; t) = h(1) - h(t) - \frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)(1-t)$ for the parameters $\alpha > 0$ and $\lambda \geq 0$, where $h(t)$ and $\gamma_{\lambda(\frac{b-a}{2})}(\alpha, s)$ are defined in Lemma 2.1. Then, for $\lambda = 0$, we have that

$$\begin{aligned}
&\Delta(0, \alpha; t) \\
&= \frac{2t(\alpha+1) - 2\alpha + 1 - 3t^{\alpha+1}}{3\alpha(\alpha+1)}.
\end{aligned} \tag{15}$$

*Proof:* From the parts (i) and (ii) in Remark 2.2, we have that

$$\begin{aligned}
&\Delta(\lambda, \alpha; t) \\
&= \gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1) - \gamma_{\lambda(\frac{b-a}{2})}(\alpha+1, 1) \\
&\quad - t\gamma_{\lambda(\frac{b-a}{2})}(\alpha, t) + \gamma_{\lambda(\frac{b-a}{2})}(\alpha+1, t) \\
&\quad - \frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)(1-t).
\end{aligned} \tag{16}$$

If we take $\lambda = 0$ in (16), then we can get the desired result. This ends the proof. ■

## III. Simpson's type inequalities for twice-differentiable functions

For the sake of simplicity, we will use the following notation in the sequel:

$$\begin{aligned}
&\mathcal{T}_F(\alpha, \lambda; a, b) \\
&:= \frac{1}{6}\left[F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right] \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^\alpha \gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)} \\
&\quad \times \left[\mathcal{I}_{a^+}^{\alpha,\lambda} F\left(\frac{a+b}{2}\right) + \mathcal{I}_{b^-}^{\alpha,\lambda} F\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

Especially, for $\lambda = 0$, we have that

$$\begin{aligned}
&\mathcal{T}_F(\alpha, 0; a, b) \\
&:= \frac{1}{6}\left[F(a) + 4F\left(\frac{a+b}{2}\right) + F(b)\right] \\
&\quad - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \\
&\quad \times \left[\mathcal{I}_{a^+}^{\alpha} F\left(\frac{a+b}{2}\right) + \mathcal{I}_{b^-}^{\alpha} F\left(\frac{a+b}{2}\right)\right].
\end{aligned}$$

For functions whose second derivatives are convex, Simpson's type inequalities will be established by using the lemmas given in Sec. II.

**Theorem 3.1:** Let us consider that assumptions of Lemma 2.1 are valid. And suppose that the mapping $|F''|$ is convex on the interval $[a, b]$. Then, for $\alpha > 0$ and $\lambda \geq 0$, we get the following inequality

$$\begin{aligned}
&|\mathcal{T}_F(\alpha, \lambda; a, b)| \\
&\leqslant \frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)} \Psi_1(\lambda, \alpha; t)\left[|F''(a)| + |F''(b)|\right],
\end{aligned} \tag{17}$$

where $\Psi_1(\lambda, \alpha; t)$ is defined by

$$\begin{aligned}
&\Psi_1(\lambda, \alpha; t) \\
&= \int_0^1 \left|h(1) - h(t) - \frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)(1-t)\right| \mathrm{d}t.
\end{aligned} \tag{18}$$

*Proof:* By taking modulus for the identity in Lemma 2.1, we have that

$$\begin{aligned}
&|\mathcal{T}_F(\alpha, \lambda; a, b)| \\
&\leqslant \frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)} \\
&\quad \times \int_0^1 \left|h(1) - h(t) - \frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha, 1)(1-t)\right| \\
&\quad \times \left[\left|F''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right| \right. \\
&\quad \left. + \left|F''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|\right] \mathrm{d}t.
\end{aligned}$$

Using convexity of the mapping $|F''|$ defined on the interval

$[a,b]$, we obtain that

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)| \\
&\leqslant \frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)} \\
&\quad\times\left\{\int_0^1 \left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|\right. \\
&\quad\times\left[\frac{1+t}{2}\left|F''(b)\right|+\frac{1-t}{2}\left|F''(a)\right|\right. \\
&\quad\left.\left.+\frac{1+t}{2}\left|F''(a)\right|+\frac{1-t}{2}\left|F''(b)\right|\right]\mathrm{d}t\right\} \\
&=\frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\left[|F''(a)|+|F''(b)|\right] \\
&\quad\times\int_0^1 \left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|\mathrm{d}t.
\end{aligned}$$

This ends the proof. ■

**Corollary 3.1:** In Theorem 3.1, if we take $\lambda=0$, then we have the following result.

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,0;a,b)| \\
&\leqslant \frac{(b-a)^2\alpha}{8}\Theta_1(\alpha)\big[\,|F''(a)|+|F''(b)|\,\big],
\end{aligned} \tag{19}$$

where $\Theta_1(\alpha)$ is defined by

$$\Theta_1(\alpha)=\begin{cases}\dfrac{1-\alpha}{3\alpha(\alpha+2)}, & \text{if } 0<\alpha\leqslant\dfrac{1}{2},\\[2ex] 2\left[\dfrac{(\xi_\alpha)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}-\dfrac{(\xi_\alpha)^2}{3\alpha}\right. \\ \left.+\dfrac{2\xi_\alpha}{3(\alpha+2)}+\dfrac{\xi_\alpha(2\alpha-1)}{3(\alpha+1)(\alpha+2)}\right] \\ +\dfrac{1-\alpha}{3\alpha(\alpha+2)}, & \text{if } \alpha>\dfrac{1}{2},\ \xi_\alpha\in[0,1].\end{cases} \tag{20}$$

*Proof:* $(i)$ Let us consider the function $\Delta(0,\alpha;t)\to\mathbb{R}$ by $\Delta(0,\alpha;t)=\frac{2t(\alpha+1)-2\alpha+1-3t^{\alpha+1}}{3\alpha(\alpha+1)}$ with $0<\alpha\le\frac{1}{2}$. Then, $\Delta(0,\alpha;t)\ge 0$ for all $t\in[0,1]$. Thus, it can be easily seen that

$$\begin{aligned}
\Psi_1(0,\alpha;t)&=\int_0^1|\Delta(0,\alpha;t)|\mathrm{d}t\\
&=\frac{1-\alpha}{3\alpha(\alpha+2)}.
\end{aligned}$$

$(ii)$ if $\alpha>\frac{1}{2}$, then there exists a real number $\xi_\alpha\in[0,1]$ such that $\Delta(0,\alpha;t)\le 0$ for $0\le t\le\xi_\alpha$ and $\Delta(0,\alpha;t)\ge 0$ for $\xi_\alpha\le t\le 1$. Therefore, we obtain that

$$\begin{aligned}
&\Psi_1(0,\alpha;t)\\
&=\int_0^1|\Delta(0,\alpha;t)|\mathrm{d}t\\
&=\int_0^{\xi_\alpha}(-\Delta(0,\alpha;t))\mathrm{d}t+\int_{\xi_\alpha}^1\Delta(0,\alpha;t)\mathrm{d}t\\
&=2\left[\frac{(\xi_\alpha)^{\alpha+2}}{\alpha(\alpha+1)(\alpha+2)}-\frac{(\xi_\alpha)^2}{3\alpha}+\frac{2\xi_\alpha}{3(\alpha+2)}\right.\\
&\quad\left.+\frac{\xi_\alpha(2\alpha-1)}{3(\alpha+1)(\alpha+2)}\right]+\frac{1-\alpha}{3\alpha(\alpha+2)}.
\end{aligned}$$

This ends the proof. ■

**Remark 3.1:** The result given in Corollary 3.1 is consistent with the result in Theorem 3 established by Hezenci et al. [33]. However, the function

$$\Delta(0,\alpha;t)=\frac{2t(\alpha+1)-2\alpha+1-3t^{\alpha+1}}{3\alpha(\alpha+1)}$$

defined in Lemma 2.2 is different from the function

$$\varpi(\tau)=\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}$$

considered in Lemma 2 by Hezenci et al. [33]. Especially, if we choose $\alpha=1$ in Corollary 3.1, then, for $\xi_\alpha=\frac{1}{3}$, we have the following inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[F(a)+4F\left(\frac{a+b}{2}\right)+F(b)\right]-\frac{1}{b-a}\int_a^b F(t)\mathrm{d}t\right|\\
&\leqslant\frac{(b-a)^2}{162}\big[\,|F''(a)|+|F''(b)|\,\big],
\end{aligned}$$

which is proved by Sarikaya et al. in [49].

**Theorem 3.2:** Let us note that assumptions of Lemma 2.1 hold. If the mapping $|F''|^q$ is convex on $[a,b]$ with $q>1$, then we have the following inequality

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)|\\
&\leqslant\frac{(b-a)^2\Psi_2(\lambda,\alpha;t)}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\left[|F''(a)|^q+|F''(b)|^q\right]^{\frac{1}{q}},
\end{aligned} \tag{21}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\Psi_2(\lambda,\alpha;t)$ is defined by

$$\begin{aligned}
&\Psi_2(\lambda,\alpha;t)\\
&=\left(\int_0^1\left|\begin{array}{l}h(1)-h(t)\\ \quad-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\end{array}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned} \tag{22}$$

*Proof:* By using the Hölder inequality, we obtain that

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)|\\
&\leqslant\frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\\
&\quad\times\left\{\left(\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\right.\\
&\quad\times\left(\int_0^1\left|F''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\\
&\quad\left.\times\left(\int_0^1\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

By using the convexity of $|F''|$, we get that

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)| \\
&\quad\leqslant \frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)} \\
&\qquad\times\left(\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\qquad\times\left\{\left[\int_0^1\left(\frac{1-t}{2}\left|F''(a)\right|^q+\frac{1+t}{2}\left|F''(b)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\right. \\
&\qquad\left.+\left[\int_0^1\left(\frac{1+t}{2}\left|F''(a)\right|^q+\frac{1-t}{2}\left|F''(b)\right|^q\right)\mathrm{d}t\right]^{\frac{1}{q}}\right\} \\
&\quad=\frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\left[|F''(a)|^q+|F''(b)|^q\right]^{\frac{1}{q}} \\
&\qquad\times\left(\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad=\frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\Psi_2(\lambda,\alpha;t)\left[|F''(a)|^q+|F''(b)|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof of Theorem 3.2. ■

**Theorem 3.3:** Let us note that assumptions of Lemma 2.1 hold. If the mapping $|F''|^q$ is convex on $[a,b]$ with $q>1$, then we have the following inequality

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)| \\
&\quad\leqslant \frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\left(\Psi_1(\lambda,\alpha;t)\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\frac{\left(\Psi_1(\lambda,\alpha;t)-\Psi_3(\lambda,\alpha;t)\right)|F''(a)|^q}{2}\right. \\
&\qquad\left.+\frac{\left(\Psi_1(\lambda,\alpha;t)+\Psi_3(\lambda,\alpha;t)\right)|F''(b)|^q}{2}\right)^{\frac{1}{q}} \\
&\qquad+\frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)}\left(\Psi_1(\lambda,\alpha;t)\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\frac{\left(\Psi_1(\lambda,\alpha;t)+\Psi_3(\lambda,\alpha;t)\right)|F''(a)|^q}{2}\right. \\
&\qquad\left.+\frac{\left(\Psi_1(\lambda,\alpha;t)-\Psi_3(\lambda,\alpha;t)\right)|F''(b)|^q}{2}\right)^{\frac{1}{q}},
\end{aligned} \tag{23}$$

where $\Psi_1(\lambda,\alpha;t)$ is defined in Theorem 3.1, and $\Psi_3(\lambda,\alpha;t)$ is defined by

$$\begin{aligned}
&\Psi_3(\lambda,\alpha;t) \\
&\quad=\int_0^1 t\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|\mathrm{d}t.
\end{aligned} \tag{24}$$

*Proof:* By applying the power-mean inequality, we obtain that

$$\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)| \\
&\quad\leqslant \frac{(b-a)^2}{8\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)} \\
&\qquad\times\left(\int_0^1\left|\begin{array}{l} h(1)-h(t) \\ \quad -\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\end{array}\right|\mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\qquad\times\left[\left(\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|\right.\right. \\
&\qquad\times\left.\left|F''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\qquad+\left(\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right|\right. \\
&\qquad\times\left.\left.\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{25}$$

Using convexity of the mapping $|F''|^q$ defined on the interval $[a,b]$, we obtain that

$$\begin{aligned}
&\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right| \\
&\qquad\times\left|F''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q\mathrm{d}t \\
&\quad\leqslant\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right| \\
&\qquad\times\left[\frac{1-t}{2}\left|F''(a)\right|^q+\frac{1+t}{2}\left|F''(b)\right|^q\right]\mathrm{d}t \\
&\quad=\frac{\left(\Psi_1(\lambda,\alpha;t)+\Psi_3(\lambda,\alpha;t)\right)|F''(b)|^q}{2} \\
&\qquad+\frac{\left(\Psi_1(\lambda,\alpha;t)-\Psi_3(\lambda,\alpha;t)\right)|F''(a)|^q}{2}.
\end{aligned} \tag{26}$$

Similarly, we have that

$$\begin{aligned}
&\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right| \\
&\qquad\times\left|F''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q\mathrm{d}t \\
&\quad\leqslant\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda(\frac{b-a}{2})}(\alpha,1)(1-t)\right| \\
&\qquad\times\left[\frac{1+t}{2}\left|F''(a)\right|^q+\frac{1-t}{2}\left|F''(b)\right|^q\right]\mathrm{d}t \\
&\quad=\frac{\left(\Psi_1(\lambda,\alpha;t)+\Psi_3(\lambda,\alpha;t)\right)|F''(a)|^q}{2} \\
&\qquad+\frac{\left(\Psi_1(\lambda,\alpha;t)-\Psi_3(\lambda,\alpha;t)\right)|F''(b)|^q}{2}.
\end{aligned} \tag{27}$$

Applying (26) and (27) to (25), we obtain the desired result of Theorem 3.3. This ends the proof. ■

**Corollary 3.2:** In Theorem 3.3, if we take $\lambda=0$, then we have the following inequality:

$$
\begin{aligned}
&|\mathcal{T}_F(\alpha,0;a,b)| \\
&\quad \leqslant \frac{(b-a)^2\alpha}{8}\left(\Theta_1(\alpha)\right)^{1-\frac{1}{q}} \\
&\quad \times\left(\frac{\left(\Theta_1(\alpha)-\Theta_2(\alpha)\right)|F''(a)|^q}{2}\right. \\
&\quad \left.+\frac{\left(\Theta_1(\alpha)+\Theta_2(\alpha)\right)|F''(b)|^q}{2}\right)^{\frac{1}{q}} \\
&\quad +\frac{(b-a)^2\alpha}{8}\left(\Theta_1(\alpha)\right)^{1-\frac{1}{q}} \\
&\quad \times\left(\frac{\left(\Theta_1(\alpha)+\Theta_2(\alpha)\right)|F''(a)|^q}{2}\right. \\
&\quad \left.+\frac{\left(\Theta_1(\alpha)-\Theta_2(\alpha)\right)|F''(b)|^q}{2}\right)^{\frac{1}{q}},
\end{aligned}
\tag{28}
$$

where $\Theta_1(\alpha)$ is defined in Corollary 3.1, and $\Theta_2(\alpha)$ is defined by

$$
\Theta_2(\alpha)=\begin{cases}
\dfrac{3-2\alpha}{18\alpha(\alpha+3)}, & \text{if } 0<\alpha\leqslant\dfrac{1}{2},\\[2ex]
\dfrac{4(\xi_\alpha)^{\alpha+3}}{\alpha(\alpha+1)(\alpha+3)}+\dfrac{2(\xi_\alpha)^{\alpha+3}}{\alpha(\alpha+3)} & \\
\quad -\dfrac{4(\xi_\alpha)^3}{9\alpha}+\dfrac{2(\xi_\alpha)^2}{3(\alpha+3)}-\dfrac{(\xi_\alpha)^2}{3\alpha(\alpha+3)} & \\
\quad +\dfrac{2(\xi_\alpha)^2(2\alpha-1)}{3\alpha(\alpha+1)(\alpha+3)} & \\
\quad +\dfrac{3-2\alpha}{18\alpha(\alpha+3)}, & \text{if } \alpha>\dfrac{1}{2},\ \xi_\alpha\in[0,1].
\end{cases}
\tag{29}
$$

*Proof:* $(i)$ Let us consider the function $\Delta(0,\alpha;t)\to\mathbb{R}$ by

$$
\Delta(0,\alpha;t)=\frac{2t(\alpha+1)-2\alpha+1-3t^{\alpha+1}}{3\alpha(\alpha+1)}
$$

with $0<\alpha\leq\frac{1}{2}$. Then, $\Delta(0,\alpha;t)\geq 0$ for all $t\in[0,1]$. Thus, it can be easily seen that

$$
\begin{aligned}
&\Psi_3(0,\alpha;t)\\
&\quad=\int_0^1 t|\Delta(0,\alpha;t)|\mathrm{d}t\\
&\quad=\frac{3-2\alpha}{18\alpha(\alpha+3)}.
\end{aligned}
$$

$(ii)$ if $\alpha>\frac{1}{2}$, then there exists a real number $\xi_\alpha\in[0,1]$ such that $\Delta(0,\alpha;t)\leq 0$ for $0\leq t\leq\xi_\alpha$ and $\Delta(0,\alpha;t)\geq 0$ for $\xi_\alpha\leq t\leq 1$. Therefore, we obtain that

$$
\begin{aligned}
&\Psi_3(0,\alpha;t)\\
&\quad=\int_0^1 t|\Delta(0,\alpha;t)|\mathrm{d}t\\
&\quad=\int_0^{\xi_\alpha} t(-\Delta(0,\alpha;t))\mathrm{d}t+\int_{\xi_\alpha}^1 t(\Delta(0,\alpha;t))\mathrm{d}t\\
&\quad=\frac{4(\xi_\alpha)^{\alpha+3}}{\alpha(\alpha+1)(\alpha+3)}+\frac{2(\xi_\alpha)^{\alpha+3}}{\alpha(\alpha+3)}-\frac{4(\xi_\alpha)^3}{9\alpha}\\
&\qquad+\frac{2(\xi_\alpha)^2}{3(\alpha+3)}-\frac{(\xi_\alpha)^2}{3\alpha(\alpha+3)}\\
&\qquad+\frac{2(\xi_\alpha)^2(2\alpha-1)}{3\alpha(\alpha+1)(\alpha+3)}+\frac{3-2\alpha}{18\alpha(\alpha+3)}.
\end{aligned}
$$

This ends the proof. ■

**Remark 3.2:** The result given in Corollary 3.2 is consistent with the result in Theorem 5 established by Hezenci et al. [33]. However, the function

$$
\Delta(0,\alpha;t)=\frac{2t(\alpha+1)-2\alpha+1-3t^{\alpha+1}}{3\alpha(\alpha+1)}
$$

defined in Lemma 2.2 is different from the function

$$
\varpi(\tau)=\frac{1-2\alpha}{3}+\frac{2(\alpha+1)}{3}\tau-\tau^{\alpha+1}
$$

considered in Lemma 2 by Hezenci et al. [33].

## IV. Examples

In this section, we provide some examples to illustrate our main results.

**Example 4.1:** Let the function $F(x)$ be defined by $F(x)=x^2$, $x\in[0,1]$. If we take $a=0, b=1$, $\alpha=1$ and $\lambda=2$, then all assumptions in Theorem 3.1 are satisfied.

The left-hand side term of the inequality established in Theorem 3.1 is

$$
\begin{aligned}
&|\mathcal{T}_F(\alpha,\lambda;a,b)|\\
&\quad=\left|\frac{1}{6}\left[F(0)+4F\left(\frac{1}{2}\right)+F(1)\right]\right.\\
&\qquad-\frac{\Gamma(1)}{\int_0^1 e^{-u}\mathrm{d}u}\left[\frac{1}{\Gamma(1)}\int_{\frac{1}{2}}^1 e^{-2\left(t-\frac{1}{2}\right)}t^2\mathrm{d}t\right.\\
&\qquad\left.\left.+\frac{1}{\Gamma(1)}\int_0^{\frac{1}{2}} e^{-2\left(\frac{1}{2}-t\right)}t^2\mathrm{d}t\right]\right|\\
&\quad\approx 0.0198.
\end{aligned}
\tag{30}
$$

The right-hand side term of the inequality established in Theorem 3.1 is

$$
\begin{aligned}
&\frac{(b-a)^2}{8\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)}\Psi_1\left[|F''(a)|+|F''(b)|\right]\\
&\quad=\frac{(b-a)^2}{8\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)}\left[|F''(a)|+|F''(b)|\right]\\
&\qquad\times\int_0^1\left|h(1)-h(t)-\frac{2}{3}\gamma_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)(1-t)\right|\mathrm{d}t\\
&\quad=\frac{1}{8\int_0^1 e^{-u}\mathrm{d}u}\left[|F''(0)|+|F''(1)|\right]\\
&\qquad\times\int_0^1\left|\int_0^1\gamma_{\lambda\left(\frac{1}{2}\right)}(\alpha,s)\mathrm{d}s\right.\\
&\qquad\left.-\int_0^t\gamma_{\lambda\left(\frac{1}{2}\right)}(\alpha,s)\mathrm{d}s-\frac{2(1-t)}{3}\int_0^1 e^{-u}\mathrm{d}u\right|\mathrm{d}t\\
&\quad=\frac{1}{8\int_0^1 e^{-u}\mathrm{d}u}\left[|F''(0)|+|F''(1)|\right]\\
&\qquad\times\int_0^1\left|\int_0^1\int_0^s e^{-u}\mathrm{d}u\mathrm{d}s\right.\\
&\qquad\left.-\int_0^t\int_0^s e^{-u}\mathrm{d}u\mathrm{d}s-\frac{2(1-t)}{3}\int_0^1 e^{-u}\mathrm{d}u\right|\mathrm{d}t\\
&\quad\approx 1.1664.
\end{aligned}
\tag{31}
$$

It is clear that $0.0198<1.1664$, which is consistent with the theoretical result described in Theorem 3.1.

**Example 4.2:** In Corollary 3.1, let us define the function $F : [0, 1] \to \mathbb{R}$ as $F(x) = 2x^4 + 3$ for all $0 < \alpha < 1$. Then, we have the following results, see TABLE I.

TABLE I: Numerical estimations of Corollary 3.1 for $F(x) = 2x^4 + 3$, $a = 0$ and $b = 1$.

| values of $\alpha$ | values of the left term | values of the right term |
|---|---|---|
| 0.1 | 0.2529 | 0.4286 |
| 0.2 | 0.2175 | 0.3636 |
| 0.3 | 0.1851 | 0.3043 |
| 0.4 | 0.1553 | 0.2500 |
| 0.5 | 0.1278 | 0.2000 |
| 0.6 | 0.1023 | 1.8569 |
| 0.7 | 0.0786 | 1.5973 |
| 0.8 | 0.0565 | 1.3152 |
| 0.9 | 0.0359 | 1.0061 |

Fig. 1 shows the visual analysis of Corollary 3.1.

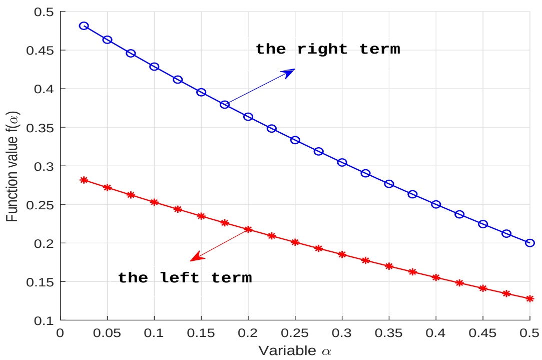


Fig. 1: The graphical representation of Corollary 3.1 for $F(x) = 2x^4 + 3$, $a = 0$, $b = 1$ and $0 < \alpha < 0.5$

From TABLE I and Fig. 1, we can intuitively observe that the value on the left part is less than the value on the right part, which is consistent with the theoretical result described in Corollary 3.1.

**Example 4.3:** In Corollary 3.2, let us define the function $F : [0, 1] \to \mathbb{R}$ as $F(x) = 3x^5 + 4$ for $\alpha = 0.1$ and $q = 1.5$, then we have the following result.

The left-hand side term of the inequality established in Corollary 3.2 is

$$\begin{aligned} &|\mathcal{T}_F(\alpha, 0; a, b)| \\ &= \left| \begin{array}{l} \frac{1}{6}\left[F(0) + 4F\left(\frac{1}{2}\right) + F(1)\right] - 2^{-0.9}\Gamma(1.1) \\ \times \left[ \begin{array}{l} \frac{1}{\Gamma(0.1)} \int_0^{\frac{1}{2}} \left(t - \frac{1}{2}\right)^{-0.9} (3t^5 + 4)\mathrm{d}t \\ + \frac{1}{\Gamma(0.1)} \int_{\frac{1}{2}}^1 \left(\frac{1}{2} - t\right)^{-0.9} (3t^5 + 4)\mathrm{d}t \end{array} \right] \end{array} \right| \\ &\approx 0.4127. \end{aligned}$$

The right-hand side term of the inequality established in Corollary 3.2 is

$$\begin{aligned} &\frac{(b-a)^2\alpha}{8} (\Theta_1(\alpha))^{1-\frac{1}{q}} \\ &\times \left( \begin{array}{l} \frac{(\Theta_1(\alpha)-\Theta_2(\alpha))\left|F''(a)\right|^q}{2} \\ + \frac{(\Theta_1(\alpha)+\Theta_2(\alpha))\left|F''(b)\right|^q}{2} \end{array} \right)^{\frac{1}{q}} \\ &+ \frac{(b-a)^2\alpha}{8} (\Theta_1(\alpha))^{1-\frac{1}{q}} \\ &\times \left( \begin{array}{l} \frac{(\Theta_1(\alpha)+\Theta_2(\alpha))\left|F''(a)\right|^q}{2} \\ + \frac{(\Theta_1(\alpha)-\Theta_2(\alpha))\left|F''(b)\right|^q}{2} \end{array} \right)^{\frac{1}{q}} \\ &\approx 1.1921. \end{aligned}$$

It is clear that $0.4127 < 1.1921$, which is consistent with the theoretical result presented in Corollary 3.2.

When $0 < \alpha < 0.5$, the correctness of the results given in Corollary 3.2 is shown in Fig. 2.

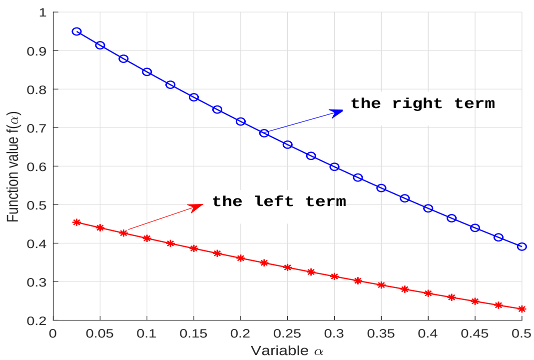


Fig. 2: The graphical representation of Corollary 3.2 for $F(x) = 3x^5 + 4$, $a = 0$, $b = 1$, $q = 1.2$ and $0 < \alpha < 0.5$

From Fig. 2, we can intuitively observe that the value on the left part is less than the value on the right part, which is consistent with the theoretical result established in Corollary 3.2.

## V. Conclusions

In the current study, we apply the $\lambda$-incomplete gamma functions to generalize a series of results, which involve the Simpson-type inequalities with respect to twice-differentiable functions. To obtain the novel results in the investigation, we develop a twice-differentiable Simpson-type identity by virtue of the tempered fractional integrals. Here, we would like to emphasize that the results obtained in the paper generalize the inequalities given by Hezenci et al. [33]. With these contributions, we believe that the approaches of the present study could be a source of enlightenment for researchers working in the Simpson-type inequalities field. In future studies, researchers can try to generalize our results by using other kinds of convex functions or different types fractional integral operators.

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