Novel Three-way Decision Models Based on Bayes Decision Theory with Neutrosophic Sets

Dongsheng Xu, Liang Zhou, Yihui Zhao, Xin Liu

Abstract—**The three-way decision paradigm is a decision-making approach that is analogous to human reasoning. It involves a neutrosophic set as an efficient tool for handling ambiguity. Using cotangent similarity, the Hamming distance, and the innovative Vicis-Wave Hedges distance for ranking, this study proposes three novel three-way decision models based on the Bayes decision. In particular, single-valued neutrosophic numbers are employed to represent each loss function. The proposed models are then condensed into an algorithm. Subsequently, an example is presented to demonstrate the validity and rationality of the proposed models. Finally, we compare the three models. This research provides a new method for decision-making problems.**

Index Terms—**three-way decision; cotangent similarity; single-valued neutrosophic set; Vicis-Wave Hedges distance; Hamming distance**

I. Introduction

THINKING in terms of threes is the primary concept underlying the three-way decision (3WD) paradigm. In contrast to the conventional two-way decision approach, the 3WD approach involves acceptance, rejection, and delayed decision-making in order to address the ambiguity of the problem.

Yao proposed 3WD in 2010 [1] and described 3WD using the trisecting–acting–outcome model in 2018 [2]. 3WD is a specific instance of the larger three-way decision space defined by Hu [3]. Yao [4] combined the 3WD framework with various nonstandard sets. Zhang [5] identified two categories of classification mistakes. Hu [6] addressed two open issues by modifying the decision parameters of the 3WD definition. Using 3WD, Lang [7] developed efficient conflict analysis techniques. An innovative 3WD model that incorporates order information was proposed by Liu and Liang [8]. By combining the notions of erosion and dilation from mathematical morphology with the principles of 3WD, Wang and Yao [9] proposed a framework of contraction-and-expansion-based three-way clustering. Jia [10] proposed a unique decision model by combining 3WD and multi-criteria decision-making. A generalized multi-granulation sequential 3WD model based on various thresholds was proposed by Qian [11]. Luo [12] presented a 3WD algorithm that can handle data scale changes. Li [13] expanded the three-way decision model originally developed for 0-1 tables to include general information tables and established a 3WD framework for such tables.

Zadeh proposed the fuzzy set [14]; subsequently, Smarandache [15] presented neutrosophic sets (NSs) based on an intuitionistic fuzzy set [16]. In particular, Wang [17] methodically introduced single-valued neutrosophic sets (SVNSs). SVNSs are more in line with human thinking. Zhang [18] investigated the inclusion relations of NSs. Xu [19] examined the applications of SVNSs and multi-attribute decision-making. An increasing number of scientists are employing 3WD and fuzzy sets in their research. Using three membership degrees of an NS, Abedel-Basset presented two rules for 3WD [20]. Singh proposed the use of a neutrosophic set to describe a three-way fuzzy concept lattice [21] and also investigated the three-way n-valued neutrosophic concept lattice at different granulation levels [22]. Jiao [23] proposed two 3WD models combined with SVNNs.

This study proposes three 3WD models that are based on cotangent similarity, the Hamming distance, and the novel Vicis-Wave Hedges distance with SVNSs. The remainder of this paper is organized as follows. Section II presents the preliminaries. Section III describes the three 3WD models. Section IV discusses applications of the models with specific cases. Finally, Section V concludes this paper.

II. Preliminaries

A. Single-valued neutrosophic set

Definition 2.1 [15]: Supposed that U is a domain. The single-valued neutrosophic set $\tilde{A}$ on the domain U is constituted by three elements: truth, indeterminacy, and false memberships. They can be represented by $T_A(x)$, $I_A(x)$ and $F_A(x)$. Each membership is a real number between [0,1]. Its form is as follows:

$$\tilde{A}=\left\{x,T_A(x),I_A(x),F_A(x):x\in U\right\}.$$

For convenience, we called the $\tilde{a}=(T_A(x),I_A(x),F_A(x))$ or $a=(T_a,I_a,F_a)$ as a single-valued neutrosophic number. The abbreviation is SVNN.

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Definition 2.2 [24]: Let n and m be two SVNNs. The addition between SVNNs and number multiplication with a real number θ are defined as follows:

(1) $e \oplus f = \left\langle T_e + T_f - T_e \cdot T_f, I_e \cdot I_f, F_e \cdot F_f \right\rangle$

(2) $\theta e = \left\langle 1 - (1 - T_e)^\theta, (I_e)^\theta, (F_e)^\theta \right\rangle, \theta > 0$

Definition 2.3 [25]: Suppose $e = (T_e, I_e, F_e)$ and $f = (T_f, I_f, F_f)$ are two SVNNs, if $T_e \le T_f, I_e \ge I_f, F_e \ge F_f$, then we say n ≤ m.

This comparison technique, however, is excessively severe and cannot be extensively applied. As an illustration, consider the neutrosophic numbers n = (0.3,0.4,0.5) and m = (0.4,0.5,0.6). Since these two neutrosophic numbers do not meet the requirements of Definition 2.3, they cannot be compared.

Definition 2.4 [26] Suppose $e = (T_e, I_e, F_e)$ and $f = (T_f, I_f, F_f)$, their cotangent similarity is defined as follows:

$$\cot_1(e, f) = \cot[\frac{\pi}{4} + \frac{\pi}{12}(|T_e - T_f| + |I_e - I_f| + |F_e - F_f|)] \quad (1)$$

To compare the sizes of two neutrosophic sets, we typically select an ideal neutrosophic set to facilitate the comparison. In general, the minimum I = (0,1,1) or the maximum T = (1,0,0) is widely employed. Assume that the minimum neutrosophic set I = (0,1,1) is used for calculation. In this case, a larger or more significant cotangent similarity, indicating greater similarity to the minimum neutrosophic set, implies a smaller neutrosophic set and vice versa.

Example 2.1 We can compare any two SVNNs using cotangent similarity. If e = (0.3,0.6,0.1), f = (0.8,0.2,0.4) and we choose I = (0,1,1), the calculation results of the cotangent similarity between e and I and between f and I are as follows:

$\cot(e, I)$ =0.3839

$\cot(f, I)$ =0.2126

because of $\cot(e, I) > \cot(f, I)$, it can be concluded that e < f.

Cotangent similarity has good operational properties. For instance, if n and m are given as e = (0.1, 0.2, 0.2) and f = (0.2, 0.4, 0.4), using the common cosine similarity, the cosine similarity of e and f will be found to be 1.But obviously $e \ne f$, the cosine similarity is not useful in this case. However, we may calculate that e and f have a 0.7673 cotangent similarity. We believe that this result is more logical than the cosine similarity.

Definition 2.5 [27] Suppose $e = (T_e, I_e, F_e)$ and $f = (T_f, I_f, F_f)$, their Hamming distance is defined as follows:

$$H(e, f) = (|T_e - T_f| + |I_e - I_f| + |F_e - F_f|) \quad (2)$$

Example 2.2 if e = (0.3,0.6,0.1), f = (0.8,0.2,0.4), and we choose I = (0,1,1), then the Hamming distance results of e and I, f and I are as follows:

$H(e, I)$ = 1.6

$H(f, I)$ = 2.2

because of $H(e, I) > H(f, I)$, it can be concluded that e<f.

Definition 2.6 Suppose that the $e = (T_e, I_e, F_e)$ and $f = (T_f, I_f, F_f)$ are two SVNNs; the Vicis-Wave Hedges distance of n and m is defined as follows:

$$V(e, f) = \frac{|T_e - T_f|}{1 + \min(T_e, T_f)} + \frac{|I_e - I_f|}{1 + \min(I_e, I_f)} + \frac{|F_e - F_f|}{1 + \min(F_e, F_f)} \quad (3)$$

Proof 2.1: Let $e = (T_e, I_e, F_e)$ and $f = (T_f, I_f, F_f)$ be two SVNNs. The above defined the SVNNs neutrosophic distance $V(e, f)$ between SVNNs e and f satisfies the following properties (1)–(4):

(1) $V(e, f) \ge 0$;

(2) $V(e, f) = 0$ if and only if $e = f$;

(3) $V(e, f) = V(e, f)$;

(4) If $e \subseteq f \subseteq g$, C is the other SVNN in X, then $V(e, g) \ge V(e, f)$ and $V(e, g) \ge V(f, g)$.

Proof: Obviously, $V(e, f)$ satisfies (1), (2), and (3). It only needs to be verified that property (4).

If $e \subseteq f \subseteq g$,i.e., $T_e \le T_f \le T_g$, $I_e \le I_f \le I_g$, $F_e \ge F_f \ge F_g$.

From the aforementioned conditions, we obtain

$\min(T_e, T_f) = T_e$, $\min(T_e, T_g) = T_e$, $\min(T_f, T_g) = T_f$,

$$|T_e - T_g| \ge |T_e - T_f|, |T_e - T_g| \ge |T_f - T_g|;$$

$$|I_e - I_g| \ge |I_e - I_f|, |I_e - I_g| \ge |I_f - I_g|;$$

$$|F_e - F_g| \ge |F_e - F_f|, |F_e - F_g| \ge |F_f - F_g|.$$

According to equation (1), we can get

$$V(e, f) = \frac{|T_e - T_f|}{T_e + 1} + \frac{|I_e - I_f|}{I_f + 1} + \frac{|F_e - F_f|}{F_f + 1}$$

$$V(e, g) = \frac{|T_e - T_g|}{T_e + 1} + \frac{|I_e - I_g|}{I_g + 1} + \frac{|F_e - F_g|}{F_g + 1}$$

$$V(f, g) = \frac{|T_f - T_g|}{T_f + 1} + \frac{|I_f - I_g|}{I_g + 1} + \frac{|F_f - F_g|}{F_g + 1}$$

It is obvious that $V(e, g) \ge V(e, f), V(e, g) \ge V(f, g)$.

Example 2.3 Similarly, if e = (0.3,0.6,0.1), f = (0.8,0.2,0.4), and we choose O= (0,1,1), then the Vicis-Wave Hedges distance results of e and I, f and I are as follows

$V(e, O)$ =1.3682

$V(f, O)$ =1.8952

because of $V(e, O) < V(f, O)$, it can be concluded that e<f.

In addition, the distance provides advantageous operating features. For example, if e = (0.2,0.4,0.6), f = (0.8,0.6,0.3), and g = (0.1,0.3,0.2), by using the common Hamming distance, we find that the Hamming distance of e and f and that of f and g are both 1.1. However, obviously e ≠ g; hence, this reflects the limitation of the Hamming distance. After calculation, the Vicis-Wave Hedges distance of e and f is 0.8304, and that of f and g is 0.9505. Thus, the results show that e is closer than g to f.

B. Three-way decision

The fundamental tenet of the 3WD is to partition the domain into three distinct sections: the positive, negative, and border. Additionally, they have established three decision-making guidelines: acceptance, rejection, and non-commitment. The 3WD is greatly congruent with how people behave and make judgments.

3WD, based on the Bayesian decision process, is composed of a state set and a behavior set. The state set is $\Omega=\{X,\neg X\}$, among them, X and $\neg X$ represents the object x in state X and not in state X. The behavior set is $\mathrm{A}=\{a_p,a_B,a_N\}$, represents the loss function when x is decided to accept, do not commit, and reject, respectively. The loss functions caused by different behaviors in different states are shown in Table 1.

TABLE I
THE LOSS FUNCTIONS

	X (positive)	$\neg X$ (negative)
a_p (acceptance)	λ_{PP}	λ_{PN}
a_B (non-commitment)	λ_{BP}	λ_{BN}
a_N (rejection)	λ_{NP}	λ_{NN}

After obtaining the loss functions, we can obtain the loss expectation of decision-makers taking different behaviors. The calculation formulas are as follows:

$$\varepsilon\left(a_\bullet\middle|[x]_D\right)=\lambda_{\bullet P}\Pr(X|[x]_D)+\lambda_{\bullet N}\Pr(\neg X|[x]_D),\bullet=P,B,N$$

Among them, $\mathrm{P}(X|[x]_R)=\frac{|X\cap[x]_R|}{|[x]_R|}$ and $\mathrm{P}(\neg X|[x]_R)=\frac{|\neg X\cap[x]_R|}{|[x]_R|}$ represent the conditional probabilities that x in the equivalence class $[x]_R$ belongs to X and $\neg X$ respectively. According to Bayesian decision theory, there are the following minimum-cost criterion decision rules:

(P)if $\varepsilon\left(a_p\middle|[x]_R\right)\leqslant\varepsilon\left(a_B\middle|[x]_R\right)$ and $\varepsilon\left(a_p\middle|[x]_R\right)\leqslant\varepsilon\left(a_N\middle|[x]_R\right)$, then $x\in Pos(X)$;

(B)if $\varepsilon\left(a_B\middle|[x]_R\right)\leqslant\varepsilon\left(a_P\middle|[x]_R\right)$ and $\varepsilon\left(a_B\middle|[x]_R\right)\leqslant\varepsilon\left(a_N\middle|[x]_R\right)$, then $x\in \mathrm{Bnd}(X)$;

(N)if $\varepsilon\left(a_N\middle|[x]_R\right)\leqslant\varepsilon\left(a_P\middle|[x]_R\right)$ and $\varepsilon\left(a_N\middle|[x]_R\right)\leqslant\varepsilon\left(a_B\middle|[x]_R\right)$, then $x\in Neg(X)$;

III. THREE-WAY DECISION MODELS

A. Three-way decision model using cotangent similarity ranking method

The 3WD model based on cotangent similarity under single-valued neutrosophic information will be presented in this part. Under single-valued neutrosophic information, the loss functions are represented by SVNNs. Table 2 shows the loss function table under single-valued neutrosophic information.

TABLE II
THE LOSS FUNCTIONS BY SVNNs

	X (positive)	$\neg X$ (negative)
a_p (acceptance)	$\lambda_{PP}=(T_{\lambda_{PP}},I_{\lambda_{PP}},F_{\lambda_{PP}})$	$\lambda_{PN}=(T_{\lambda_{PN}},I_{\lambda_{PN}},F_{\lambda_{PN}})$
a_B (non-commitment)	$\lambda_{BP}=(T_{\lambda_{BP}},I_{\lambda_{BP}},F_{\lambda_{BP}})$	$\lambda_{BN}=(T_{\lambda_{BN}},I_{\lambda_{BN}},F_{\lambda_{BN}})$
a_N (rejection)	$\lambda_{NP}=(T_{\lambda_{NP}},I_{\lambda_{NP}},F_{\lambda_{NP}})$	$\lambda_{NN}=(T_{\lambda_{NN}},I_{\lambda_{NN}},F_{\lambda_{NN}})$

Accordingly, for the object x in X, take actions a_p, a_B and a_N, then the loss expectations are

$$\varepsilon\left(a_\bullet\middle|[x]_D\right)=\lambda_{\bullet P}\Pr(X|[x]_D)+\lambda_{\bullet N}\Pr(\neg X|[x]_D),\bullet=P,B,N$$

Concretely, by Definition 2.2, it can be got that

$$\begin{aligned}\varepsilon\left(a_p\middle|[x]_R\right)=(&1-(1-T_{\lambda_{PP}})^{\Pr(X|[x]_R)}(1-T_{\lambda_{PN}})^{\Pr(\neg X|[x]_R)},\\&(I_{\lambda_{PP}})^{\Pr(X|[x]_R)}(I_{\lambda_{PN}})^{\Pr(\neg X|[x]_R)},\\&(F_{\lambda_{PP}})^{\Pr(X|[x]_R)}(F_{\lambda_{PN}})^{\Pr(\neg X|[x]_R)})\end{aligned}\tag{4}$$

$$\begin{aligned}\varepsilon\left(a_B\middle|[x]_R\right)=(&1-(1-T_{\lambda_{BP}})^{\Pr(X|[x]_R)}(1-T_{\lambda_{BN}})^{\Pr(\neg X|[x]_R)},\\&(I_{\lambda_{BP}})^{\Pr(X|[x]_R)}(I_{\lambda_{BN}})^{\Pr(\neg X|[x]_R)},\\&(F_{\lambda_{BP}})^{\Pr(X|[x]_R)}(F_{\lambda_{BN}})^{\Pr(\neg X|[x]_R)})\end{aligned}\tag{5}$$

$$\begin{aligned}\varepsilon\left(a_N\middle|[x]_R\right)=(&1-(1-T_{\lambda_{NP}})^{\Pr(X|[x]_R)}(1-T_{\lambda_{NN}})^{\Pr(\neg X|[x]_R)},\\&(I_{\lambda_{NP}})^{\Pr(X|[x]_R)}(I_{\lambda_{NN}})^{\Pr(\neg X|[x]_R)},\\&(F_{\lambda_{NP}})^{\Pr(X|[x]_R)}(F_{\lambda_{NN}})^{\Pr(\neg X|[x]_R)}).\end{aligned}\tag{6}$$

Since $\lambda_{\bullet\bullet}$ are all single-valued neutrosophic numbers, it is obvious that all loss expectations are SVNNs, and $0\le\Pr(X|[x]_R),\Pr(\neg X|[x]_R)\le1$. Then, according to the definition of cotangent similarity and the minimum cost decision rules, we can get

(P1)if $\cot\left(\mathrm{O},\varepsilon(a_p|[x]_R)\right)\ge\cot\left(\mathrm{O},\varepsilon(a_B|[x]_R)\right)$ and $\cot\left(\mathrm{O},\varepsilon(a_p|[x]_R)\right)\ge\cot\left(\mathrm{O},\varepsilon(a_N|[x]_R)\right)$, then $x\in Pos(X)$;

(B1)if $\cot\left(\mathrm{O},\varepsilon(a_B|[x]_R)\right)\ge\cot\left(\mathrm{O},\varepsilon(a_P|[x]_R)\right)$ and $\cot\left(\mathrm{O},\varepsilon(a_B|[x]_R)\right)\ge\cot\left(\mathrm{O},\varepsilon(a_N|[x]_R)\right)$, then $x\in Bnd(X)$;

(N1)if $\cot\left(\mathrm{O},\varepsilon(a_N|[x]_R)\right)\ge\cot\left(\mathrm{O},\varepsilon(a_P|[x]_R)\right)$ and $\cot\left(\mathrm{O},\varepsilon(a_N|[x]_R)\right)\ge\cot\left(\mathrm{O},\varepsilon(a_B|[x]_R)\right)$, then $x\in Neg(X)$.

The newly generated decision rules (P1) - (N1) provide a specific and clear scheme to use the three-way decision method for decision making.

B. Three-way decision model using Hamming distance ranking method

Now we apply the Hamming distance ranking method to three-way decisions. The loss expectations are identical to the previous chapter. Then, according to the Hamming distance, the decision rules $(P1^H)$ - $(N1^H)$ can be re-represented as:

$(P1^H)$ if $H\left(\mathrm{O}, \varepsilon(a_p|[x]_R)\right) \le H\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right)$ and $H\left(\mathrm{O}, \varepsilon(a_p|[x]_R)\right) \le H\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right)$, then $x \in Pos(X)$;

$(B1^H)$ if $H\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right) \le H\left(\mathrm{O}, \varepsilon(a_P|[x]_R)\right)$ and $H\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right) \le H\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right)$, then $x \in Bnd(X)$;

$(N1^H)$ if $H\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \le H\left(\mathrm{O}, \varepsilon(a_P|[x]_R)\right)$ and $H\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \le H\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right)$, then $x \in Neg(X)$.

C. Three-way decision model using Vicis-Wave Hedges distance ranking method

Now we apply the Vicis-Wave Hedges distance ranking method to three-way decisions. The loss expectations are identical to the previous chapter. Then, according to the Hamming distance, the decision rules $(P1^V)$ - $(N1^V)$ can be re-represented as:

$(P1^V)$ if $V\left(\mathrm{O}, \varepsilon(a_p|[x]_R)\right) \le V\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right)$ and $V\left(\mathrm{O}, \varepsilon(a_p|[x]_R)\right) \le V\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right)$, then $x \in Pos(X)$;

$(B1^V)$ if $V\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right) \le V\left(\mathrm{O}, \varepsilon(a_P|[x]_R)\right)$ and $V\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right) \le V\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right)$, then $x \in Bnd(X)$;

$(N1^V)$ if $V\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \le V\left(\mathrm{O}, \varepsilon(a_P|[x]_R)\right)$ and $V\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \le V\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right)$, then $x \in Neg(X)$.

In conclusion, we may outline the 3WD process method under single-valued neutrosophic information based on cotangent similarity, Hamming distance, and Vicis-Wave Hedges distance. These are the steps in the algorithm.

First step : It is primary to give the decision table $(U, C \bigcup D, V)$, where the object set is $U = \{x_1, x_2, \cdots, x_n\}$, the condition attribute set is $C = \{a_1, a_2, \cdots, a_m\}$, the decision attribute set is $D = \{d\}$, and V is the attribute value set. According to the decision attribute, the two states X and $\neg X$,and the equivalence class $[x]_R = \{y \in U \mid R_{a_i}(x) = R_{a_i}(y), i = 1, 2, \cdots, m\}$ of the object are obtained; among them $R_{a_i}(x)$ is the attribute value of x concerning the attribute.

Second step: The $\mathrm{P}(X|[x]_R) = \dfrac{|X \bigcap [x]_R|}{|[x]_R|}$ and $\mathrm{P}(\neg X|[x]_R) = \dfrac{|\neg X \bigcap [x]_R|}{|[x]_R|}$ are calculated

Third step: The expected losses of each equivalence class $\varepsilon\left(a_\bullet|[x]_R\right)(\bullet = P, B, N)$ are calculated by (4) ~ (6).

Fourth step: According to definition 2.4, definition 2.5, and definition 2.6, the cotangent similarities, Hamming distances, and Vicis-Wave Hedges distances corresponding to each loss expectation are computed.

Fifth step: decision results are finally gained according to the decision rules.

IV. Application with Case

The model is then applied to an antique collection situation [28]. The pastime of collecting antiques is well-known and has been around for a long time. Table 3 displays the current selection of 15 antiques, each with 4 conditional and 1 decisional features. $U = \{x_1, x_2, \cdots, x_{15}\}$ stands for the fifteen antiques. $C = \{a_1, a_2, a_3, a_4\}$ represents the four condition attributes. The market value is represented by a_1,and the values of it are α, β and γ, representing high, medium, and low respectively; a_2 represents the storage needs, which contain the terms harsh, medium, and loose; and a_3 represents the purchase price, which contains the terms expensive, suitable, and cheap. a_4, which includes 1, 2, and 3, representing big, general, and tiny, respectively, was used to indicate value-added space. The decision attribution set is $D = \{d\}$, in which the attribution value has Y, N and means 'buy' or 'do not buy.'

TABLE III
THE DECISIION TABLE

U	a_1	a_2	a_3	a_4	d
x_1	β	L	E	1	Y
x_2	α	M	S	1	N
x_3	β	L	E	1	N
x_4	γ	L	C	3	Y
x_5	β	M	S	3	Y
x_6	α	H	C	2	N
x_7	β	L	E	1	N
x_8	γ	L	C	3	N
x_9	β	M	S	3	Y
x_{10}	β	L	E	1	Y
x_{11}	γ	L	C	3	N
x_{12}	β	M	S	3	N
x_{13}	α	H	C	2	Y
x_{14}	β	L	E	1	Y
x_{15}	γ	L	C	3	N

TABLE IV
THE LOSS FUNCTIONS

loss fuction	$[x_1]_R$	$[x_2]_R$	$[x_4]_R$	$[x_5]_R$	$[x_6]_R$
λ_{PY}	(0.81,0.09,0.15)	(0.14,0.65,0.76)	(0.7,0.82,0.43)	(0.48,0.27,0.49)	(0.75,0.95,0.84)
λ_{BY}	(0.90,0.27,0.97)	(0.42,0.03,0.74)	(0.04,0.69,0.38)	(0.44,0.67,0.96)	(0.25,0.54,0.26)
λ_{NY}	(0.13,0.54,0.96)	(0.92,0.84,0.39)	(0.27,0.31,0.76)	(0.64,0.65,0.34)	(0.50,0.13,0.82)
λ_{PN}	(0.91,0.95,0.48)	(0.79,0.93,0.65)	(0.05,0.95,0.79)	(0.70,0.16,0.59)	(0.69,0.14,0.24)
λ_{BN}	(0.63,0.96,0.80)	(0.95,0.67,0.17)	(0.39,0.03,0.18)	(0.75,0.12,0.22)	(0.89,0.26,0.92)
λ_{NN}	(0.47,0.69,0.03)	(0.95,0.12,0.71)	(0.47,0.93,0.84)	(0.51,0.26,0.90)	(0.72,0.86,0.75)

The 15 antiques can be divided into 5 equivalence classes. $\lambda_{\cdot Y}$ and $\lambda_{\cdot N}$ indicate the loss of taking action a_P , a_B ,and a_N in the case of purchase and non-purchase. The values are shown in Table IV.

From Table 3, two state sets can be obtained:

X=$\{x_1, x_4, x_5, x_9, x_{10}, x_{13}, x_{14}\}$,

┐X=$\{x_2, x_3, x_6, x_9, x_7, x_8, x_{11}, x_{12}, x_{15}\}$,

And the equivalence classes of each object are:

$[x_1]_R = \{x_1, x_3, x_7, x_{10}, x_{14}\}$,

$[x_2]_R = \{x_2\}$,

$[x_4]_R = \{x_4, x_8, x_{11}, x_{15}\}$,

$[x_5]_R = \{x_5, x_9, x_{12}\}$,

$[x_6]_R = \{x_6, x_{13}\}$.

The conditional probabilities $\mathrm{P}(*|[x]_R)$ can be calculated (as shown in Table 5).

TABLE V
CONDITIONAL PROBABILITIES

	$[x_1]_R$	$[x_2]_R$	$[x_4]_R$	$[x_5]_R$	$[x_5]_R$
$\mathrm{P}(X\|[x]_R)$	0.60	0	0.25	0.67	0.50
$\mathrm{P}(\neg X\|[x]_R)$	0.40	1	0.75	0.33	0.50

Tables 4, 5, and Equation (1) show the expected loss of taking action a_p on the object in the equivalence class $[x_1]_R$ is $\varepsilon(a_p\,|[x_1]_R) = (0.86, 0.23, 0.24)$. Similarly, the expected losses of different behaviors of other equivalence classes can be taken, as shown in Table 6.

TABLE VI
EXPECTED LOSS

	$\varepsilon(a_p\,\|[x]_R)$	$\varepsilon(a_B\,\|[x]_R)$	$\varepsilon(a_N\,\|[x]_R)$
$[x_1]_R$	(0.86,0.23,0.24)	(0.83,0.45,0.90)	(0.29,0.60,0.24)
$[x_2]_R$	(0.21,0.93,0.65)	(0.95,0.67,0.17)	(0.95,0.12,0.71)
$[x_4]_R$	(0.29,0.92,0.68)	(0.08,0.07,0.22)	(0.43,0.71,0.82)
$[x_5]_R$	(0.57,0.23,0.52)	(0.57,0.38,0.59)	(0.61,0.48,0.47)
$[x_6]_R$	(0.72,0.36,0.45)	(0.71,0.37,0.49)	(0.63,0.33,0.78)

A. *Application with cotangent similarity for ranking*

According to Definition 2.4, we can calculate the cotangent similarity between each loss expectation and the ideal neutrosophic number, respectively, as shown in Table 7. Consider the equivalence class $[x_1]_R$ as an example. As shown in Table 7, due to $\cot\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \geq \cot\left(\mathrm{O}, \varepsilon(a_P|[x]_R)\right)$ and $\cot\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \geq \cot\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right)$, the equivalence class A should be distributed to the negative region, meaning that the items $x_1, x_3, x_7, x_{10}, x_{14}$ are inappropriate for purchase. According to an analogy, equivalence classes $[x_2]_R$ and $[x_4]_R$ should be separated into a positive region.

Similarly, equivalence classes $[x_5]_R$ should be separated into a boundary region, and equivalence classes $[x_6]_R$ into a negative region. In particular, antiques $x_2, x_4, x_8, x_{11}, x_{15}$ are wise selections and worthwhile purchases; antiques x_5, x_9, x_{12} require further consideration; and antiques $x_1, x_3, x_6, x_7, x_{10}, x_{13}, x_{14}$ are inappropriate for purchase.

B. *Application with Hamming distance for ranking*

The Hamming distance of each loss expectation and ideal neutrosophic number may be determined using Definition 2.5, as shown in Table VIII.

Consider the equivalence class $[x_1]_R$ as an example. As can be seen from Table 8, due to $H\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \leq H\left(\mathrm{O}, \varepsilon(a_P|[x]_R)\right)$ and $H\left(\mathrm{O}, \varepsilon(a_N|[x]_R)\right) \leq H\left(\mathrm{O}, \varepsilon(a_B|[x]_R)\right)$, the equivalence class $[x_1]_R$ should be distributed to the negative region, meaning that antiques $x_1, x_3, x_7, x_{10}, x_{14}$ are inappropriate for purchase. According to an analogy, equivalence classes $[x_2]_R$ should be separated into a positive region, equivalence classes $[x_4]_R$ into a positive region, equivalence classes $[x_5]_R$ into a boundary region, and equivalence classes $[x_6]_R$ into a negative region. In particular, we think that antiques $x_2, x_4, x_8, x_{11}, x_{15}$ are

TABLE VII
COTANGENT SIMILARITIES

equivalence class	$\cot_1\left(\mathrm{O},\varepsilon(a_p\|[x]_R)\right)$	$\cot_1\left(\mathrm{O},\varepsilon(a_B\|[x]_R)\right)$	$\cot_1\left(\mathrm{O},\varepsilon(a_N\|[x]_R)\right)$
$[x_1]_R$	0.1611	0.4204	0.4296
$[x_2]_R$	0.7146	0.2373	0.2345
$[x_4]_R$	0.6911	0.3278	0.6128
$[x_5]_R$	0.3191	0.3839	0.3659
$[x_6]_R$	0.2934	0.3105	0.4081

TABLE VIII
HAMMING DISTANCE

equivalence class	$H\left(\mathrm{O},\varepsilon(a_P\|[x]_R)\right)$	$H\left(\mathrm{O},\varepsilon(a_B\|[x]_R)\right)$	$H\left(\mathrm{O},\varepsilon(a_N\|[x]_R)\right)$
$[x_1]_R$	2.39	1.48	1.45
$[x_2]_R$	0.63	2.11	2.12
$[x_4]_R$	0.69	1.79	0.9
$[x_5]_R$	1.82	1.6	1.66
$[x_6]_R$	1.91	1.85	1.52

TABLE IX
VICIS-WAVE HEDGES DISTANCE

equivalence class	$V\left(\mathrm{O},\varepsilon(a_P\|[x]_R)\right)$	$V\left(\mathrm{O},\varepsilon(a_B\|[x]_R)\right)$	$V\left(\mathrm{O},\varepsilon(a_N\|[x]_R)\right)$
$[x_1]_R$	2.0989	1.2619	1.1529
$[x_2]_R$	0.4587	1.8570	1.9638
$[x_4]_R$	0.5221	1.5885	0.8119
$[x_5]_R$	1.5118	1.2771	1.3219
$[x_6]_R$	1.5699	1.5121	1.2574

wise selections and worthwhile purchases; antiques x_5, x_9, x_{12} require more thought; and antiques $x_1, x_3, x_6, x_7, x_{10}, x_{13}, x_{14}$ are inappropriate for purchase.

C. Application with Vicis-Wave Hedges for ranking

The Vicis-Wave Hedges distance of each loss expectation and ideal neutrosophic number may be determined using Definition 2.5, as shown in Table IX.

Consider the equivalence class $[x_1]_R$ as an example. As can be seen from Table IX, due to $V\left(\mathrm{O},\varepsilon(a_N|[x]_R)\right) \le V\left(\mathrm{O},\varepsilon(a_P|[x]_R)\right)$ and $V\left(\mathrm{O},\varepsilon(a_N|[x]_R)\right) \le V\left(\mathrm{O},\varepsilon(a_B|[x]_R)\right)$, the equivalence class $[x_1]_R$ should be distributed to the negative region, meaning that antiques $x_1, x_3, x_7, x_{10}, x_{14}$ are inappropriate for purchase. According to an analogy, equivalence classes $[x_2]_R$ should be separated into a positive region, equivalence classes $[x_4]_R$ into a positive region, equivalence classes $[x_5]_R$ into a boundary region, and equivalence classes $[x_6]_R$ into a negative region. In particular, we think that antiques $x_2, x_4, x_8, x_{11}, x_{15}$ are wise selections and worthwhile purchases; antiques x_5, x_9, x_{12} require more thought; and antiques $x_1, x_3, x_6, x_7, x_{10}, x_{13}, x_{14}$ are inappropriate for purchase.

D. Comparative analysis of these methods

Section III presented different 3WD models that employ various distance measurement techniques. In Section IV, these models were simultaneously applied to an ancient collection case. A comparative study shows that even though these models use distinct calculating techniques, the outcomes of their decisions are identical. The decision results of these methods indicate that antiques $x_2, x_4, x_8, x_{11}, x_{15}$ lie in the positive region, making them suitable for purchase; antiques x_5, x_9, x_{12} lie in the boundary region, requiring further thought; and antiques $x_1, x_3, x_6, x_7, x_{10}, x_{13}, x_{14}$ lie in the negative region, making them unsuitable for purchase.

Nevertheless, the three approaches differ significantly even though their decision outcomes are identical. The cotangent similarity is calculated using the cotangent function, which is more complex and time-consuming albeit more standardized as the result lies within [0,1]. The Hamming distance and the Vicis-Wave Hedges distance employ the easier-to-calculate absolute value and minimum value functions, and their computation results do not always lie between 0 and 1.

V. Conclusion

This study proposed three 3WD models using single-valued neutrosophic information. Specifically, cotangent similarity, the Hamming distance, and the Vicis-Wave Hedges distance were employed to compare the sizes of two neutrosophic sets and determine how similar the two sets are to each other. The three-way decision model, which is more in line with human thinking, enhances delayed decision-making compared to the conventional binary decision models.

Before considering the Vicis-Wave Hedges distance measuring technique, the knowledge of the pertinent single-valued neutrosophic sets was first reviewed. Subsequently, three 3WD models were proposed. Finally, the proposed models were applied to a scenario of an antique collection, which sufficiently demonstrated their rationality and feasibility.

In future studies, we can consider using other forms of neutrosophic sets to combine with 3WD. There is still a lot of work to be done.

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References

[1] Y. Yao, "Three-way decisions with probabilistic rough sets," *Information Sciences*, vol. 180, no. 3, pp. 341-35, 2010.
[2] Y. Yao, "Three-way decision and granular computing. *International Journal of Approximate Reasoning*," vol.103, pp. 107-123, 2018.
[3] B. Hu, Three-way decisions space and three-way decisions. *Information Sciences*, vol. 281, pp.21-52, 2014.
[4] Y. Yao, Set-theoretic models of three-way decision. *Granular Computing*, vol. 6, pp. 133-148, 2021.
[5] Q. Zhang, D. Xia, G. Wang, "Three-way decision model with two types of classification errors," *Information Sciences*, vol. 420, pp. 431-453.
[6] B. Hu, H. Wong, K. C. Yiu, "On two novel types of three-way decisions in three-way decision spaces," *International Journal of Approximate Reasoning*, vol. 82, pp. 285-306.
[7] G. Lang, D. Miao, M. Cai, "Three-way decision approaches to conflict analysis using decision-theoretic rough set theory," *Information Sciences*, vol. 406, pp. 185-207, 2017.
[8] D. Liu, D. Liang. "Three-way decisions in ordered decision system," *Knowledge-Based Systems*, vol. 137, pp. 182-195, 2017.
[9] P. Wang, Y. Yao, "CE3: a three-way clustering method based on mathematical morphology," *Knowl Based Syst*, vol. 155, pp. 54-65, 2018.
[10] F. Jia, P. Liu, "A novel three-way decision model under multiple-criteria environment," Information Sciences, vol. 471, pp. 29-51, 2019.
[11] J. Qian, C. Liu, X. Xu, "Multigranulation sequential three-way decisions based on multiple thresholds," *International Journal of Approximate Reasoning*, vol. 105, pp. 396-416, 2019.
[12] C. Luo, T. Li, Y. Huang, H. Fujita, "Updating three-waydecision in incomplete multi-scale information systems," *Information Science*, vol.476, pp. 274-289, 2019.
[13] X. Li, X. Wang, B. Sun, Y. She, L. Zhao, "Three-way decision on information tables," *Information Sciences*, vol. 545, pp. 25-43,2021.
[14] L.A. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338-353, 1996.
[15] F. Smarandache, "Neutrosophic probability, set and logic." *Bull. Transilv. Univ. Braşov Ser. B (N.S.)*, vol.6, pp. 41-48, 1999.
[16] T. Krassimir, "Intuitionistic fuzzy sets," *Fuzzy sets and systems*, vol. 20, no. 1, pp. 87-96, 1986.
[17] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, "Single valued neutrosophic sets," *Review of the Air Force Acedemy*, vol.1, no.1, 2010.
[18] X. Zhang, C. Bo, F. Smarandache, J. Dai. "New inclusionrelation of neutrosophic sets with applications and related latticestructure," *International Journal of Machine Learning and Cybernetics*, vol. 9, no. 10, pp. 1753-1763, 2018.
[19] D. Xu, X. Lu, L. Peng, "An Extended TODIM Based on Cumulative Prospect Theory for Single-Valued Neutrosophic Multi-Attribute Decision-Making," *Engineering Letters*, vol.30, no.2, pp.752-756, 2022.
[20] M. Abdel-basset, G. Manogara, M. Mohamedm, N. Chilamkurti, "three-way decisions based onneutrosophic sets and AHP-QFD framework for supplier selection problem," *Future generation comput-systems*, vol. 89, no. 0, pp. 19-30, 2018.
[21] P. Singh, "Three-way fuzzy concept lattice represen-tation using neutrosophic set," *International journalof machine learning and cybernetics*, vol. 8, no.1, pp. 69-79, 2017
[22] P. Singh, "Three- way n-valued neutrosophic conceptlattice at different granulation," *International Journalof Machine Learning and Cybernetics*, vol. 9, no. 11, pp. 1839-1855, 2018.
[23] L. Jiao, H. Yang, S. Li, "Three-way decision based on decision-theoretic rough sets with single-valued neutrosophic information", *International Journal of Machine Learning and Cybernetics*, vol. 11, pp. 657-665, 2020.
[24] J. Peng, J. Wang, H. Zhang, X. Chen, "An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets," *Applied Soft Computing*, vol. 25, pp. 336-346, 2014.
[25] R. Borzooei，H. Farahani，M. Moniri, "Neutrosophic deductive filters on BL-algebras," *Journal of Intelligent &. Fuzzy Systems*, vol. 26, no. 6, pp. 2993-3004，2014
[26] J. Ye, "Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine," *Soft Computing - A Fusion of Foundations, Methodologies & Applications*, Vol. 21, no. 3, pp. 817-825, 2017.
[27] P. Majumdar, S. Samanta, "On similarity and entropy of neutrosophic sets," *Journal of Intelligent and Fuzzy Systems*, vol.26, no.3, pp. 1064-1246, 2014.
[28] H. Yang, H. Ren, L. Jiao, "A novel three-way decision model under single-valued neutrosphic information," *Journal of Shanxi Normal University (Natural Science Edition)*, vol. 50, no. 3, pp. 1-6, 2022.