Comments on a Simple Method to Compute Economic Order Quantities

Te-Yuan Chiang, Liqiu Zhou, Zhigao Luo, Hailin Li

***Abstract*—This study examines several papers to study inventory models with back orders to provide a simpler solution procedure for EOQ/EPQ inventory models by using Arithmetic Geometric Mean inequality or Cauchy Bunyakovsky Schwarz inequality. We will point out that a paper implicitly adopted the fill rate from a published article without proper citation to convert two-variable minimum problems into one-variable problems. Hence, his simpler solution procedure is standing on the shoulder of a giant. We also provide detailed examinations for other three related papers that was related to our studied article to point out their contributions and questionable results.**

***Index Terms*—Inventory models, Without calculus, Fill rate, Arithmetic Geometric Mean inequality**

I. Introduction

In the revolution of academic literature, if a author can provide a simplified solution process to replace previously established lengthy and hard-working procedure, then his breaking through achievement will have great impact on scholastic society. Teng [1] announced that he created a new solution approach that is superior to Wu and Ouyang [2], Wee and Chung [3], Sphicas [4], Ronald et al. [5], Grubbström and Erdem [6], and Chang et al. [7]. This astonished assertion arouses our attention which deserves a careful study. Until now, there are 46 papers that had been referred Teng [1] in their References. We divide them into the next three parts.

(A) There are three comprehensive literature survey papers: Andriolo et al. [8], Drake and Marley [9] and Shekarian et al. [10] such that Teng [1] did not examined in detail within these three reviewing articles.

(B) Those 40 papers only mentioned Teng [1] in their Introduction and then they concentrated on their new inventory models: Leung [11], Teng and Goyal [12], Chang and Ho [13], Dellino et al. [14], Widyadana and Wee [15], Cárdenas-Barrón [16], Cárdenas-Barrón et al. [17], Chang and Ho [18], Pasandideh et al. [19], Teng et al. [20], Widyadana et al. [21], Chang [22], Chung [23], Li et al. [24], Yadav et al. [25, 26], Chung [27, 28], Gambini et al. [29], Lou and Wang [30], Teng et al. [31], Wee et al. [32], Yadav et al. [33], Chang [34], Chen et al. [35], Chern et al. [36], Gambini et al. [37], Liao et al. [38], Teng et al. [39], Nagoor Gani and Raja Dharik [40], Teng and Hsu [41], Chen et al. [42], Kumar and Aarya [43], Niknamfar and Niaki [44], Gong et al. [45], Goyal et al. [46], Kojić [47], Liao et al. [48], Rajan and Uthayakumar [49] and Vidal-Carreras et al. [50]. These 40 articles did not consider Teng [1] to provide further study.

(C) These three papers are really related to Teng [1]: Cárdenas-Barrón [51, 52] and Leung [53]. We will check these three papers in detail to crystallize their interrelationship with Teng [1].

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II. Assumptions and Notation

Because Teng [1] is the source article, we use the same assumptions and notation as his paper.

Assumptions

(1) $TC_1(Q)$ is the EOQ model without backorders.

(2) $TC_2(Q)$ is the EOQ model with backorders.

(3) $TC_3(Q)$ is the EPQ model with backorders.

(4) Both the initial and the ending inventory levels are zero so that there is no salvage value.

(5) There is no quantity discount.

(6) The lead time is zero so that replenishment is instantaneous.

Notation

v is the backorder cost per unit per unit of time.
T is the planning horizon for one replenishment.
r is the fill rate, that is the ratio between the inventory period and one replenishment cycle.
Q is the order quantity per replenishment.
Q^* is the optimal order quantity.
p is the production rate per unit of time.
d is the constant demand per unit of time.
h is the holding cost per unit and per unit of time.
A is the ordering cost per replenishment.

III. Recap of the Previous Results

Teng [1] mentioned that he used the Arithmetic Geometric Mean inequality, which is

$$(a+b)/2 \geq \sqrt{ab}\,, \qquad (3.1)$$

for any two positive real numbers. The equality holds if and only if a = b.

Tang [1] considered three inventory models: (a) EOQ model without backorders, (b) EOQ model with backorders, and (c) EPQ model with backorders.

For EOQ model without backorders, the objective function is expressed as follows,

$$TC_1(Q)=\frac{Ad}{Q}+\frac{hQ}{2}. \tag{3.2}$$

Teng [1] applied Arithmetic Geometric Mean inequality to derive

$$TC_1^*(Q)=\sqrt{2Adh}, \tag{3.3}$$

and

$$Q^*=\sqrt{2Ad/h}. \tag{3.4}$$

For EOQ model with backorders, Teng [1] cited the objective function from Wee et al. [54],

$$TC_2(Q)=\frac{Ad}{Q}+\frac{Q}{2}\left[v(1-r)^2+hr^2\right], \tag{3.5}$$

Teng [1] applied Arithmetic Geometric Mean inequality to derive

$$TC_2^*(Q)=\sqrt{2Ad\left[v(1-r)^2+hr^2\right]}, \tag{3.6}$$

and

$$Q^*=\sqrt{2Ad/\left[v(1-r)^2+hr^2\right]}. \tag{3.7}$$

For EPQ model with backorders, Teng [1] cited the objective function from Wee et al. [54],

$$TC_3(Q)=\frac{Ad}{Q}+\frac{Q}{2}\left(\frac{p-d}{p}\right)\left[v(1-r)^2+hr^2\right], \tag{3.8}$$

Teng [1] applied Arithmetic Geometric Mean inequality to derive that

$$TC_3^*(Q)=\sqrt{2(p-d)dA\left[v(1-r)^2+hr^2\right]/p}, \tag{3.9}$$

and

$$Q^*=\sqrt{2dA/\left\{\left[v(1-r)^2+hr^2\right](p-d)\right\}}. \tag{3.10}$$

Teng [1] mentioned that his solution approach is easy to apply and simple to understand. We quote his assertion in his Conclusions "By using the proposed method, we also can obtain the global minimum solutions much easier and simpler than the method of computing perfect square established by Chang et al. [7], Grubbström and Erdem [6], Ronald et al. [5], Sphicas [4], Wee and Chung [3], Wu and Ouyang [2], and others."

IV. Our Comments for Teng

For Economic Ordering Quantity model with backorders, the objective function of Wee et al. [54] is expressed as follows,

$$TC_2(Q,r)=\frac{Ad}{Q}+\frac{Q}{2}\left[hr^2+v(1-r)^2\right], \tag{4.1}$$

and then Wee et al. [54] applied Cost Difference Comparison Method to derive that

$$r^*=v/(h+v). \tag{4.2}$$

In Teng [1], he never mentioned the result of Equation (4.2) that is a severe violation of academic integrity.

Ronald et al. [5] and Grubbström and Erdem [6] considered EOQ model with backorders, in different expression,

$$C(Q,B)=\frac{D}{B+Q}\left(\frac{b}{2D}B^2+\frac{h}{2D}Q^2+K\right). \tag{4.3}$$

We transferred their expressions to the same notation as Wee et al. [54], then

$$C(rQ,(1-r)Q)=\frac{Ad}{Q}+\frac{h}{2}(rQ)^2+\frac{v}{2}[Q(1-r)]^2. \tag{4.4}$$

Ronald et al. [5] and Grubbström and Erdem [6] were facing two-variable minimum problem. Teng [1] adopted results from Wee et al. [54] to convert his two-variable problem into one-variable problem, and then Teng [1] asserted that his solution approach is simpler than Ronald et al. [5] and Grubbström and Erdem [6]. It is a bias assertion, because Teng [1] overlooked the contribution of Wee et al. [54].

For EPQ model with backorders, Chang et al. [7] considered the following minimum problem, in different expression,

$$C(B,Q)=\frac{D}{B+Q}\left(\frac{b}{2D}B^2+\frac{h}{2D}Q^2+K\rho\right)+cD, \tag{4.5}$$

We transferred their expressions of Equation (4.5) to the same notation as Wee et al. [54], then

$$C(rQ,(1-r)Q)=\frac{Ad}{Q}+\frac{(p-d)hr^2Q^2}{2p}, +cd+\frac{(p-d)v(1-r)^2Q^2}{p}, \tag{4.6}$$

such that Chang et al. [7] was facing two-variable minimum problem.

In Wee et al. [54], they did not apply Cost Difference Comparison Method to find r^* for EPQ model with backorders. Instead, Wee et al. [54] claimed that by the similar argument for Economic Ordering Quantity model with backorders, they can derive $r^*=v/(h+v)$.

Teng [1] implicitly quoted the results from Wee et al. [54] to simplify $TC_3(Q,r)$ with two variables: Q and r, to $TC_3(Q)$ with one variable, Q. However, Teng [1] did not mention the relation of Equation (4.2).

Consequently, the superiority of Teng [1] with $TC_3(Q)$ of Equation (3.8) to Chang et al. [7] with $C(rQ,(1-r)Q)$ of Equation (4.6) is a false assertion, because Teng [1] overlooked the key step that was solved by Wee et al. [54].

V. Review of Early Paper of Cárdenas-Barrón

Cárdenas-Barrón [51] extended the Arithmetic Geometric Mean inequality mentioned by Teng [1] to a more general setting, such that $a_1, a_2, \dots, a_n$ are real and positive numbers, then the Arithmetic Geometric Mean inequality is defined as follows,

$$(a_1+a_2+\dots+a_n)/n\ge\sqrt[n]{a_1a_2\dots a_n}, \tag{5.1}$$

with equality if and only if $a_1=a_2=\dots=a_n$, which is also known as Cauchy Bunyakovsky Schwarz inequality.

Cárdenas-Barrón [51] considered the same Economic Ordering Quantity model with backorders, but in different expression, where B is the maximum backorders level and Q is the order quantity such that $Q-B$ is the maximum inventory level,

$$TC(B,Q)=\frac{vB^2}{2Q}+\frac{h(Q-B)^2}{2Q}+\frac{Ad}{Q}, \tag{5.2}$$

Cárdenas-Barrón [51] rewrote Equation (5.2) as

$$TC(B,Q)=\left\{\left[\sqrt{h}\left(1-\frac{B}{Q}\right)\right]^2+\left[\sqrt{v}\frac{B}{Q}\right]^2\right\}\frac{Q}{2}+\frac{Ad}{Q}. \quad (5.3)$$

Cárdenas-Barrón [51] adopted a genius formula

$$1=\left\{\left(\sqrt{v/(v+h)}\right)^2+\left(\sqrt{h/(h+v)}\right)^2\right\}, \quad (5.4)$$

to further rewrite Equation (5.4) as

$$\mathrm{TC}(B,Q)=\left\{\left[\sqrt{h}\left(1-\frac{B}{Q}\right)\right]^2+\left(\sqrt{v}\frac{B}{Q}\right)^2\right\},$$
$$\times\left[\left(\sqrt{\frac{v}{v+h}}\right)^2+\left(\sqrt{\frac{h}{v+h}}\right)^2\right]\frac{Q}{2}+\frac{dA}{Q}. \quad (5.5)$$

Cárdenas-Barrón [51] applied the Cauchy Bunyakovsky Schwarz inequality to Equation (5.5) to derive that

$$\mathrm{TC}(B,Q)\geq\left\{\left[\sqrt{h}\left(1-\frac{B}{Q}\right)\right]\sqrt{\frac{v}{v+h}},\right.$$
$$\left.+\sqrt{v}\frac{B}{Q}\sqrt{\frac{h}{v+h}}\right\}^2\frac{Q}{2}+\frac{dA}{Q}. \quad (5.6)$$

The equality holds in Equation (5.6) if and only if

$$\frac{\sqrt{h}\left(1-\frac{B}{Q}\right)}{\sqrt{\frac{v}{v+h}}}=\frac{\sqrt{v}\frac{B}{Q}}{\sqrt{\frac{h}{v+h}}}. \quad (5.7)$$

Cárdenas-Barrón [51] simplified Equation (5.7) to derive that

$$hQ=(v+h)B. \quad (5.8)$$

Cárdenas-Barrón [51] rewrote Equation (5.6) to imply that

$$\mathrm{TC}(B,Q)\geq\sqrt{\frac{vh}{v+h}}\frac{Q}{2}+\frac{dA}{Q}. \quad (5.9)$$

Based on Equation (5.9), Cárdenas-Barrón [51] showed that

$$Q^*=\sqrt{2dA(v+h)/hv}, \quad (5.10)$$

with

$$B^*=\frac{h}{v+h}Q^*$$
$$=\sqrt{2dAh/(v+h)v}, \quad (5.11)$$

and

$$\mathrm{TC}(Q^*,B^*)=\sqrt{2vhdA/(v+h)}. \quad (5.12)$$

We can claim that Cárdenas-Barrón [51] provided a patchwork for Teng [1] for applying Cauchy Schwarz inequality and Arithmetic Geometric Mean inequality to solve Economic Ordering Quantity model with backorders.

Moreover, for Economic Production Quantity model with backorders,

$$TC(B,Q)=\frac{vB^2}{2Q\delta}+\frac{h(Q\delta-B)^2}{2Q\delta}+\frac{Ad}{Q}, \quad (5.13)$$

with an abbreviation,

$$\delta=1-(d/p), \quad (5.14)$$

Cárdenas-Barrón [51] repeated his solution approach for Economic Ordering Quantity models to solve his Economic Production Quantity models. Cárdenas-Barrón [51] overlooked Sphicas [4] already used $\delta=1-(d/p)$ of Equation (5.14) and another two abbreviations,

$$\hat{Q}=Q\delta, \quad (5.15)$$

and

$$\hat{A}=A\delta, \quad (5.16)$$

to rewrite Equation (5.13) as

$$TC(\hat{Q},B)=\frac{\hat{A}d}{\hat{Q}}+\frac{h(\hat{Q}-B)^2}{2\hat{Q}}+\frac{vB^2}{2\hat{Q}}. \quad (5.17)$$

If we neglect "cap" of Equation (5.17), then Equation (5.17) is identical to Equation (5.2) such that findings from Economic Ordering Quantity model can be directly applied for Economic Production Quantity model, without the lengthy repeated computation proposed by Cárdenas-Barrón [51].

Based on our above examination of Cárdenas-Barrón [51], we claimed that the solution procedure of Cárdenas-Barrón [51] with respect to the Economic Ordering Quantity model with backorders by the Cauchy Bunyakovsky Schwarz inequality is an excellent finding. However, his solution process with respect to the Economic Production Quantity model with backorders is tiresome and unnecessary.

VI. Review of Later Paper of Cárdenas-Barrón

Cárdenas-Barrón [52] mentioned that there are three restrictions that must be hold, before applying Arithmetic Geometric Mean inequality to solve minimum problems as applied by Teng [1], and then Cárdenas-Barrón [52] criticized Wee et al. [54] did not find the backorders level.

Cárdenas-Barrón [52] referred to the fill rate

$$r=v/(v+h), \quad (6.1)$$

derived by Wee et al. [54] by Cost Difference Comparison Method to claim that

$$B=Q(1-r)=\left(\frac{h}{h+v}\right)Q, \quad (6.2)$$

$$Q=\sqrt{\frac{2Ad(v+h)}{vh}}, \quad (6.3)$$

and

$$\mathrm{TC}=\sqrt{2vhdA/(v+h)}. \quad (6.4)$$

However, in Cárdenas-Barrón [52], he did not inform us how to derive Equations (6.3) and (6.4). Similar problems occur for Economic Production Quantity model with backorders of Cárdenas-Barrón [52].

We find that Wee et al. [54] did not write the backorder level, B for their Economic Ordering Quantity and Economic Production Quantity models. However, Wee et al. [54] already found Q^* and r^* such that the derivation of the maximum backorder level as

$$B^*=Q^*(1-r^*), \quad (6.5)$$

becomes a trivial exercise.

The similar problem occurs for his criticism for the backorder level of Wee et al. [54] for Economic Production Quantity model with backorders. Cárdenas-Barrón [52] provided a patch work for the missing discussion for fill rate in Wee et al. [54] for Economic Production Quantity models and then Cárdenas-Barrón [52] still obtained the fill rate $r=v/(h+v)$ as Equation (6.1). However, the optimal order quantity for Economic Production Quantity models,

$$Q=\sqrt{\frac{2Ad(h+v)}{(1-(d/p))hv}}, \tag{6.6}$$

still suddenly appeared in Cárdenas-Barrón [52] without any explanation.

Based on our above discussion, we claimed that Cárdenas-Barrón [52] did not provide any meaning improvement for solving inventory systems.

VII. Review of Leung

There are two comments proposed by Leung [53] with respect to Teng [1]. For the first comment, Leung [53] noticed that in Teng [1], the fill rate, r, is not treated. Instead to cite from Wee et al. [54] for

$$r=v/(h+v), \tag{7.1}$$

Leung [53] considered to minimize the last term of Equation (3.5) used by Teng [1], to denote it as a new expression, say $\eta(r)$, with

$$\eta(r)=v(1-r)^2+hr^2, \tag{7.2}$$

to derive

$$\eta(r)=(v+h)\left(r-\frac{v}{h+v}\right)^2+\frac{vh}{v+h}. \tag{7.3}$$

Hence, Leung [53] obtained the optimal fill rate,

$$r^*=v/(v+h), \tag{7.4}$$

as Equation (4.2) or Equation (7.1) proposed by Wee et al. [54].

For the second comment, Leung [53] mentioned that for Economic Ordering Quantity model with backorders, then the objective function should contained two variables: Q and r such that Leung [53] changed Equation (3.5) to the following expression,

$$TC(Q,r)=\frac{Ad}{Q}+\frac{Q}{2}\left[hr^2+v(1-r)^2\right]. \tag{7.5}$$

Leung [53] pointed out that the solution procedure should be divided into three parts: Situation (i): $0<r<1$, Situation (ii): $r=0$, and Situation (iii): $r=1$.

Under three different situations, Leung [53] found that

$$\mathrm{TC}_{\mathrm{i}}^*(\mathrm{r}^*,\mathrm{Q}^*)=\sqrt{2\mathrm{Avhd}/(\mathrm{v}+\mathrm{h})}, \tag{7.6}$$

$$\mathrm{TC}_{\mathrm{ii}}^*(\mathrm{Q}_{\mathrm{ii}}^*,\mathrm{r}=0)=\sqrt{2\mathrm{Adv}}, \tag{7.7}$$

and

$$\mathrm{TC}_{\mathrm{iii}}^*(\mathrm{Q}_{\mathrm{iii}}^*,\mathrm{r}=1)=\sqrt{2\mathrm{Adh}}. \tag{7.8}$$

Leung [53] compared findings of Equations (7.6-7.8) to imply that

$$\mathrm{TC}_{\mathrm{ii}}^*(\mathrm{Q}_{\mathrm{ii}}^*,\mathrm{r}=0)>\mathrm{TC}_{\mathrm{i}}^*(\mathrm{r}^*,\mathrm{Q}^*), \tag{7.9}$$

and

$$\mathrm{TC}_{\mathrm{iii}}^*(\mathrm{Q}_{\mathrm{iii}}^*,\mathrm{r}=1)>\mathrm{TC}_{\mathrm{i}}^*(\mathrm{r}^*,\mathrm{Q}^*), \tag{7.10}$$

to conclude that $\mathrm{TC}_{\mathrm{i}}^*(\mathrm{r}^*,\mathrm{Q}^*)$ is the global minimum.

We will show that the second comment of Leung [53] is unnecessary. We recall the first comment of Leung [53] by algebraic method to derive $r^*=v/(h+v)$ for $0\le r\le 1$ such that there is unnecessary to divide $0\le r\le 1$ into three different cases as Case (i): $0<r<1$, Case (ii): $r=0$, and Case (iii): $r=1$.

We can offer a reasonable explanation why did Leung [53] assume three different situations: Situation (i): $0<r<1$, Situation (ii): $r=0$, and Situation (iii): $r=1$.

If researchers used calculus to find $r^*=v/(h+v)$, because calculus only handles interior points that is $0<r<1$ and then researchers will have three different objective functions: one for interior points, and two for two boundaries, $r=0$ and $r=1$. For these three objective functions, each one have its own local minimums and then researchers have to compare three local minimums to decide which one is the global minimum.

However, Leung [53] applied algebraic method without calculus such that there is only one objective function that is suitable for $0\le r\le 1$ to consider three different situations is tedious and useless.

Moreover, after the first challenge proposed by Leung [53] to Teng [1], Leung [53] already the optimal solution of r^* as Equation (7.4). Consequently, the optimal solution of r is already decided. Leung [53] further divided the domain of r with $0\le \mathrm{r}\le 1$, into three different situations: Situation (i): $0<r<1$, Situation (ii): $r=0$, and Situation (iii): $r=1$. This partition of $0\le \mathrm{r}\le 1$ is not only unnecessary but also confuse ordinary practitioners.

Based on our above discussion, we first pointed out that Leung [53] overlooked the existing results of Wee et al. [54] to derive the identical finding as Wee et al. [54], and then using his repeated derivation to challenge Teng [1] that is a severe violation in academic society. For the second comment proposed by Leung [53] to challenge Teng [1] is another questionable assertion. Based on the first comment proposed by Leung [53] to challenge Teng [1], we know that Leung [53] already obtained the optimal solution of r^* as

Equation (7.4). However, Leung [53] separated the domain of r into three parts: the interior, $0<r<1$, and two boundaries, $r=0$ and $r=1$. Moreover, Leung [53] proved that two local minimums along two boundaries is greater than the local minimum in the interior. We conclude that his partitions were unnecessary and his proof is right but other two repeated derivations of already known results.

VIII. A Related Inventory Model

In this section, we will apply geometric program approach to solve inventory systems that was used by Cheng [55]. We recalled the traditional economic ordering quantity inventory model of Equation (3.2) as follows,

$$TC_1(Q)=\frac{Ad}{Q}+\frac{hQ}{2}. \tag{8.1}$$

Based on geometric program approach, there are two steps for the solution procedure. In the first step, we assumed that

$$\alpha_1=\frac{Ad}{Q}, \tag{8.2}$$

and

$$\alpha_1=\frac{hQ}{2}, \tag{8.3}$$

and then we tried to solve the following minimum problem,

$$f(b_1,b_2)=\left(\frac{\alpha_1}{b_1}\right)^{b_1}\left(\frac{\alpha_2}{b_2}\right)^{b_2}. \tag{8.4}$$

We plugged Equations (8.2) and (8.3) into Equation (8.4) to derive that

$$f(b_1,b_2)=\left(\frac{Ad}{b_1}\right)^{b_1}\left(\frac{h}{2b_2}\right)^{b_2}Q^{b_2-b_1}, \tag{8.5}$$

under the conditions of (i)

$$b_1+b_2=0, \tag{8.6}$$

and (ii) the exponential index of Q equals to zero. Hence, we obtain that

$$b_2-b_1=0. \tag{8.7}$$

According to Equations (8.6) and (8.7), we derived that

$$b_1=1/2. \tag{8.8}$$

and

$$b_2=1/2. \tag{8.9}$$

In the second step, geometric program approach assumed that

$$\frac{\alpha_1}{b_1}=\frac{\alpha_2}{b_2}. \tag{8.10}$$

We plugged the findings of Equations (8.8) and (8.9) into Equation (8.10) to show that

$$Q=\sqrt{2Ad/h}, \tag{8.11}$$

which is identical to the result of Equation (3.4).
Moreover, we compute $f(b_1^*,b_2^*)$, then

$$f(b_1^*,b_2^*)=\sqrt{2Adh}, \tag{8.12}$$

which is identical to the result of Equation (3.3).

Based on the above discussion, we demonstrate that geometric program approach can solve the traditional economic ordering quantity inventory model without back orders.
We may predict that applying geometric program approach to solve the traditional economic ordering quantity inventory model with back orders as Equation (4.3) will be an interesting research topic.

IX. A Similar Approach with Back Orders

In this section, we followed Teng [1] to cite the results of Wee et al. [54] to use the objective function as Equation (3.5), and then we know the inventory system with bask orders,

$$TC_2(Q)=\frac{Ad}{Q}+\frac{Q}{2}\left[v(1-r)^2+hr^2\right], \tag{9.1}$$

We begin to apply geometric program approach to solve inventory model of Equation (9.1).
We assumed that

$$\beta_1=\frac{Ad}{Q}, \tag{9.2}$$

and

$$\beta_2=\frac{Q}{2}\left[v(1-r)^2+hr^2\right], \tag{9.3}$$

such that we will execute the first of geometric program approach to solve the next minimum problem,

$$g(c_1,c_2)=\left(\frac{\alpha_1}{c_1}\right)^{c_1}\left(\frac{\alpha_2}{c_2}\right)^{c_2}. \tag{9.4}$$

We plugged Equations (9.2) and (9.3) into Equation (9.4) to derive that

$$g(c_1,c_2)=\left(\frac{Ad}{c_1}\right)^{c_1}\left(\frac{v(1-r)^2+hr^2}{2c_2}\right)^{c_2}Q^{c_2-c_1}, \tag{9.5}$$

under the conditions of (i)

$$c_1+c_2=0, \tag{9.6}$$

and (ii) the exponential index of Q equals to zero. Hence, we obtained that

$$c_2-c_1=0. \tag{9.7}$$

According to Equations (9.6) and (9.7), we derived that

$$c_1=1/2. \tag{9.8}$$

and

$$c_2=1/2. \tag{9.9}$$

In the second step, we recalled that geometric program approach obtained that

$$\frac{\beta_1}{c_1}=\frac{\beta_2}{c_2}. \tag{9.10}$$

We plugged the findings of Equations (9.8) and (9.9) into Equation (9.10) to show that

$$Q=\sqrt{2Ad/(v(1-r)^2+hr^2)} \tag{9.11}$$

which is identical to the result of Equation (3.7).
Moreover, we compute $g(c_1^*,c_2^*)$, then

$$g(c_1^*,c_2^*)=\sqrt{2Ad(v(1-r)^2+hr^2)}, \tag{9.12}$$

which is identical to the result of Equation (3.6).

The above derivations point out that if we followed Teng [1] standing on the shoulder of Wee et al. [54] implicitly used the result of Equation (4.2), and then geometric program approach can solve the inventory system with back orders.

X. Direction for Future Study

In this section, we cited several recently published articles to help researchers to find possible future study topics. Referring to salvage value and shortages, Liu [56] developed inventory systems with deterioration products. Based on a class experiment in the Makassar City with eight grade purples of state junior high schools, Side et al. [57] studied obsession of internet video matches by analysis and optimal control. According to stability analysis and analytical solutions, Suksamran et al. [58] examined COVID 19 inflection and immunity by vaccine for diffusion reaction. Considering integral over determination, Batiha et al. [59]

constructed a time fractional parabolic equation to examine a super linear question. Applying algebraic methods, Wu and Liu [60] studied inventory systems. For shortages cost under a setting of fuzzy conditions, Liu [61] improved a stochastic newsboy problem. Under neutrosophic dual topological families, Das et al. [62] defined new open spaces for neutrosophic relationship under dual environment. For enhanced wild string, Dapo et al. [63] executed tentative exploration to study various fiber effect. For dimensionless mathematical system under similar configuration, Manilam and Pochai [64] applied finite different procedure to study unrestricted establishment. For several energetic networks under various facet, Yan et al. [65] extended function project to deal with short time lag management. Referring to graphical theorem, Medini et al. [66] examined multiple face problems with respect to cipher memorandum. Umilasari et al. [67] considered administrated family to derive local solutions to search for product graphs. Applying the Lp norm to deal with facial pattern recognition environment, Wang et al. [68] constructed a extended categorization. According to optimization process with iterative skills, Paul et al. [69] developed a representation solidity system. Based on our above examinations, in the future, practitioners can locate interested research trend for their developments.

XI. Conclusion

Teng [1] adopted findings of Wee et al. [54] to change two-variable minimum problems into one-variable minimum problems, and then Teng [1] claimed that his solution approach is simpler than several papers dealt with two-variable setting. Teng [1] overlooked that he had already used the findings of Wee et al. [54], such that the comparison of his solution approach with those previously published papers of Wu and Ouyang [2], Wee and Chung [3], Sphicas [4], Ronald et al. [5], Grubbström and Erdem [6], and Chang et al. [7], is unfair and misleading.

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