Existence of Positive Solution for Fractional Differential Equation with Switched Nonlinearity

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***Abstract*—In this study, by application of the theory of two mixed monotonic operators and their properties, the author propose an existence theorem of the unique positive solution for the switched system. Among them, the equation is of fractional order, and the boundary conditions are multipoint. Then, based on the obtained results, we bring a specific fractional order switching system to obtain more general results.**

***Index Terms*—Unique positive solution; Switched system; Fractional differential equation; theory of two mixed monotonic operators and their properties.**

I. Introduction

HYBRID system is a system that allows for the interaction both discrete and continuous events. Switched system belongs to hybrid systems, it is currently a very popular research field in the scientific community [1-4,19-20]. It includes a set of nonlinear subsystems and a logical rule. As we all know, the intelligent control are based on the concept of switching between different controllers [5], this is also why people discuss switching systems. The actual system was initially generated by several dynamic systems with multiple models that needed to be expressed, and their behavior depends on many environmental factors [6]. Switched systems arise in many fields like chemical processes, communication industries. And computer controlled systems and transportation systems also involve switching systems. In recent years, the design methodology and stability analysis of the switched system are considered in [7-11]. [12-14] demonstrate existence theorem of positive function which satisy some switched system.

In [12], the authors concerned with positive solution for integral boundary condition model which nonlinear term with switching signal

$$v''(s) + f_{\varrho(s)}(s, v(s)) = 0, \qquad s \text{ within } J = [0,1],$$

$$v(0) = 0, \qquad v(1) = \int_0^1 a(\zeta)v(\zeta)d\zeta,$$

in this the constant function $\varrho(s)$ is piecewise constant function, and it maps $[0,1]$ to $\{1,2,\cdots,N\}$. Also, $\varrho(t)$ is a finite switching signal.

In [13], Guo discussed the unique solution to fractional p-Laplacian operator model which nonlinear term with switching signal

$$D^{\beta}_{0+}\varphi_p(D^{\alpha}_{0+}v(t)) = f_{\varrho(t)}(t, v(t), D^{\gamma}_{0+}v(t)),$$

$$t \text{ within the set } [0,1],$$

$$v(0) = a\int_0^1 v(\zeta)d\zeta + \lambda v(\mu),$$

$$D^{\alpha}_{0+}v(0) = bD^{\alpha}_{0+}v(\nu), \quad \mu,\nu \in [0,1].$$

Lv and Chen [14] demonstrate the existence theorem for the system model which nonlinear term with switching signal

$${}^{C}D^{\alpha}_{0+}y(t) + f_{\varrho(t)}(t, y(t)) + g_{\varrho(t)}(t, y(t)) = 0,$$

$$t \text{ lies in the interval from 0 to 1},$$

$$y(0) = y''(0) = 0, \quad y(1) = \int_0^1 y(\zeta)d\zeta.$$

However, there is no result on the existence theorem for the fractional model with switched nonlineartity

$$\begin{aligned} &D^{\beta}_{0+}\mu(t) + f_{\varrho(t)}(t, \mu(t), \mu(t)) \\ &+g_{\varrho(t)}(t, \mu(t), \mu(t)) = 0, \\ &t \text{ lies in the interval from 0 to 1}, \end{aligned} \tag{1}$$

$$\mu(0) = 0, \quad D^{\gamma}_{0+}\mu(1) = \sum_{i=1}^{m-2}\alpha_i D^{\gamma}_{0+}\mu(\eta_i), \tag{2}$$

where D^{β}_{0+} represents the fractional derivative of Riemann-Liouville type, $1 < \beta \le 2,\ 0 < \gamma \le 1,\ 0 < \beta-\gamma-1,\ \alpha_i, \eta_i$ are constants between the interval 0 and 1, i equals $1,2,\cdots,m$, and $\sum_{i=1}^{m-2}\alpha_i\eta_i^{\beta-\gamma-1} < 1$, here $\varrho(t) : J \to \{1,2,\cdots,N\}$ refers to piecewise constant function and $f_i, g_i \in C[J \times R^+ \times R^+, R^+]$, i equals $1,2,\cdots,N$. Corresponding to the switching signal $\varrho(t)$, there have the switching sequence

$$\{(i_0,t_0),\cdots,(i_j,t_j),\cdots,(i_k,t_k)|i_j \text{ within } \{1,2,\cdots,N\},$$

$$j \text{ equals } 0,1,2,\cdots,k\},$$

which means that the i_jth nonlinearity is activated when t lies in the interval $[t_j, t_{j+1})$ and the i_kth nonlinearity is activated when t lies in the interval $[t_k, 1]$. Here $x_0 = 0,\ t_0 = 0$. The objective of this study is to establish an iterative scheme for approximating the unique solution for the model to be studied in this article.

II. The preliminary lemmas

Let us begin this article with a few fundamental definitions, useful Lemmas.

Definition 2.1 [15] The β order Riemann and Liouville fractional integral form about function h is like

$$I^{\beta}_{0+}h(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\zeta)^{\beta-1}h(\zeta)d\zeta, \quad \beta > 0.$$

Definition 2.2 [15] For continuous function h, the $\beta > 0$ order Riemann-Liouville fractional differentiator is expressed like

$$D^{\beta}_{0+}h(t) = \frac{1}{\Gamma(n-\beta)}(\frac{d}{dt})^n\int_0^t (t-\zeta)^{n-\beta-1}h(\zeta)d\zeta.$$

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Assume E represents a real Banach space, it is sorted of the size relationship of the elements in the cone, we say $\nu - \mu \in P$ if and only if the set $P \subset E$ and satisfy $\mu \leq \nu$. We write $\mu < \nu$ or $\nu > \mu$, if $\mu \leq \nu$ and μ not equals to ν. θ is the zero element in E. Choose μ, ν with in the set E, $\theta \leq \mu \leq \nu$, we get $\|\mu\| \leq N\|\nu\|$, here N is a positive constant, we call P a normal cone. The interval between ν_1 and ν_2, which we denote $[\nu_1, \nu_2] = \{\nu \in E | \nu_1 \leq \nu \leq \nu_2\}$ for $\nu_1, \nu_2 \in E$.If $\mu \leq \nu$ implies $T\mu \leq T\nu$ or $(T\mu \geq T\nu)$, we call T maps E to E is increasing or (decreasing) operator.

For μ, ν within the set E, if the positive constants μ and λ satisfying $\lambda\mu \leq \nu \leq \mu\mu$, we denote $\mu \sim \nu$. From this definition, $\sim$ stands an equivalence relation. Given a positive function h, we write $P_h = \{\mu \in E | \mu \sim h\}$. Then P_h is a subset of P.

Definition 2.3 [16] For $\mu_i, \nu_i (i = 1, 2)$ within P, if $\mu_1 \leq \mu_2,\ \nu_1 \geq \nu_2$, we have $T(\mu_1,\ \nu_1) \leq T(\mu_2,\ \nu_2)$, then we say the operator T is a mixed monotone operator. If μ within P, $T(\mu, \mu)$ equals μ implies μ a fixed point of T.

Definition 2.4 [16] For μ within P, if λ lies in the interval $(0, 1)$, we have

$$T(\lambda\mu) \geq \lambda T(\mu),$$

then we say operator $T:\ P \to P$ is sub-homogeneous.

Definition 2.5 [16] For μ within P, if λ lies in the interval $(0, 1)$, and $0 \leq \beta < 1$, we have

$$T(\lambda\mu) \geq \lambda^{\beta} T(\mu),$$

then we say operator $T : P \to P$ is $\beta-$ concave.

Lemma 2.6 [17] We define P which is a normal cone and two mixed monotone type operators C, D which map $P \times P \to P$, suppose C, D satisfies the subsequent relation:
(i) for any λ lies in the interval $(0, 1)$, and every μ, ν within P, if $\psi(\lambda) \in (\lambda, 1]$ we have

$$C(\lambda\mu, \lambda^{-1}\nu) \geq \psi(\lambda) C(\mu, \nu);$$

(ii) for λ lies in the interval $(0, 1)$, and every μ, ν within P, there have $D(\lambda\mu, \lambda^{-1}\nu) \geq \lambda D(\mu, \nu)$;
(iii) for h within $P, h > \theta$, we have $C(h, h)$ belong to the set P_h, $D(h, h)$ belong to the set P_h;
(iv) for $\delta > 0$, every μ, ν within P, the relation $C(\mu, \nu) \geq \delta D(\mu, \nu)$ holds:
Thus we can find a $\mu^* \in P_h$ that satisfy $C(\mu, \mu) + D(\mu, \mu) = \mu$. Moreover, if we establish an iterative sequence

$$\mu_n = C(\mu_{n-1}, \nu_{n-1}) + D(\mu_{n-1}, \nu_{n-1}),$$

$$\nu_n = C(\nu_{n-1}, \mu_{n-1}) + D(\nu_{n-1}, \mu_{n-1}), \quad n \text{ equals } 1, 2, \cdots,$$

here $\mu_0, \nu_0 \in P_h$, thus we have if $n \to \infty$, we have μ_n converges to μ^* and also ν_n converges to μ^*.

Lemma 2.7 [18] For every h in the function space $C[0, 1]$, the system is presented as follows

$$D^{\beta}_{0+}\mu(t) + h(t) = 0, \quad 0 < t < 1,\ 1 < \beta \leq 2, \tag{3}$$

$$\mu(0) = 0, \quad D^{\gamma}_{0+}\mu(1) = \sum_{i=1}^{m-2} \alpha_i D^{\gamma}_{0+}\mu(\eta_i),\ 0 < \gamma \leq 1, \tag{4}$$

then the following function $\mu(t)$

$$\mu(t) = \int_0^1 G(t, \zeta) h(\zeta) d\zeta, \tag{5}$$

satisfies the equation (3)(4), here

$$G(t, \zeta) = G_1(t, \zeta) + G_2(t, \zeta), \tag{6}$$

in this

$$G_1(t, \zeta) = \begin{cases} \dfrac{t^{\beta-1}(1-\zeta)^{\beta-\gamma-1} - (t-\zeta)^{\beta-1}}{\Gamma(\beta)}, \\ \qquad\qquad 0 \leq \zeta \leq t \leq 1, \\ \dfrac{t^{\beta-1}(1-\zeta)^{\beta-\gamma-1}}{\Gamma(\beta)}, \\ \qquad\qquad 0 \leq t \leq \zeta \leq 1, \end{cases} \tag{7}$$

$$G_2(t, \zeta) = \begin{cases} \dfrac{t^{\beta-1}}{D\Gamma(\beta)} \displaystyle\sum_{0 \leq \zeta \leq \eta_i} \alpha_i [\eta_i^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1} \\ \qquad -(\eta_i - \zeta)^{\beta-\gamma-1}], \quad 0 \leq t \leq 1, \\ \dfrac{t^{\beta-1}}{D\Gamma(\beta)} \displaystyle\sum_{\eta_i \leq \zeta \leq 1} \alpha_i \eta_i^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1}, \\ \qquad\qquad 0 \leq t \leq 1, \end{cases} \tag{8}$$

in which

$$D = 1 - \sum_{i=1}^{m-2} \alpha_i \eta_i^{\beta-\gamma-1}.$$

Lemma 2.8 [18] If the constants α_i, η_i, here the value of i ranging from 1 to $m-2$ satisfy $\sum\limits_{i=1}^{m-2} \alpha_i \eta_i^{\beta-\gamma-1} < 1$, then the expression $G(t, \zeta)$ possesses the following relationship:
(i) the expression $G(t, \zeta)$ is a continuous function about variable $t, \zeta \in [0, 1]$;
(ii) the expression $G(t, \zeta)$ is a positive function for the variables ζ, t in the interval $(0, 1)$;
(iii) the expression $G(t, \zeta) \leq \left[\dfrac{(1-\zeta)^{\beta-\gamma-1}}{\Gamma(\beta)} + \dfrac{\sum\limits_{i=1}^{m-2} \alpha_i \eta_i^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1}}{D\Gamma(\beta)}\right] h(t)$ for $\zeta, t \in [0, 1]$;
(iv) the expression $G(t, \zeta) \geq \dfrac{h(t)}{D} \sum\limits_{i=1}^{m-2} \alpha_i \overline{G_2}(\eta_i, \zeta)$ for $\zeta, t \in [0, 1]$, where

$$\overline{G_2}(t, \zeta) = \begin{cases} \dfrac{t^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1} - (t-\zeta)^{\beta-\gamma-1}}{\Gamma(\beta)}, \\ \qquad\qquad 0 \leq \zeta \leq t \leq 1, \\ \dfrac{t^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1}}{\Gamma(\beta)}, \\ \qquad\qquad 0 \leq t \leq \zeta \leq 1, \end{cases}$$

$h(t) = t^{\beta-1}$.

III. Theorem

The Banach space E standing for all continuous function from 0 to 1 is considered and the norm $\|\mu\| = \max\limits_{0 \leq t \leq 1} |\mu(t)|$ is assigned to E. We write the set P just like $P = \{\mu \text{ within } E | \mu(t) \geq 0, \text{ and } t \text{ within the interval } [0, 1]\}$, we know the cone P is normal.

Theorem 3.1 Suppose that

(H_1) functions f_i, g_i map $[0, 1] \times [0, +\infty) \times [0, +\infty)$ to $[0, +\infty)$, here i equals $1, 2, \cdots, N$, and for all t within the interval $[0, 1]$, $g_i(t, 0, 1) \neq 0$;

(H_2) when variables t within the interval $[0, 1]$ and ν within the interval $[0, +\infty)$ are fixed, functions

$f_i(t,\mu,\nu), g_i(t,\mu,\nu)$ are increasing about the variable μ within the interval $[0,+\infty)$; when variables t within the interval $[0,1]$ and μ belonging to the interval $[0,+\infty)$ are fixed, functions $f_i(t,\mu,\nu), g_i(t,\mu,\nu)$ are decreasing about the variable ν belonging to the interval $[0,+\infty)$ for $i = 1,2,\cdots,N$;

(H_3) for λ belonging to the interval $(0,1)$, and t belonging to the interval $[0,1]$, x,y belonging to the interval $[0,+\infty)$, if $\psi(\lambda)$ belongs to the interval $(\lambda,1)$, we can get

$$f_i(t,\lambda x,\lambda^{-1}y) \geq \psi(\lambda) f_i(t,x,y),$$
$$g_i(t,\lambda x,\lambda^{-1}y) \geq \lambda g_i(t,x,y),\ i=1,2,\cdots,N;$$

(H_4) if $\delta > 0$, and t belongs to the interval $[0,1]$, and x,y belonging to the interval $[0,+\infty)$, we obtain

$$f_i(t,x,y) \geq \delta g_i(t,x,y),$$

when i equals to $1,2,\cdots,N$. It can be deduced that the problem (1)(2) has a unique positive solution, denoted as μ^* belonging to P_h, here $h(t)=t^{\beta-1},\ t\in[0,1]$. If we define the iterative sequences as follows

$$\mu_{n+1}(t) = \int_0^1 G(t,\zeta)[f_{\varrho(\zeta)}(\zeta,\mu_n(\zeta),\nu_n(\zeta)) + g_{\varrho(\zeta)}(\zeta,\mu_n(\zeta),\nu_n(\zeta))]d\zeta,\ n=0,1,2,\cdots,$$
$$\nu_{n+1}(t) = \int_0^1 G(t,\zeta)[f_{\varrho(\zeta)}(\zeta,\nu_n(\zeta),\mu_n(\zeta)) + g_{\varrho(\zeta)}(\zeta,\nu_n(\zeta),\mu_n(\zeta))]d\zeta,\ n=0,1,2,\cdots,$$

in this place μ_0,ν_0 belongs to the P_h. Letting $n\to\infty$, we can obtain that $\|\mu_n-\mu^*\|$ approaches 0 and $\|\nu_n-\mu^*\|$ approaches 0.

Proof: In light of Lemma 2.2, given

$$\mu(t) = \int_0^1 G(t,\zeta)[f_{\varrho(\zeta)}(\zeta,\mu(\zeta),\mu(\zeta)) + g_{\varrho(\zeta)}(\zeta,\mu(\zeta),\mu(\zeta))]d\zeta,$$

where the Green function $G(t,\zeta)$ is expressed by (6). Consequently, the boundary value problem possesses a solution $\mu=\mu(t)$ precisely $\mu(t)$ satisfy the above integral equation.

We now specify two operator $C,D:\ P\times P\to E$ in the following manner:

$$C(\mu,\nu)(t) = \int_0^1 G(t,\zeta) f_{\varrho(\zeta)}(\zeta,\mu(\zeta),\nu(\zeta))d\zeta,$$
$$D(\mu,\nu)(t) = \int_0^1 G(t,\zeta) g_{\varrho(\zeta)}(\zeta,\mu(\zeta),\nu(\zeta))d\zeta.$$

Consequently, the switched system (1)(2) possesses a solution $\mu=\mu(t)$ precisely $\mu(t)$ satisfy the relationship $\mu = C(\mu,\mu)+D(\mu,\mu)$.

By condition (H_1), we have C maps $P\times P$ to P and D maps $P\times P$ to P. Thus we set out to prove that all conditions of Lemma 1 are satisfied by C,D.

Firstly, two mixed monotone operators of C and D will be proven. Take any μ_1,μ_2,ν_1,ν_2 with in P, moreover, $\mu_1(t)\geq\mu_2(t),\ \nu_1(t)\leq\nu_2(t)$, which imply $\mu_1(t)\geq\mu_2(t)$ and $\nu_1(t)\leq\nu_2(t)$, when t belongs to the interval $[0,1]$. From (H_2), one has

$$\int_0^1 G(t,\zeta) f_i(\zeta,\mu_1(\zeta),\nu_1(\zeta))d\zeta \geq \int_0^1 G(t,\zeta) f_i(\zeta,\mu_2(\zeta),\nu_2(\zeta))d\zeta,\ i=1,2,\cdots,N,$$
$$\int_0^1 G(t,\zeta) g_i(\zeta,\mu_1(\zeta),\nu_1(\zeta))d\zeta \geq \int_0^1 G(t,\zeta) g_i(\zeta,\mu_2(\zeta),\nu_2(\zeta))d\zeta,\ i=1,2,\cdots,N,$$

this equation is equivalent to

$$C(\mu_1,\nu_1)(t)\geq C(\mu_2,\nu_2)(t),$$
$$D(\mu_1,\nu_1)(t)\geq D(\mu_2,\nu_2)(t),$$

this is also

$$C(\mu_1,\nu_1)\geq C(\mu_2,\nu_2),\quad D(\mu_1,\nu_1)\geq D(\mu_2,\nu_2).$$

Secondly, we prove that assumption (i) of Lemma 2.1 holds.
For any λ lies in the interval $(0,1)$, and μ,ν with in P, from (H_3), one has

$$\begin{array}{l}\int_0^1 G(t,\zeta) f_i(\zeta,\lambda\mu(\zeta),\lambda^{-1}\nu(\zeta))d\zeta \\ \geq \psi(\lambda)\int_0^1 G(t,\zeta) f_i(\zeta,\mu(\zeta),\nu(\zeta))d\zeta, \\ i=1,2,\cdots,N,\end{array} \tag{9}$$

this equation is equivalent to

$$C(\lambda\mu,\lambda^{-1}\nu)(t)\geq\psi(\lambda)C(\mu,\nu)(t),$$

this is also,

$$C(\lambda\mu,\lambda^{-1}\nu)\geq\psi(\lambda)C(\mu,\nu) \text{ for } \lambda$$

lies in the interval $(0,1),\ \mu,\nu$ within P.

Thirdly, we prove that assumption (ii) of Lemma 2.1 holds. For any λ lies in the interval from 0 to 1, and μ,ν within the cone, condition (H_3) implies that

$$\begin{array}{l}\int_0^1 G(t,\zeta) g_i(\zeta,\lambda\mu(\zeta),\lambda^{-1}\nu(\zeta))d\zeta \\ \geq \lambda\int_0^1 G(t,\zeta) g_i(\zeta,\mu(\zeta),\nu(\zeta))d\zeta, \\ i=1,2,\cdots,N,\end{array} \tag{10}$$

this equation is equivalent to

$$D(\lambda\mu,\lambda^{-1}\nu)(t)\geq\lambda D(\mu,\nu)(t),$$

this is also,

$$D(\lambda\mu,\lambda^{-1}\nu)\geq\lambda D(\mu,\nu) \text{ for } \lambda\in(0,1),\ \mu,\nu\in P.$$

Next, we prove that assumption (iii) of Lemma 2.1 holds. In light of condition (H_2) and Lemma 2.3, about any t lies in the interval $[0,1],\ i=1,2,\cdots,N$, we get

$$\begin{array}{l}\int_0^1 G(t,\zeta) f_i(\zeta,h(\zeta),h(\zeta))d\zeta \\ \leq \dfrac{h(t)}{D\Gamma(\beta)}\displaystyle\int_0^1 (D(1-\zeta)^{\beta-\gamma-1} \\ +\sum_{i=1}^{m-2}\alpha_i\eta_i^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1}) f_i(\zeta,1,0)d\zeta.\end{array} \tag{11}$$

On the other hand, in light of (H_2) and Lemma 2.3, for every t lies in the interval $[0,1]$, then

$$\begin{array}{l}\int_0^1 G(t,\zeta) f_i(\zeta,h(\zeta),h(\zeta))d\zeta \\ \geq \dfrac{h(t)}{D}\displaystyle\int_0^1\sum_{i=1}^{m-2}\alpha_i\overline{G_2}(\eta_i,\zeta) f_i(\zeta,0,1)d\zeta, \\ i=1,2,\cdots,N.\end{array} \tag{12}$$

For i equals $1,2,\cdots,N$, denote

$$m_i=\frac{1}{D\Gamma(\beta)}\int_0^1(D(1-\zeta)^{\beta-\gamma-1}+\sum_{i=1}^{m-2}\alpha_i\eta_i^{\beta-\gamma-1}(1-\zeta)^{\beta-\gamma-1})f_i(\zeta,1,0)d\zeta, \tag{13}$$

$$n_i=\frac{1}{D}\int_0^1\sum_{i=1}^{m-2}\alpha_i\overline{G_2}(\eta_i,\zeta)f_i(\zeta,0,1)d\zeta. \tag{14}$$

It follows from (H_1) and (H_4), one has

$$\int_0^1 f_i(\zeta,1,0)d\zeta\geq\int_0^1 f_i(\zeta,0,1)d\zeta\geq\delta\int_0^1 g_i(\zeta,0,1)d\zeta,$$

condition $g_i(t,0,1)\neq 0$ implies that $\int_0^1 f_i(\zeta,1,0)d\zeta$ $>0,\int_0^1 f_i(\zeta,0,1)d\zeta>0$.
Thus, the constants

$$m_i,\quad n_i>0,\quad i \text{ equals from 1 to N}. \tag{15}$$

Choose $\min\{n_i,\ i \text{ equals from } 1 \text{ to } N\}$, we denote by n, and $\max\{m_i,\ i \text{ equals from } 1 \text{ to } N\}$, we denote by m, we have $n>0$ and $m>0$. Therefore,

$$nh(t)\leq C(h,h)\leq mh(t), \tag{16}$$

this means that

$$C(h,h) \text{ lies in the set } P_h. \tag{17}$$

In the same way, we have

$$D(h,h)\in P_h. \tag{18}$$

Finally, we prove that assumption (iv) of Lemma 2.1 holds. For t lying in the interval $J, i=1,2,\cdots,N$, and μ,ν within P, utilizing (H_4), we have

$$\int_0^1 G(t,\zeta)f_i(\zeta,\mu(\zeta),\nu(\zeta))d\zeta \geq\delta\int_0^1 G(t,\zeta)g_i(\zeta,\mu(\zeta),\nu(\zeta))d\zeta, \tag{19}$$

which implies that

$$C(\mu,\nu)\geq\delta D(\mu,\nu). \tag{20}$$

Considering Lemma 2.1, the unique nonnegative function denoted by μ^* satisfies $C(\mu,\mu)+D(\mu,\mu)=\mu$. If we construct the sequence

$$\mu_n=C(\mu_{n-1},\nu_{n-1})+D\mu_{n-1},$$

$$\nu_n=C(\nu_{n-1},\mu_{n-1})+D\nu_{n-1},\quad n \text{ equals } 1,2,\cdots,$$

in this place $\mu_0,\nu_0\in P_h$, letting $n\to\infty$, we obtain $\mu_n\to\mu^*$ and $\nu_n\to\mu^*$, which consequently means that

$$\mu_{n+1}(t)=\int_0^1 G(t,\zeta)[f_{\varrho(\zeta)}(\zeta,\mu_n(\zeta),\nu_n(\zeta)) +g_{\varrho(\zeta)}(\zeta,\mu_n(\zeta),\nu_n(\zeta))]d\zeta,\ n \text{ equals } 0,1,2,\cdots,$$

$$\nu_{n+1}(t)=\int_0^1 G(t,\zeta)[f_{\varrho(\zeta)}(\zeta,\nu_n(\zeta),\mu_n(\zeta)) +g_{\varrho(\zeta)}(\zeta,\nu_n(\zeta),\mu_n(\zeta))]d\zeta,\ n \text{ equals } 0,1,2,\cdots,$$

satisfy $\|\mu_n-\mu^*\|$ approaching 0 and $\|\nu_n-\mu^*\|$ approaching 0 as n approaching∞. ■

IV. Applications

Example 4.1 Let's give the system with switched signal:

$$D_{0+}^{\frac{3}{2}}\mu(t)+f_{\varrho(t)}(t,\mu(t),\mu(t)) +g_{\varrho(t)}(t,\mu(t),\mu(t))=0,\ t\in(0,1), \tag{21}$$

$$\mu(0)=0,$$

$$D_{0+}^{\frac{1}{4}}\mu(1)=\frac{1}{100}D_{0+}^{\frac{1}{4}}\mu(\frac{1}{3})+\frac{12}{1000}D_{0+}^{\frac{1}{4}}\mu(\frac{2}{3}), \tag{22}$$

in this place $\varrho(t):\ J\to M=\{1,2\}$,

$$f_1(t,\mu,\nu)=2+\mu^{\frac{1}{3}}+\nu^{-\frac{1}{2}};$$

$$f_2(t,\mu,\nu)=4+sint+\mu^{\frac{1}{4}}+\nu^{-\frac{1}{6}};$$

$$g_1(t,\mu,\nu)=\frac{\mu}{1+\mu}t^2+1+\nu^{-\frac{1}{2}};$$

$$g_2(t,\mu,\nu)=\frac{\mu}{(1+t^2)(1+\mu)}+2+\nu^{-\frac{1}{6}}.$$

Obviously, $\beta-\gamma-1=\frac{1}{4}>0$, $\sum\limits_{i=1}^{m-2}\alpha_i\eta_i^{\beta-\gamma-1}=\alpha_1\eta_1^{\beta-\gamma-1}+\alpha_2\eta_2^{\beta-\gamma-1}=\frac{1}{100}\times(\frac{1}{3})^{\frac{1}{4}}+\frac{12}{1000}\times(\frac{2}{3})^{\frac{1}{4}}=0.01884<1$.

In the following, we prove
(1) It is obvious that continuous functions f_i,g_i map $[0,1]\times[0,+\infty)\times[0,+\infty)$ to $[0,+\infty)$, i equals $1,2$, and $g_1(t,0,1)=2\neq 0,\ g_2(t,0,1)=3\neq 0$;
(2) For the first variable, functions f_i,g_i, here i equaling $1,2$ are increasing, for the second variable, functions f_i,g_i, here i equaling $1,2$ are are decreasing;
(3) on the other hand, for λ belonging to the interval $(0,1)$, t within $[0,1]$, and the elements x,y within $[0,+\infty)$, choosing $\psi(\lambda)=\lambda^{\frac{1}{2}}\in(0,1)$, thus

$$\begin{aligned}f_1(t,\lambda\mu,\lambda^{-1}\nu)&=2+(\lambda\mu)^{\frac{1}{3}}+(\lambda^{-1}\nu)^{-\frac{1}{2}}\\&\geq\lambda^{\frac{1}{2}}(2+\mu^{\frac{1}{3}}+\nu^{-\frac{1}{2}})\\&=\psi(\lambda)f_1(t,\mu,\nu).\end{aligned}$$

$$\begin{aligned}f_2(t,\lambda\mu,\lambda^{-1}\nu)&=4+sint+(\lambda\mu)^{\frac{1}{4}}+(\lambda^{-1}\nu)^{-\frac{1}{6}}\\&\geq\lambda^{\frac{1}{4}}(4+sint+\mu^{\frac{1}{4}}+\nu^{-\frac{1}{6}})\\&\geq\lambda^{\frac{1}{2}}(4+sint+\mu^{\frac{1}{4}}+\nu^{-\frac{1}{6}})\\&=\psi(\lambda)f_2(t,\mu,\nu).\end{aligned}$$

Similarly, for all λ lies in the interval $(0,1)$, t within $[0,1]$, x,ylies in the interval $[0,+\infty)$, one has

$$\begin{aligned}g_1(t,\lambda\mu,\lambda^{-1}\nu)&=\tfrac{\lambda\mu}{1+\lambda\mu}t^2+1+(\lambda^{-1}\nu)^{-\frac{1}{2}}\\&=\tfrac{\lambda\mu}{1+\lambda\mu}t^2+1+\lambda^{\frac{1}{2}}\nu^{-\frac{1}{2}}\\&>\lambda(\tfrac{\mu}{1+\lambda\mu}t^2+\tfrac{1}{\lambda}+\nu^{-\frac{1}{2}})\\&>\lambda(\tfrac{\mu}{1+\mu}t^2+1+\nu^{-\frac{1}{2}})\\&=\lambda g_1(t,\mu,\nu).\end{aligned}$$

$$\begin{aligned}g_2(t,\lambda\mu,\lambda^{-1}\nu)&=\tfrac{\lambda\mu}{(1+t^2)(1+\lambda\mu)}+2+(\lambda^{-1}\nu)^{-\frac{1}{6}}\\&=\tfrac{\lambda\mu}{(1+t^2)(1+\lambda\mu)}+2+\lambda^{\frac{1}{6}}\nu^{-\frac{1}{6}}\\&>\lambda(\tfrac{\mu}{(1+t^2)(1+\lambda\mu)}+\tfrac{2}{\lambda}+\nu^{-\frac{1}{6}})\\&>\lambda(\tfrac{\mu}{(1+t^2)(1+\mu)}+2+\nu^{-\frac{1}{6}})\\&=\lambda g_2(t,\mu,\nu).\end{aligned}$$

(4) Taking $\delta_0=1$, for each t within the interval from 0 to 1, x,y belonging to $[0,+\infty)$, one has

$$\begin{aligned}f_1(t,\mu,\nu)&=2+\mu^{\frac{1}{3}}+\nu^{-\frac{1}{2}}\\&\geq\tfrac{\mu}{1+\mu}t^2+1+\nu^{-\frac{1}{2}}=g_1(t,\mu,\nu).\end{aligned}$$

$$\begin{aligned} f_2(t,\mu,\nu) \quad &= 4 + sint + \mu^{\frac{1}{4}} + \nu^{-\frac{1}{6}} \\ &\geq 3 + \mu^{\frac{1}{4}} + \nu^{-\frac{1}{6}} \\ &\geq \frac{\mu}{(1+t^2)(1+\mu)} + 2 + \nu^{-\frac{1}{6}} \\ &= g_2(t,\mu,\nu). \end{aligned}$$

It follows that all necessary assumptions of Theorem 3.1 have been fulfilled. Thus, (21)(22) possesses a unique fixed point $\mu^* \in P_h$, at this $h(t) = t^{\beta-1},\ t \in [0,1]$.

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