# New Inner Bounds for the Extreme Eigenvalues of Real Symmetric Matrices

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*Abstract*—**In this paper, we advocate a new technique to determine inner bounds for the extreme eigenvalues of real symmetric matrices. Our method involves the matrix elements and compares favourably with existing methods. We also show how the bounds can be optimized.**

*Index Terms*—**positive definite matrix, eigenvalues, bounds.**

## I. Introduction

Knowledge of eigenvalues and eigenvectors are crucial in almost all spheres of engineering and science. The determination of the eigenvalues by solving the characteristic $n_{th}$ degree polynomial equation $\det(\lambda\mathbf{I}-\mathbf{A})=0$ is challenging, especially for large values of the matrix dimension $n$. Thus several methods have evolved that determine the eigenvalues together with the eigenvectors. However, in some cases only bounds for the extremal eigenvalues may suffice. These are particulary important in approximation theory, especially the spread $sp(\mathbf{A})$, spectral radius $\rho(\mathbf{A})$ and condition number $|\frac{\lambda_1}{\lambda_n}|$, where $\lambda_1$ and $\lambda_n$ are the dominant and least dominant eigenvalues of a symmetric matrix $\mathbf{A}$, in the absolute sense. For symmetric matrices especially $\mathbf{A}\in\mathbb{R}^{n\times n}$, the existence of real eigenvalues and real eigenvectors simplify the eigenvalue problem tremendously. However, the task is still daunting for large $n$. Weinstein bounds [3] depend on an approximate eigenpair and bounds a portion of the spectrum. Kato bounds [7] is an improvement of the latter. It is well known that the Temple quotient [14] provides a lower bound for the smallest eigenvalue, and is a special case of Lehmann's method [8]. Brauer bounds [1], using the interlacing property for Hermitian matrices and Rayleigh's quotient [6] give better results. For positive definite symmetric matrices Dembo bounds [4] arise by examining the characteristic equation of $\mathbf{A}$ and depends on bounds of a principal submatrix. Sun [13] bounded the minimal eigenvalue of positive definite matrices, improving on Dembo bounds. However, we stress that the prior mentioned methods all require some additional information regarding $\sigma(\mathbf{A})$. The following methods rely only on the entries of the matrix, though they can be very effective. Some crude methods of this type are based on Gerschgorin disks and the ovals of Cassini [2]. The latter two methods are particularly useful for sparse matrices, say tridiagonal, especially when a disk or oval is disjoint from the rest. Mirskey [9], Brauer and Mewbom [1] used traces to bound $sp(\mathbf{A})$. Wolcowicz and Styan [15] used a statistical approach for the extremal bounds which resulted in the employment of trace bounds. Sharma et al. [10] extended and improved the work of Wolcowiz and Styan. Singh et al. [11], [12] generalized the work of Sharma, and Wolcowicz and Styan, by employing functions of the matrix $\mathbf{A}$. Trace bounds are elegant as they are functions only of the diagonal entries of a matrix and its associated powers. Here we shall utilize more information from all the matrix entries. Whilst here we discuss real symmetric matrices, we must bear in mind that these are important in bounding certain forms of block $2\times 2$ matrices. For example in [16], the spectral bounds of a preconditioned block matrix, depends on the bounds of each of the component matrices. Here the first matrix on the diagonal is real positive definite, while the second matrix on the diagonal is real positive semi-definite.

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## II. Theory

Let $\mathbf{A}=(a_{ij})\in\mathbb{R}^{n\times n}$ be a symmetric matrix, with spectrum $\sigma(\mathbf{A})=\{\lambda_i\}_{i=1}^n$ and associated normalized eigenvectors denoted by $\{\mathbf{u}_i\}_1^n$ . It is well known that $\mathbf{A}$ is unitarily diagonalizable [6]. Thus it follows from the spectral theorem that

$$\mathbf{I}=\sum_{i=1}^{n}\mathbf{G}_i$$
$$\mathbf{A}=\sum_{i=1}^{n}\lambda_i\mathbf{G}_i. \quad (1)$$

Here $\mathbf{G}_i$ is the orthogonal projector onto the nullspace $N(\mathbf{A}-\lambda_i\mathbf{I})$ along the range $R(\mathbf{A}-\lambda_i\mathbf{I})$, $\mathbf{G}_i=\mathbf{u}_i\mathbf{u}_i^t$ and satisfies $\mathbf{G}_i\mathbf{G}_j=\delta_{ij}\mathbf{I}$, where $\delta_{ij}$ denotes the well known Kronecker delta symbol. Assume that the eigenvalues are arranged in the order

$$\lambda_n\le\lambda_{n-1}\le\cdots\lambda_2\le\lambda_1. \quad (2)$$

Let $\langle\cdot,\cdot\rangle$ denote the standard innerproduct in $\mathbb{R}^n$ .

*Theorem 2.1:* Let $\mathbf{A}\in\mathbb{R}^{n\times n}$ be symmetric with the eigenvalues arranged as in (2). We have for $\mathbf{x}\in\mathbb{R}^n$, $\|\mathbf{x}\|_2=1$ that

$$\lambda_n\le\langle\mathbf{A}\mathbf{x},\mathbf{x}\rangle\le\lambda_1$$

*Proof:*

$$\langle\mathbf{A}\mathbf{x},\mathbf{x}\rangle=\left\langle\sum_{i=1}^{n}\lambda_i\mathbf{G}_i\mathbf{x},\mathbf{x}\right\rangle=\sum_{i=1}^{n}\lambda_i\langle\mathbf{G}_i\mathbf{x},\mathbf{x}\rangle \quad (3)$$

Since $\lambda_n \le \lambda_i \le \lambda_1$, for all $i$, it follows from (3) that

$$\lambda_n \le \lambda_n \left\langle \sum_{i=1}^{n} \mathbf{G}_i\mathbf{x}, \mathbf{x} \right\rangle \le \langle \mathbf{Ax}, \mathbf{x} \rangle \le \lambda_1 \left\langle \sum_{i=1}^{n} \mathbf{G}_i\mathbf{x}, \mathbf{x} \right\rangle \le \lambda_1.$$

■

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be orthonormal vectors and consider the two dimensional subspace span$\{\mathbf{u}, \mathbf{v}\}$. Let

$$\mathbf{x} = \frac{\mathbf{u} + \alpha\mathbf{v}}{\sqrt{1+\alpha^2}}, \tag{4}$$

where $\alpha \in \mathbb{R}$ is a parameter, then $\|\mathbf{x}\|_2 = 1$. We shall optimize the quantity $\lambda(\alpha)$ given by

$$\begin{aligned} \lambda(\alpha) &= \langle \mathbf{Ax}, \mathbf{x} \rangle \\ &= \frac{\langle \mathbf{A}(\mathbf{u} + \alpha\mathbf{v}), \mathbf{u} + \alpha\mathbf{v} \rangle}{1+\alpha^2} \end{aligned} \tag{5}$$

From (5) we have

$$(1+\alpha^2)\lambda(\alpha) = \langle \mathbf{A}(\mathbf{u} + \alpha\mathbf{v}), \mathbf{u} + \alpha\mathbf{v} \rangle \tag{6}$$

Differentiate (6) with respect to $\alpha$ and set $\lambda'(\alpha) = 0$ to obtain

$$\alpha\lambda(\alpha) = \langle \mathbf{Av}, \mathbf{u} + \alpha\mathbf{v} \rangle \tag{7}$$

$$\langle \mathbf{A}(\mathbf{u} + \alpha\mathbf{v}), \mathbf{u} + \alpha\mathbf{v} \rangle = (1+\alpha^2)\langle \mathbf{Av}, \mathbf{u} + \alpha\mathbf{v} \rangle$$

$$\alpha^2\langle \mathbf{Au}, \mathbf{v} \rangle + \alpha[\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle] - \langle \mathbf{Au}, \mathbf{v} \rangle = 0$$

$$\alpha^2 D + \alpha C - D = 0, \tag{8}$$

where $D = \langle \mathbf{Au}, \mathbf{v} \rangle$ and $C = \langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle$. Hence

$$\alpha = \frac{-C \pm \sqrt{C^2 + 4D^2}}{2D}. \tag{9}$$

From (7)

$$\lambda(\alpha) = \langle \mathbf{Av}, \mathbf{v} \rangle + \frac{\langle \mathbf{Av}, \mathbf{u} \rangle}{\alpha} \tag{10}$$

$$= \langle \mathbf{Av}, \mathbf{v} \rangle + \frac{D}{\alpha} \tag{11}$$

From (9)

$$\frac{1}{\alpha} = \frac{C \pm \sqrt{C^2 + 4D^2}}{2D}, \tag{12}$$

so that (11) simplifies to

$$\begin{aligned} \lambda_\pm &= \langle \mathbf{Av}, \mathbf{v} \rangle + \frac{1}{2}\left[C \pm \sqrt{C^2 + 4D^2}\right] \\ &= \frac{1}{2}\left[\langle \mathbf{Au}, \mathbf{u} \rangle + \langle \mathbf{Av}, \mathbf{v} \rangle \right. \\ &\quad \left. \pm\sqrt{(\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle)^2 + 4\langle \mathbf{Au}, \mathbf{v} \rangle^2}\right] \end{aligned} \tag{13}$$

Here $\lambda_+$ and $\lambda_-$ correspond to the plus and minus signs in (13), respectively. Clearly $\lambda_-$ is an upper bound for $\lambda_n$ while $\lambda_+$ is a lower bound for $\lambda_1$. The magnitude $\delta_\pm$ of the corresponding residual is given by

$$\begin{aligned} \delta_\pm^2 &= \|\mathbf{Ax} - \lambda_\pm\mathbf{x}\|_2^2 \\ &= \langle \mathbf{Ax} - \lambda_\pm\mathbf{x}, \mathbf{Ax} - \lambda_\pm\mathbf{x} \rangle \\ &= \|\mathbf{Ax}\|_2^2 - \lambda_\pm^2. \end{aligned} \tag{14}$$

Thus

$$\delta_\pm = \sqrt{\|\mathbf{Ax}\|_2^2 - \lambda_\pm^2}.$$

## III. Results

## IV. Case dimension n even

For this case we choose

$$\mathbf{u} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{e}_i,$$

$$\mathbf{v} = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{\frac{n}{2}}\mathbf{e}_i - \sum_{i=\frac{n}{2}+1}^{n}\mathbf{e}_i\right), \tag{15}$$

where $\mathbf{e}_i$ are the standard basis vectors in $\mathbb{R}^n$. The matrix $\mathbf{A}$ is partitioned as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

Let $S_{ij}$ be the sum of the elements of $\mathbf{A}_{ij}$ where $i, j \in \{1, 2\}$ or explicitly

$$\begin{aligned} S_{11} &= \sum_{i=1}^{\frac{n}{2}}\sum_{j=1}^{\frac{n}{2}} a_{ij} \\ S_{12} &= \sum_{i=1}^{\frac{n}{2}}\sum_{j=\frac{n}{2}+1}^{n} a_{ij} \\ S_{21} &= S_{12} \\ S_{22} &= \sum_{i=\frac{n}{2}+1}^{n}\sum_{j=\frac{n}{2}+1}^{n} a_{ij} \end{aligned}$$

and $S = S_{11} + 2S_{12} + S_{22}$. Then

$$\begin{aligned} \langle \mathbf{Au}, \mathbf{u} \rangle &= \frac{S}{n} \\ &= \frac{S_{11} + 2S_{12} + S_{22}}{n}, \\ \langle \mathbf{Av}, \mathbf{v} \rangle &= \frac{S_{11} + S_{22} - 2S_{12}}{n}, \\ \langle \mathbf{Au}, \mathbf{v} \rangle &= \frac{S_{11} - S_{22}}{n}. \end{aligned}$$

Thus from (13) we have that

$$\lambda_\pm = \frac{1}{2n}\left[2S_{11} + 2S_{22} \pm \sqrt{16S_{12}^2 + 4\left(S_{11} - S_{22}\right)^2}\right]$$

$$\lambda_\pm = \frac{1}{n}\left[S_{11} + S_{22} \pm \sqrt{4S_{12}^2 + \left(S_{11} - S_{22}\right)^2}\right]$$

## V. Case dimension n odd

For this case we choose $\mathbf{u}$ as before and

$$\mathbf{v} = \frac{1}{\sqrt{n-1}}\left(\sum_{i=1}^{\frac{n-1}{2}}\mathbf{e}_i - \sum_{i=\frac{n+3}{2}}^{n}\mathbf{e}_i\right), \tag{16}$$

and partition $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{b} & \mathbf{A}_{12} \\ \mathbf{b}^t & \beta & \mathbf{c}^t \\ \mathbf{A}_{21} & \mathbf{c} & \mathbf{A}_{22}, \end{bmatrix}$$

where $\mathbf{A}_{ij}$ are order $\frac{n-1}{2}$ matrices and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^{\frac{n-1}{2}}$, with $\beta = a_{\frac{n+1}{2},\frac{n+1}{2}}$. Explicitly

$$S_{11} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=1}^{\frac{n-1}{2}} a_{ij}$$
$$S_{12} = \sum_{i=1}^{\frac{n-1}{2}} \sum_{j=\frac{n+3}{2}}^{n} a_{ij}$$
$$S_{21} = S_{12}$$
$$S_{22} = \sum_{i=\frac{n+3}{2}}^{n} \sum_{j=\frac{n+3}{2}}^{n} a_{ij}$$

Then

$$\langle \mathbf{Au}, \mathbf{u} \rangle = \frac{S}{n} \tag{17}$$
$$= \frac{S_{11} + 2S_{12} + S_{22} + 2(S_b + S_c + \frac{\beta}{2})}{n},$$
$$\langle \mathbf{Av}, \mathbf{v} \rangle = \frac{S_{11} + S_{22} - 2S_{12}}{n-1}, \tag{18}$$
$$\langle \mathbf{Au}, \mathbf{v} \rangle = \frac{S_{11} - S_{22} + S_b - S_c}{\sqrt{n(n-1)}}, \tag{19}$$

where $S_b$ and $S_c$ are the sum of the elements of $\mathbf{b}$ and $\mathbf{c}$ respectively. Thus from (17) and (18) we have that

$$\langle \mathbf{Au}, \mathbf{u} \rangle + \langle \mathbf{Av}, \mathbf{v} \rangle = \frac{(2n-1)(S_{11} + S_{22}) - 2S_{12} + 2(n-1)(S_b + S_c + \frac{\beta}{2})}{n(n-1)}$$

$$\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle = \frac{2(2n-1)S_{12} - (S_{11} + S_{22}) + 2(n-1)(S_b + S_c + \frac{\beta}{2})}{n(n-1)}$$
$$= \frac{2\left[(2n-1)S_{12} - \frac{S_{11}+S_{22}}{2} + (n-1)\left(S_b + S_c + \frac{\beta}{2}\right)\right]}{n(n-1)}.$$

Thus from (13) we have that

$$\lambda_{\pm} = \frac{1}{n(n-1)} \left[ (n - \tfrac{1}{2})(S_{11} + S_{22}) - S_{12} + (n-1)\left(S_b + S_c + \frac{\beta}{2}\right) \pm \sqrt{\Delta} \right]$$

where

$$\Delta = \left[(2n-1)S_{12} - \frac{S_{11}+S_{22}}{2} + (n-1)\left(S_b + S_c + \frac{\beta}{2}\right)\right]^2 + n(n-1)\left[S_{11} - S_{22} + S_b - S_c\right]^2.$$

## VI. Further optimization

We shall refer to the choice of $\mathbf{v}$ in equations (15) and (16) as $\mathbf{v_s}$. It is obvious that any permutation of the elements of $\mathbf{v}$ will suffice in (4) and (13) as $\mathbf{u}$ and $\mathbf{v}$ will still maintain orthogonality.

*Theorem 6.1:* For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we have that

$$\text{(a) } \lambda_- \leq \langle \mathbf{Au}, \mathbf{u} \rangle \leq \lambda_+ \tag{20}$$
$$\text{(b) } \lambda_- \leq \langle \mathbf{Av}, \mathbf{v} \rangle \leq \lambda_+$$

*Proof:* We shall only prove (a) as (b) is proved in a similar manner. From

$$\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle \leq \sqrt{(\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle)^2 + 4\langle \mathbf{Au}, \mathbf{v} \rangle^2}$$
$$\langle \mathbf{Au}, \mathbf{u} \rangle \leq \langle \mathbf{Av}, \mathbf{v} \rangle + \sqrt{(\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle)^2 + 4\langle \mathbf{Au}, \mathbf{v} \rangle^2}$$
$$2\langle \mathbf{Au}, \mathbf{u} \rangle \leq \langle \mathbf{Au}, \mathbf{u} \rangle + \langle \mathbf{Av}, \mathbf{v} \rangle + \sqrt{(\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle)^2 + 4\langle \mathbf{Au}, \mathbf{v} \rangle^2}$$

from which the upper bound in (20) follows. The lower bound is proved by considering

$$-\langle \mathbf{Au}, \mathbf{u} \rangle + \langle \mathbf{Av}, \mathbf{v} \rangle \leq \sqrt{(\langle \mathbf{Au}, \mathbf{u} \rangle - \langle \mathbf{Av}, \mathbf{v} \rangle)^2 + 4\langle \mathbf{Au}, \mathbf{v} \rangle^2}$$

■

In the limit when $\langle \mathbf{Au}, \mathbf{v} \rangle \to 0$ in (13) we would have either $\langle \mathbf{Au}, \mathbf{u} \rangle$ or $\langle \mathbf{Av}, \mathbf{v} \rangle$ as inner bounds (from a one dimensional subspace). Thus from theorem 6.1 it is only natural to increase the value of $|\langle \mathbf{Au}, \mathbf{v} \rangle|$ in order to further optimize the inner bounds. We may choose to make $\langle \mathbf{Au}, \mathbf{v} \rangle$ large negative or positive, however, we choose the latter approach. Let $r_i = \sum_{j=1}^{n} a_{ij}$ be the row sums of $\mathbf{A}$ and note that $r_i = (\mathbf{Au})_i$. Arrange the set $\{|r_i|\}_{i=1}^{n}$ in descending order, say $|r_{\sigma_1}| \geq |r_{\sigma_2}| \geq \cdots |r_{\sigma_n}|$, where $\sigma_i \in \{1, 2, \cdots, n\}$. Choose

$$\mathbf{v}_{\sigma_1} = \begin{cases} \frac{+1}{\sqrt{n}} & \text{if } r_{\sigma_1} > 0 \\ \frac{-1}{\sqrt{n}} & \text{if } r_{\sigma_1} < 0 \end{cases}$$

Continue in this manner for $r_{\sigma_2}, r_{\sigma_3}, \cdots$ until $\frac{n}{2}$ (assuming $n$ is even) entries are $\frac{+1}{\sqrt{n}}$ or $\frac{-1}{\sqrt{n}}$, whichever comes first. Then fill the remaining entries of $\mathbf{v}$ by elements of the opposite sign. For the case of odd $n$, we repeat the procedure for the even case. However, say we have first assigned $n_2 = \frac{n-1}{2}$ elements as $\frac{-1}{\sqrt{n-1}}$ and $p < n_2$ as $\frac{+1}{\sqrt{n-1}}$. We are still left with the position of the 0 element and $n_2 - p$ elements $\frac{+1}{\sqrt{n-1}}$. We then examine $r_{\sigma_{n_2+p+k}}$, $k = 1, 2, \cdots, n_2 + 1 - p$. Let $k = q$ be the first index where $r_{\sigma_{n_2+p+k}} < 0$, then set $\mathbf{v}_{\sigma_{n_2+p+q}} = 0$ and fill the remaining positions with $\frac{+1}{\sqrt{n-1}}$. If $r_{\sigma_{n_2+p+k}} > 0$ for all $k$, then set $\mathbf{v}_n = 0$ and fill the remaining positions with $\frac{+1}{\sqrt{n-1}}$. For now let us designate the vector from this procedure by $\mathbf{v}'$ It is obvious that there is a permutation matrix $\mathbf{P}$ such that $\mathbf{v}' = \mathbf{Pv}$, where the original $\mathbf{v} = \mathbf{v_s}$ is from (15) or (16). Now

$$\langle \mathbf{Au}, \mathbf{v}' \rangle = \langle \mathbf{Au}, \mathbf{P^t v} \rangle = \langle \mathbf{PAP^t u}, \mathbf{v} \rangle = \langle \mathbf{A'u}, \mathbf{v} \rangle,$$

where $\mathbf{A}' = \mathbf{PAP^t}$ and obviously $\mathbf{P^t u} = \mathbf{u}$. Similarly

$$\langle \mathbf{Au}, \mathbf{u} \rangle = \langle \mathbf{A'u}, \mathbf{u} \rangle$$
$$\langle \mathbf{Av}, \mathbf{v} \rangle = \langle \mathbf{A'v}, \mathbf{v} \rangle$$

Thus using $\mathbf{u}$, $\mathbf{A}$ and $\mathbf{v}'$ in (13) is equivalent to using $\mathbf{u}$, $\mathbf{A}'$ and $\mathbf{v}$ in (13). Thus an alternative, though equivalent, approach is to permute the rows and columns of $\mathbf{A}$ and use the original $\mathbf{v} = \mathbf{v_s}$. It is obvious that there are many other choices for $\mathbf{v}$. Thus we shall refer to $\mathbf{v}'$ as $\mathbf{v_{opt}}$ for reasons that will be clear later on. The one other choice of $\mathbf{v}$ that interests us is given by

$$\mathbf{v_{alt}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^i \mathbf{e}_i, \; n \text{ even}$$

$$\mathbf{v_{alt}} = \frac{1}{\sqrt{n-1}} \left[ \sum_{i=1}^{\frac{n-1}{2}} (-1)^i \mathbf{e}_i + \sum_{i=\frac{n+3}{2}}^{n} (-1)^i \mathbf{e}_i \right], \; n \text{ odd.}$$

For comparison with our results we summarize the bounds of [15].

$$\lambda_- = m - \frac{S_d}{\sqrt{n-1}}, \quad (21)$$

$$\lambda_+ = m + \frac{S_d}{\sqrt{n-1}}, \quad (22)$$

where

$$m = \frac{\text{tr}(\mathbf{A})}{n},$$

$$S_d = \sqrt{\frac{\text{tr}(\mathbf{A}^2)}{n} - m^2},$$

are the average of the eigenvalues and the corresponding standard deviation. We shall refer to these bounds as **tr1** bounds. Singh etal. [11] have shown that if $f(\lambda)$ is continuous on the spectrum $\sigma(\mathbf{A})$, and $\mathbf{B} = f(\mathbf{A}) - m\mathbf{I}$, then

$$f(\lambda_1) \geq m + \frac{\text{tr}(\mathbf{B}^2)}{n} \left[ \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1}\text{tr}(\mathbf{B}^{2r})} \right]^{\frac{1}{2r}} \quad (23)$$

$$f(\lambda_n) \leq m - \frac{\text{tr}(\mathbf{B}^2)}{n} \left[ \frac{1 + (n-1)^{2r-1}}{(n-1)^{2r-1}\text{tr}(\mathbf{B}^{2r})} \right]^{\frac{1}{2r}} \quad (24)$$

Here, $r \geq 1$ is an integer. When $f(x) = x$, then equations (23)-(24) reduce to (21)-(22). We shall choose $r = 2$ and $f(x) = x$ for comparison with our newly advocated method. We shall refer to these bounds as **tr2** bounds. It must be noted that with $f(x) = x^k, k \geq 2$ better bounds are obtained from (23)-(24), however this is computationally much more expensive. We next present few examples, all matrices are taken from [5]. We shall summarize the choices for $\mathbf{v_s}$, $\mathbf{v_{opt}}$ and $\mathbf{v_{alt}}$ as well as the row sums of the matrices which is denoted by the vector $\mathbf{r}$.

*Example 6.2:* Consider the test matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & -2 & 0 & -2 & 5 \\ 1 & 6 & -3 & 2 & 0 & 6 \\ -2 & -3 & 8 & -5 & -6 & 0 \\ 0 & 2 & -5 & 5 & 1 & -2 \\ -2 & 0 & -6 & 1 & 6 & -3 \\ 5 & 6 & 0 & -2 & -3 & 8 \end{bmatrix}.$$

Here the parameters are summarized in Table I. Note how $|\langle \mathbf{Au}, \mathbf{v} \rangle|$ has been optimized by suitable choice of $\mathbf{v_{opt}}$

TABLE I
PARAMETERS:EXAMPLE 6.2

| $\sqrt{6}\mathbf{v_s}$ | $\mathbf{r}$ | $\sqrt{6}\mathbf{v_{opt}}$ | $\sqrt{6}\mathbf{v_{alt}}$ |
|---|---|---|---|
| 1 | 7 | 1 | −1 |
| 1 | 12 | 1 | 1 |
| 1 | −8 | −1 | −1 |
| −1 | 1 | −1 | 1 |
| −1 | −4 | −1 | −1 |
| −1 | 14 | 1 | 1 |

The corresponding bounds are summarized in Table II

TABLE II
BOUNDS:EXAMPLE 6.2

| Method | $\lambda_-$ | $\lambda_+$ |
|---|---|---|
| **tr1** | 3.040243 | 9.626424 |
| **tr2** | 2.073495 | 10.593172 |
| $\mathbf{v_s}$ | 3.666667 | 3.666667 |
| $\mathbf{v_{opt}}$ | −1.055364 | 15.055364 |
| $\mathbf{v_{alt}}$ | −0.497474 | 10.497474 |
| **exact** | −1.598734 | 16.142745 |

*Example 6.3:* For the matrix given below of odd order, the parameters are summarized in Table III and the results are summarized in Table IV

$$\mathbf{A} = \begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 6 & 5 & 4 & 3 & 2 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

TABLE III
PARAMETERS:EXAMPLE 6.3

| $\sqrt{6}\mathbf{v_s}$ | $\mathbf{r}$ | $\sqrt{6}\mathbf{v_{opt}}$ | $\sqrt{6}\mathbf{v_{alt}}$ |
|---|---|---|---|
| 1 | 28 | 1 | −1 |
| 1 | 27 | 1 | 1 |
| 1 | 25 | 1 | −1 |
| 0 | 22 | 0 | 0 |
| −1 | 18 | −1 | 1 |
| −1 | 13 | −1 | −1 |
| −1 | 7 | −1 | 1 |

TABLE IV
BOUNDS:EXAMPLE 6.3

| Method | $\lambda_-$ | $\lambda_+$ |
|---|---|---|
| **tr1** | 0.837722 | 7.162278 |
| **tr2** | 0.823352 | 7.176648 |
| $\mathbf{v_s}$ | 2.29452 | 22.372147 |
| $\mathbf{v_{opt}}$ | 2.29452 | 22.372147 |
| $\mathbf{v_{alt}}$ | 0.428228 | 20.238439 |
| **exact** | 0.261295 | 22.880783 |

*Example 6.4:* For the following pentadiagonal matrix of odd order the parameters are summarized in Table V and the

results are summarized in Table VI.

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 & 1 & & & & & \\ 2 & 6 & 3 & 1 & 1 & & & & \\ 1 & 3 & 6 & 3 & 1 & 1 & & & \\ 1 & 1 & 3 & 6 & 3 & 1 & 1 & & \\ & 1 & 1 & 3 & 6 & 3 & 1 & 1 & \\ & & 1 & 1 & 3 & 6 & 3 & 1 & 1 \\ & & & 1 & 1 & 3 & 6 & 3 & 1 \\ & & & & 1 & 1 & 3 & 6 & 2 \\ & & & & & 1 & 1 & 2 & 5 \end{bmatrix}.$$

TABLE V
PARAMETERS:EXAMPLE 6.4

| $\sqrt{8}\mathbf{v_s}$ | $\mathbf{r}$ | $\sqrt{8}\mathbf{v_{opt}}$ | $\sqrt{8}\mathbf{v_{alt}}$ |
|---|---|---|---|
| 1 | 9 | −1 | −1 |
| 1 | 13 | −1 | 1 |
| 1 | 15 | 1 | −1 |
| 1 | 16 | 1 | 1 |
| 0 | 16 | 1 | 0 |
| −1 | 16 | 1 | −1 |
| −1 | 15 | 0 | 1 |
| −1 | 13 | −1 | −1 |
| −1 | 9 | −1 | 1 |

TABLE VI
BOUNDS:EXAMPLE 6.4

| Method | $\lambda_-$ | $\lambda_+$ |
|---|---|---|
| **tr1** | 4.326937 | 7.228618 |
| **tr2** | 3.953128 | 7.602428 |
| $\mathbf{v_s}$ | 10.500000 | 13.555556 |
| $\mathbf{v_{opt}}$ | 6.535751 | 14.269805 |
| $\mathbf{v_{alt}}$ | 2.5 | 13.555556 |
| **exact** | 0.736124 | 14.499944 |

*Example 6.5:* This example is for the matrix as in Example 6.4, except that we add another row of $[\cdots 1\,1\,3\,6\,3\,1\,1 \cdots]$, to increase the order to even. Corresponding parameters and results are summarized in Table VII and Table VIII.

TABLE VII
PARAMETERS:EXAMPLE 6.5

| $\sqrt{10}\mathbf{v_s}$ | $\mathbf{r}$ | $\sqrt{10}\mathbf{v_{opt}}$ | $\sqrt{10}\mathbf{v_{alt}}$ |
|---|---|---|---|
| 1 | 9 | −1 | −1 |
| 1 | 13 | −1 | 1 |
| 1 | 15 | 1 | −1 |
| 1 | 16 | 1 | 1 |
| 1 | 16 | 1 | −1 |
| −1 | 16 | 1 | 1 |
| −1 | 16 | 1 | −1 |
| −1 | 15 | −1 | 1 |
| −1 | 13 | −1 | −1 |
| −1 | 9 | −1 | 1 |

TABLE VIII
BOUNDS:EXAMPLE 6.5

| Method | $\lambda_-$ | $\lambda_+$ |
|---|---|---|
| **tr1** | 4.411156 | 7.188844 |
| **tr2** | 3.993205 | 7.606795 |
| $\mathbf{v_s}$ | 10.500000 | 13.555556 |
| $\mathbf{v_{opt}}$ | 7.194449 | 14.405551 |
| $\mathbf{v_{alt}}$ | 1.000000 | 13.800000 |
| **exact** | 0.615828 | 14.749186 |

*Example 6.6:* The following matrix has 3 eigenvalues, each of algebraic multiplicity 2. Parameters are summarized in Table IX and the results are summarized in Table X.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 0 & 1 & 2 \\ 2 & 4 & 5 & -1 & 0 & 3 \\ 3 & 5 & 6 & -2 & -3 & 0 \\ 0 & -1 & -2 & 1 & 2 & 3 \\ 1 & 0 & -3 & 2 & 4 & 5 \\ 2 & 3 & 0 & 3 & 5 & 6 \end{bmatrix}.$$

TABLE IX
PARAMETERS:EXAMPLE 6.6

| $\sqrt{6}\mathbf{v_s}$ | $\mathbf{r}$ | $\sqrt{6}\mathbf{v_{opt}}$ | $\sqrt{6}\mathbf{v_{alt}}$ |
|---|---|---|---|
| 1 | 9 | 1 | −1 |
| 1 | 13 | 1 | 1 |
| 1 | 9 | −1 | −1 |
| −1 | 3 | −1 | 1 |
| −1 | 9 | −1 | −1 |
| −1 | 19 | 1 | 1 |

TABLE X
BOUNDS:EXAMPLE 6.6

| Method | $\lambda_-$ | $\lambda_+$ |
|---|---|---|
| **tr1** | 0.877800 | 6.455533 |
| **tr2** | 0.059076 | 7.274257 |
| $\mathbf{v_s}$ | 10.333333 | 10.333333 |
| $\mathbf{v_{opt}}$ | −1.289321 | 11.289321 |
| $\mathbf{v_{alt}}$ | 0.81326 | 10.520073 |
| **exact** | −1.696323 | 12.411336 |

## VII. Discussion

From Table II we see that $\mathbf{v_s}$ gives both bounds as equal. This is the case as the radical in (13) is zero. Using $\mathbf{v_{opt}}$ results in the best inner bounds for both $\lambda_1$ and $\lambda_n$, whilst $\mathbf{v_{alt}}$ gives a reasonable upper bound for $\lambda_n$. From Table IV, $\mathbf{v_s}$ and $\mathbf{v_{opt}}$ give the same results as $\mathbf{v_s} = \mathbf{v_{opt}}$ in this case. The best inner bounds are given by $\mathbf{v_{opt}}$ , while a reasonable upper bound is given by $\mathbf{v_{alt}}$. From Table VI and VIII, the best lower bound is given by $\mathbf{v_{opt}}$ and the best upper bound by $\mathbf{v_{alt}}$. Example 6.6 illustrates that $\mathbf{v_{opt}}$ gives the best results. In fact it is the only lower bound that is negative. Our inference is that it suffices to use $\mathbf{v_{alt}}$ and $\mathbf{v_{opt}}$ only, to determine the best bounds. It is also clear from all tables that **tr2** bounds are better than **tr1** bounds. However they are worse than the bounds derived in this paper. It must also be noted that tr$(\mathbf{B}^{2r})$ is computationally very expensive to compute for values of $r \geq 2$, as it involves $r$ matrix-matrix multiplications. We can easily identify the best bounds from our tables.

## VIII. Conclusion

We have presented a reasonably inexepensive technique to approximate the inner bounds for the extreme eigenvalues of real symmetric matrices. If we are willing to spend the effort to generate $\mathbf{v_{opt}}$, then we can get a good inner bound for $\lambda_1$. We can use $\mathbf{v_{alt}}$ to identify a good inner bound for $\lambda_n$. However, in some cases $\mathbf{v_{alt}}$ yields very good results for both the bounds.

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