Stochastic Dynamics for the Viral Diffusion of Products in Digital Age

Hongjie Fan, Mingqi He, Wenqiao Qu and Wenbin Hao

Abstract—**Great changes have taken place in the speed and mode of information dissemination in the digital age. In this paper, combined Word of mouth (WOM) and Internet word of mouth (IWOM), we further propose a stochastic SI_1I_2AD transmission model. WOM and IWOM are two channel to influence the consumers' decisions. We investigate the existence and uniqueness of the solution to the stochastic differential equations model. Then the basic reproduction numbers are obtained, and the dynamical behaviors of the individuals are investigated. In particular, we obtain the sufficient conditions of two types of information dissemination exist alone and together, respectively. Theoretical results show that greater environmental uncertainty poses a risk to the dissemination of product information, resulting in the disappearance of the disseminating population, and that lower noise intensity favours the continued presence of the product in the market. Moreover, only when ξ, the rate at which people from WOM to IWOM, remains low, the two modes of information diffusion could co-exist.**

Index Terms—**Information dissemination, Internet word of mouth, White noise, Dynamic analysis.**

I. Introduction

THE dual development pattern, that is, the domestic great cycle as the main body, and the domestic and international double cycle, has been proposed in China in 2020. This is to drive the troika of economic development: consumption, investment and exports. Further, the utility brought by products directly affects the behavior of potential consumers from the perspective of the consumer theory of microeconomic, . Timely dissemination of product information, and then matching people's needs is an important way to expand the consumer market. At the same time, the advent of the digital economy era has provided greater opportunities for this.

Word of mouth is a traditional way of product information dissemination, that is, through consumers' own experience to further share with people around them. Internet word of mouth is a information dissemination manner in the new era, such as e-commerce online delivery. In particular, the international import and export of digital products is more about sharing product information and looking for potential consumers through online information platforms. Product information dissemination will have a significant impact on consumers' attitudes and final decisions.

In 1969 the first mathematical model of new products was proposed by Bass [1]. After that, mathematical models play an important role in the study about the production dissemination of WOM, and this topic attracts the attention of more and more scholars, [2]–[4]. Fibich considered the Bass model and SIR model, then combined both to describe the diffusion of the new products [5], [6]. Rodrigues and Fonseca think that the message is viral through person-to-person transmission and proposed the SIR model with the standard incidence rate [7]. Jiang et al considered a S_1S_2IR epidemic model to investigate the behaviors of consumers based on the stability theory about Ordinary Differential Equations [8]. In addition, some new useful models were derived and proposed, such as $SEIR$, which are analysed by using the Hamiltonian function [9]. [10] proposed a new $SIALS$ epidemiological model to describe the characteristics of the information dissemination for the products.

As the rapid development of the digital technology, many shopping platforms and communication platforms have been used more frequently, such as JD.com and Twitter. The emergence of IWOM communication has, to a certain extent, got rid of the shackles of traditional WOM communication and broadened the communication mode of product market [11], [12]. In 2023, Qiao and Hu [13] consider the $SIAD$ (Susceptible-Infected1-Infected2-Agreeable-Disagreeable) model with IWOM. Moreover, there are many uncertain factors to influence the dynamical behaviors of the system in the real world, such as climate environment, personal emotion, and so on [14]–[17]. And the stochastic system has been widely concerned and studied [18]–[20]. Based on the above work and stochastic differential equations, we further propose and investigate the viral product diffusion model with the white noise, which is defined as follows:

$$
\begin{aligned}
dS(t) =&[\Lambda - \alpha S(t)I_1(t) - \beta S(t)I_2(t) - \mu S(t)]dt \\
&+ \sigma_1 S(t)dB_1(t), \\
dI_1(t) =&[\alpha S(t)I_1(t) + \xi I_1(t)I_2(t) - (\mu + \delta_1 + \delta_2)I_1(t)]dt \\
&+ \sigma_2 I_1(t)dB_2(t), \\
dI_2(t) =&[\beta S(t)I_2(t) - \xi I_1(t)I_2(t) - (\mu + \epsilon_1 + \epsilon_2)I_2(t)]dt \\
&+ \sigma_3 I_2(t)dB_3(t), \\
dA(t) =&[\delta_1 I_1(t) + \epsilon_1 I_2(t) - \mu A(t)]dt + \sigma_4 A(t)dB_4(t), \\
dD(t) =&[\delta_2 I_2(t) + \epsilon_2 I_2(t) - \mu D(t)]dt + \sigma_5 D(t)dB_5(t),
\end{aligned} \tag{1}
$$

where S is the susceptible class of individuals tending to purchase the products, I_1 is the infected 1 class of individuals

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that have purchased the product and spread the relevant information through IWOM, I_2 is the infected 2 classe that have purchased the product and spread the information through WOM, A is the agreeable class of individuals that is agreeable with the shopping, D is the disagreeable class of individuals that is disagreeable with the shopping, $B_i(t), i = 1,2,3,4,5$ are the independent standard Brownian motions, and $\sigma_i(t), i = 1,2,3,4,5$ are the intensity of the white nioses. Λ is the number of immigrants in the social consumer system. α and β are the transmission rate of S contacting I_1 and I_2, respectively. μ is the removal rate of the classes. δ_1 and ϵ_1 are the propotion of WOM information spreading individuals and IWOM information spreading individuals to be agreeable with the shopping production information, respectively. δ_2 and ϵ_2 are the propotion of WOM information spreading individuals and IWOM information spreading individuals to be disagreeable with the shopping production information, respectively.

In this paper, we focus on the dynamic behaviors of Infected 1 who spread the information of products by WOM and Infected 2 who spread by IWOM. And we improve the model of [13] from the following:

• As more and more consumers are exposed to digital platforms, we use bilinear incidence rate $\xi I_1(t)I_2(t)$ to present the nonlinear phenomenon, which is more suitable and realistic.

• The complex environmental noises are presented by the Brownian motion and studied by stochastic differntial equations.

It is organized as following. Section 2 is devoted to the existence and uniqueness of the solution to stochastic model (1). Dynamic behaviors are discussed and the basic reproduction number is obtained in Section 3, where the extinction, persistence in mean of I_1 or I_2, and persistence in mean of co-infections are analysed and investigated. Finally, the conclusion is presented in Section 4.

In this paper, we define $\mathbb{R}_+^n = \{(a_1, ...a_n) \in \mathbb{R}^n : a_1 > 0, a_i > 0, i = 2,3,...,n\}$, and denote $\varkappa_1 \vee \varkappa_2 = \max\{\varkappa_1, \varkappa_2\}$ and $\varkappa_1 \wedge \varkappa_2 = \min\{\varkappa_1, \varkappa_1\}$. For the function $F(t)$ is integrable on $[0, +\infty)$, we define

$$\frac{1}{t}\int_0^t F(u)du = \langle F(t)\rangle, \text{ for } t \geq 0.$$

II. Existence and uniqueness of the globally positive solution to stochastic model

In this section, we discuss the existence and uniqueness of the positive solution to model (1) by constructing the suitable Lyapunov function.

Lemma 1. [17] Let $\widetilde{N}(t) = S(t) + I_1(t) + I_2(t) + A(t) + D(t)$, then we can obtain the positively invariant set

$$\Gamma = \Big\{(S(t), I_1(t), I_2(t), A(t), D(t)) \in \mathbb{R}_+^5 : \\ S(t) + I_1(t) + I_2(t) + A(t) + D(t) \leq \frac{A}{\mu}\Big\}. \tag{2}$$

Theorem 1. For any initial value $(S(0), I_1(0), I_2(0), A(0), D(0)) \in \mathbb{R}_+^5$, there is a unique positive solution $(S(t), I_1(t), I_2(t), A(t), D(t))$ of model (1) for $t \geq 0$, which will remain in $\mathbb{R}_+^5$ with probability one.

Proof. Firstly, we can see that there is a unique local solution on $[0, \tau_e]$, because coefficients of model are locally Lipschitz continuous on $\mathbb{R}_+$. Then we only need to proof $\tau_e = +\infty$ for the global existence of the solution. In general, the proofs for this part are similar, the difference is only in constructing the appropriate Lyapunov equation.

Define a function $\Upsilon : \mathbb{R}_+^5 \to \mathbb{R}_+$ by

$$\Upsilon(S, I_1, I_2, A, D) = S - 1 - \ln S + I_1 - 1 - \ln I_1 \\ + I_2 - 1 - \ln I_2 + A - 1 - \ln A + D - 1 - \ln D.$$

Since $(u - 1 - \ln u) \geq 0$ for all $u > 0$, then the function $\Upsilon(S, I_1, I_2, A, D)$ is nonnegative.

By Itô's lemma, we could obtain

$$\begin{aligned} d\Upsilon \leq &\Bigg(\Lambda - \mu N + 5\mu + \delta_1 + \delta_2 + \epsilon_1 + \epsilon_2 + \frac{A}{\mu}(\alpha \vee \beta) \\ &+ \frac{1}{2}\sum_{i=1}^{5}\sigma_i^2\Bigg)dt + \left(1 - \frac{1}{S}\right)\sigma_1 S dB_1(t) \\ &+ \left(1 - \frac{1}{I_1}\right)\sigma_2 I_1 dB_2(t) + \left(1 - \frac{1}{I_2}\right)\sigma_3 I_2 dB_3(t) \\ &+ \left(1 - \frac{1}{A}\right)\sigma_4 A dB_4(t) + \left(1 - \frac{1}{D}\right)\sigma_5 D dB_5(t) \\ :=& K dt + \left(1 - \frac{1}{S}\right)\sigma_1 S dB_1(t) \\ &+ \left(1 - \frac{1}{I_1}\right)\sigma_2 I_1 dB_2(t) + \left(1 - \frac{1}{I_2}\right)\sigma_3 I_2 dB_3(t) \\ &+ \left(1 - \frac{1}{A}\right)\sigma_4 A dB_4(t) + \left(1 - \frac{1}{D}\right)\sigma_5 D dB_5(t). \end{aligned}$$

The remaining proof can be referred to Reference [17].

III. Dynamic analysis

In this section, we discuss the sufficient conditions for the extinction and persistence in mean of $I_1(t)$ and $I_2(t)$ in the stochastic model (1). Define the basic reproduction numbers of Infect 1 and Infect 2 as following

$$\mathcal{R}_s^* = \frac{\beta\Lambda}{\mu\left(\mu + \epsilon_1 + \epsilon_2 + \frac{1}{2}\sigma_3^2\right)}, \mathcal{R}_0^* = \frac{\alpha\Lambda}{\mu\left(\mu + \delta_1 + \delta_2 + \frac{1}{2}\sigma_2^2\right)}.$$

For the sake of simplify, let $\Psi(t)$ denote $\frac{S(t)-S(0)}{t} + \frac{I_1(t)-I_1(0)}{t} + \frac{I_2(t)-I_2(0)}{t}$.

A. Extinction

Theorem 2. For the unique positive solution $(S(t), I_1(t), I_2(t), A(t), D(t))$ of model (1) for $t \geq 0$,

(a). if $\mathcal{R}_s^* < 1$, the infected $I_2(t)$ will be extinct, and $\lim_{t\to+\infty}\langle I_2(t)\rangle = 0$;

(b). if $\mathcal{R}_0^* < 1$, the infected $I_1(t)$ will be extinct, and $\lim_{t\to+\infty}\langle I_1(t)\rangle = 0$.

Proof. Integrating both sides of the first three equations form 0 to t and dividing t

$$\begin{aligned} \Psi(t) =& \frac{S(t) - S(0)}{t} + \frac{I_1(t) - I_1(0)}{t} + \frac{I_2(t) - I_2(0)}{t} \\ =& \Lambda - \frac{1}{t}\int_0^t \mu S(v)dv - \frac{1}{t}\int_0^t (\mu + \delta_1 + \delta_2)I_1(v)dv \\ &- \frac{1}{t}\int_0^t (\mu + \epsilon_1 + \epsilon_2)I_2(v)dv + \frac{1}{t}\int_0^t \sigma_1 S(v)dB_1(v) \\ &+ \frac{1}{t}\int_0^t \sigma_2 I_1(v)dB_2(v) + \frac{1}{t}\int_0^t \sigma_3 I_2(v)dB_3(v). \end{aligned}$$

Letting

$$M(t) = \sigma_1 \int_0^t S(v)dB_1(v) + \sigma_2 \int_0^t I_1(v)dB_2(v) + \sigma_3 \int_0^t I_2(v)dB_3(v),$$

then we have

$$\begin{aligned}\Psi(t) =& \Lambda - \frac{1}{t}\int_0^t \mu S(v)dv - \frac{1}{t}\int_0^t (\mu+\delta_1+\delta_2)I_1(v)dv \\ &- \frac{1}{t}\int_0^t (\mu+\epsilon_1+\epsilon_2)I_2(v)dv + \frac{M(t)}{t} \\ =& \Lambda - \mu\langle S(t)\rangle - (\mu+\delta_1+\delta_2)\langle I_1(t)\rangle \\ &- (\mu+\epsilon_1+\epsilon_2)\langle I_2(t)\rangle + \frac{M(t)}{t}.\end{aligned}$$

In further,

$$\begin{aligned}\langle S(t)\rangle =& \frac{\Lambda}{\mu} - \frac{\mu+\delta_1+\delta_2}{\mu}\langle I_1(t)\rangle \\ &- \frac{\mu+\epsilon_1+\epsilon_2}{\mu}\langle I_2(t)\rangle + \frac{M(t)}{\mu t} - \frac{\Psi(t)}{\mu}\end{aligned} \tag{3}$$

By It$\hat{o}$'s formula,

$$\begin{aligned}d\ln I_2(t) =& \frac{1}{I_2(t)}\left[\beta S(t)I_2(t) - \xi I_1(t)I_2(t)\right. \\ &\left.-(\mu+\epsilon_1+\epsilon_2)I_2(t) + \sigma_3 I_2(t)dB_3(t)\right]dt - \frac{1}{2}\sigma_3^2 dt \\ =& \beta S(t) - \xi I_1(t) - \left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right) \\ &+ \sigma_3 dB_3(t).\end{aligned} \tag{4}$$

Integrating equation (4) and dividing by t, we have

$$\begin{aligned}&\frac{\ln I_2(t)}{t} - \frac{\ln I_2(0)}{t} \\ =& \beta\frac{1}{t}\int_0^t S(v)dv - \xi\frac{1}{t}\int_0^t I_1(v)dv \\ &- \left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right) + \frac{1}{t}\int_0^t \sigma_3 dB_3(v) \\ \le& \beta\langle S(t)\rangle - \left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right) + \frac{1}{t}\int_0^t \sigma_3 dB_3(v).\end{aligned}$$

Submitting equation (3) into the above equation,

$$\begin{aligned}&\frac{\ln I_2(t)}{t} - \frac{\ln I_2(0)}{t} \\ \le& \frac{\beta\Lambda}{\mu} - \frac{\beta(\mu+\delta_1+\delta_2)}{\mu}\langle I_1(t)\rangle \\ &- \frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu}\langle I_2(t)\rangle + \frac{\beta M(t)}{\mu t} - \beta\frac{\Psi(t)}{\mu} \\ &- \left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right) + \frac{1}{t}\int_0^t \sigma_3 dB_3(v)\end{aligned} \tag{5}$$

Letting $t \to +\infty$,

$$\begin{aligned}&\frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu}\lim_{t\to+\infty}\langle I_2(t)\rangle \\ \le& \lim_{t\to+\infty}\frac{\ln I_2(0)}{t} - \lim_{t\to+\infty}\frac{\ln I_2(t)}{t} + \frac{\beta\Lambda}{\mu} - \lim_{t\to+\infty}\beta\frac{\Psi(t)}{\mu} \\ &- \frac{\beta(\mu+\delta_1+\delta_2)}{\mu}\lim_{t\to+\infty}\langle I_1(t)\rangle + \lim_{t\to+\infty}\frac{\beta M(t)}{\mu t} \\ &- \left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right) + \lim_{t\to+\infty}\frac{1}{t}\int_0^t \sigma_3 dB_3(v)\end{aligned}$$

By the strong law of the large numbers for the continuous local martingale and the bounded of the individuals, we can have

$$\begin{aligned}&\lim_{t\to+\infty}\frac{\ln I_2(0)}{t} - \lim_{t\to+\infty}\frac{\ln I_2(t)}{t} + \lim_{t\to+\infty}\frac{\beta M(t)}{\mu t} \\ &- \lim_{t\to+\infty}\beta\frac{\Psi(t)}{\mu} + \lim_{t\to+\infty}\frac{1}{t}\int_0^t \sigma_3 dB_3(v) = 0 \ a.s..\end{aligned} \tag{6}$$

Therefore

$$\begin{aligned}&\lim_{t\to+\infty}\langle I_2(t)\rangle \\ \le& \frac{\mu}{\beta(\mu+\epsilon_1+\epsilon_2)} \\ \le& \frac{\mu}{\beta(\mu+\epsilon_1+\epsilon_2)}\left[\frac{\beta\Lambda}{\mu} - \left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right)\right]\end{aligned}$$

This implies $I_2(t)$ will be extinct eventually when $\mathcal{R}_s^* < 1$.

By It$\hat{o}$'s formula,

$$\begin{aligned}&d\ln I_1(t) \\ =& \frac{1}{I_1(t)}\left[\alpha S(t)I_1(t) + \xi I_1(t)I_2(t)\right. \\ &\left.-(\mu+\delta_1+\delta_2)I_1(t) + \sigma_2 I_1(t)dB_2(t)\right]dt - \frac{1}{2}\sigma_2^2 dt \\ =& \alpha S(t) + \xi I_2(t) - \left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) + \sigma_2 dB_2(t).\end{aligned} \tag{7}$$

Integrating equation (7) and dividing by t, we have

$$\begin{aligned}&\frac{\ln I_1(t)}{t} - \frac{\ln I_1(0)}{t} \\ =& \alpha\frac{1}{t}\int_0^t S(v)dv + \xi\frac{1}{t}\int_0^t I_2(v)dv \\ &- \left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) + \frac{1}{t}\int_0^t \sigma_2 dB_2(v) \\ \le& \alpha\langle S(t)\rangle + \xi\langle I_2(t)\rangle \\ &- \left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) + \frac{1}{t}\int_0^t \sigma_2 dB_2(v).\end{aligned}$$

Submitting the equation (3) into the above equation,

$$\begin{aligned}&\frac{\ln I_1(t)}{t} - \frac{\ln I_1(0)}{t} \\ =& \frac{\alpha\Lambda}{\mu} - \frac{\alpha(\mu+\delta_1+\delta_2)}{\mu}\langle I_1(t)\rangle \\ &- \frac{\alpha(\mu+\epsilon_1+\epsilon_2)}{\mu}\langle I_2(t)\rangle + \frac{\alpha M(t)}{\mu t} - \alpha\frac{\Psi(t)}{\mu} \\ &+ \xi\langle I_2(t)\rangle - \left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) + \frac{1}{t}\int_0^t \sigma_2 dB_2(v).\end{aligned}$$

Be similar with the equation (6), we kown

$$\begin{aligned}&\frac{\ln I_1(t)}{t} - \frac{\ln I_1(0)}{t} \\ =& \frac{\alpha\Lambda}{\mu} - \left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) - \frac{\alpha(\mu+\delta_1+\delta_2)}{\mu}\langle I_1(t)\rangle \\ &- \left(\frac{\alpha(\mu+\epsilon_1+\epsilon_2)}{\mu} - \xi\right)\langle I_2(t)\rangle.\end{aligned} \tag{8}$$

If $\alpha(\mu+\epsilon_1+\epsilon_2) \geq \mu\xi$,

$$\begin{aligned}&\lim_{t\to+\infty}\langle I_1(t)\rangle \leq \\ &\frac{\mu}{\alpha(\delta_1+\mu+\delta_2)}\left[\frac{\alpha\Lambda}{\mu}-\left(\delta_1+\mu+\delta_2+\frac{1}{2}\sigma_2^2\right)\right] \\ &\leq \frac{\mu}{\alpha(\delta_1\mu++\delta_2)\left(\delta_1+\mu+\delta_2+\frac{1}{2}\sigma_2^2\right)}(\mathcal{R}_0^*-1),\end{aligned}$$

that implies $I_1(t)$ will extinct eventually when $\mathcal{R}_0^*<1$. ■

Remark 1. From the formula of $\mathcal{R}_0^*$ and $\mathcal{R}_s^*$, we can find that the environmental factor will make effects on the dynamical behaviors of the system, and the individuals who spread the information will disappear if the intensity of the environmental noise is large enough.

B. Persistence in mean of $I_1(t)$ or $I_2(t)$

In this subsection, the sufficient conditions for the persistence of $I_1(t)$ or $I_2(t)$ of the stochastic SI_1I_2AD model (1) are investigated.

Theorem 3. For the unique positive solution $(S(t), I_1(t), I_2(t), A(t), D(t))$ of model (1) for $t \geq 0$, we can obtain the following.

(**A1**). (a). $\mathcal{R}_0^*<1$ and $\alpha(\mu+\epsilon_1+\epsilon_2)<\mu\xi$;
(b). $\mathcal{R}_s^*>1$ and $\mathcal{R}_0^*<1$.
If conditions (a) is satisfied, then $I_2(t)$ will persist in mean with

$$\begin{aligned}&\liminf_{t\to+\infty}\langle I_2(t)\rangle \geq \\ &\frac{\mu}{[\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi]\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)}(\mathcal{R}_0^*-1),\end{aligned}$$

If conditions (b) is satisfied, then $I_2(t)$ will persist in mean with

$$\begin{aligned}&\liminf_{t\to+\infty}\langle I_2(t)\rangle \geq \\ &\frac{\mu}{\beta(\mu+\epsilon_1+\epsilon_2)\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right)}(\mathcal{R}_s^*-1)\end{aligned}$$

(**A2**). (c). $\mathcal{R}_0^*>1$ and $\alpha(\mu+\epsilon_1+\epsilon_2)<\mu\xi$;
(d). $\mathcal{R}_0^*>1$ and $\mathcal{R}_s^*<1$.
If one of the conditions (c) and (d) is satisfied, then $I_1(t)$ will persist in mean with

$$\begin{aligned}&\liminf_{t\to+\infty}\langle I_1(t)\rangle \geq \\ &\frac{\mu}{\alpha(\mu+\delta_1+\delta_2)\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)}(\mathcal{R}_0^*-1)\end{aligned}$$

Proof. Case (a). From equation (6), let $t\to+\infty$ and we can have

$$\begin{aligned}&\frac{\alpha(\mu+\delta_1+\delta_2)}{\mu}\lim_{t\to+\infty}\langle I_1(t)\rangle = \\ &\frac{\alpha\Lambda}{\mu}-(\mu+\delta_1+\delta_2 \\ &+\frac{1}{2}\sigma_2^2\Big)-\frac{\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi}{\mu}\lim_{t\to+\infty}\langle I_2(t)\rangle.\end{aligned} \tag{9}$$

If $\alpha(\mu+\epsilon_1+\epsilon_2)<\mu\xi$,

$$\begin{aligned}&\lim_{t\to+\infty}\langle I_2(t)\rangle = \\ &\frac{\mu}{\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi}\left[\frac{\alpha\Lambda}{\mu}-\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)\right] \\ &-\frac{\mu}{\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi}\frac{\alpha(\mu+\delta_1+\delta_2)}{\mu}\lim_{t\to+\infty}\langle I_1(t)\rangle \\ &\geq \frac{\mu}{\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi}\left[\frac{\alpha\Lambda}{\mu}-\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)\right] \\ &\geq \frac{\mu}{[\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi]\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)}(\mathcal{R}_0^*-1),\end{aligned}$$

which implies that $I_2(t)$ will persist in mean, when $\mathcal{R}_0^*<1$.

Case (b). We have

$$\begin{aligned}d\ln I_2(t) = \beta S(t)-\xi I_1(t)-\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right) \\ +\sigma_3 dB_3(t)\end{aligned} \tag{10}$$

by (4), which describes Itô's formula of the function $\ln I_2(t)$. Since $\mathcal{R}_0^*<1$, the $I_1(t)$ will extinct. From Theorem 3 we can have

$$\lim_{t\to+\infty} I_1(t) = 0 \ a.s..$$

So for all ε_1 and t large, $0 \leq I_1(t) < \varepsilon_1$. Substituting equation (3) into equation(10),

$$\begin{aligned}&\frac{\ln I_2(t)}{t}-\frac{\ln I_2(0)}{t} \geq \frac{\beta\Lambda}{\mu}-\frac{\beta(\delta_1+\mu+\delta_2)+\mu\xi}{\mu}\varepsilon_1 \\ &-\frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu}\langle I_2(t)\rangle+\frac{\beta M(t)}{\mu t}-\beta\frac{\Psi(t)}{\mu} \\ &-\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right)+\frac{1}{t}\int_0^t \sigma_3 dB_3(v).\end{aligned} \tag{11}$$

Then let $t\to+\infty$, by the strong law of the large numbers for the continuous local martingale and using equation (6),

$$\begin{aligned}&\frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu}\liminf_{t\to+\infty}\langle I_2(t)\rangle \\ &\geq\frac{\beta\Lambda}{\mu}-\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right)-\frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu}\varepsilon_1.\end{aligned} \tag{12}$$

For ε_1 small enough,

$$\begin{aligned}&\liminf_{t\to+\infty}\langle I_2(t)\rangle \geq \\ &\frac{\mu}{\beta(\mu+\epsilon_1+\epsilon_2)\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_3^2\right)}(\mathcal{R}_s^*-1).\end{aligned}$$

That implies that $I_1(t)$ will persist in mean, when $\mathcal{R}_s^*>1$ and $\mathcal{R}_0^*<1$.

Case (c). Based on equation (9), we can also have

$$\begin{aligned}&\liminf_{t\to+\infty}\langle I_1(t)\rangle \geq \\ &\frac{\mu}{\alpha(\delta_1+\mu+\delta_2)}\left[\frac{\alpha\Lambda}{\mu}-\left(\delta_1+\mu+\delta_2+\frac{1}{2}\sigma_2^2\right)\right] \\ &\geq \frac{\mu}{\alpha(\delta_1+\mu+\delta_2)\left(\delta_1+\mu+\delta_2+\frac{1}{2}\sigma_2^2\right)}(\mathcal{R}_0^*-1),\end{aligned}$$

which implies that $I_1(t)$ will persist in mean, when $\mathcal{R}_0^*>1$.

Case (d). From equation (7), we have that

$$\begin{aligned}d\ln I_1(t) =&\alpha S(t)+\xi I_2(t)-\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) \\ &+\sigma_2 dB_2(t).\end{aligned} \tag{13}$$

Since $\mathcal{R}_s^* < 1$, $I_2(t)$ will extinct. From Theorem 3 we can have

$$\lim_{t\to+\infty}\langle I_2(t)\rangle = 0 \ a.s..$$

So for all ε_2 and t large, $0 \le I_2(t) < \varepsilon_2$. Under the condition of $\alpha(\mu+\epsilon_1+\epsilon_2) > \mu\xi$, substituting equation (3) into equation (10),

$$\begin{aligned}
&\frac{\ln I_1(t)}{t} - \frac{\ln I_1(0)}{t}\\
\ge&\frac{\alpha\Lambda}{\mu} - \frac{\alpha(\mu+\delta_1+\delta_2)}{\mu}\langle I_1(t)\rangle\\
&-\frac{\alpha(\mu+\epsilon_1+\epsilon_2)-\mu\xi}{\mu}\varepsilon_2 + \frac{\alpha M(t)}{\mu t} - \alpha\frac{\Psi(t)}{\mu}\\
&-\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right) + \frac{1}{t}\int_0^t \sigma_2 dB_2(v).
\end{aligned}$$

Be Similar with equation (12), and let ε_2 small enough,

$$\begin{aligned}
&\liminf_{t\to+\infty}\langle I_1(t)\rangle \ge\\
&\frac{\mu}{\alpha(\mu+\delta_1+\delta_2)\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)}(\mathcal{R}_0^*-1),
\end{aligned}$$

Under the condition of $\alpha(\mu+\delta_1+\delta_2) > \mu\xi$, $I_2(t)$ will persist in mean, when $\mathcal{R}_s^* < 1$ and $\mathcal{R}_0^* > 1$. ■

Remark 2. From the above theorem, we can see that the value of transport rate ξ can have an important impact on the dynamics of the information dissemination. This implies that if the transport rate ξ from I_2 to I_1 is small enough, the infected 2 can be persist to spread the information of the production, although the infected 1 are extinct.

Remark 3. By the theoretical proof of Theorem 1, Theorem 2 and Theorem 3, we can obtain that $\mathcal{R}_0^*$ and $\mathcal{R}_s^*$ are the basic reproduction numbers of the infected 1 and infected 2, respectively. That also implies, the smaller the intensity of the environmental noise, the better the spread of the information.

C. Persistence in mean of co-infections

Theorem 4. For the unique positive solution $(S(t), I_1(t), I_2(t), A(t), D(t))$ of model (1) for $t \ge 0$, if $\alpha(\mu+\epsilon_1+\epsilon_2) > \mu\xi$, $\mathcal{R}_0^* > 1$ and $\mathcal{R}_s^* > 1$, we can obtain that $I_1(t)$ and $I_2(t)$ will be persist together with

$$\begin{aligned}
\liminf_{t\to+\infty}[\langle I_1(t)\rangle + \langle I_2(t)\rangle] \ge& \frac{\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)(\mathcal{R}_0^*-1)}{(\Lambda_1\vee\Lambda_2)}\\
&+\frac{\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_2^2\right)(\mathcal{R}_s^*-1)}{(\Lambda_1\vee\Lambda_2)}.
\end{aligned}$$

Proof. Let $U(t) = \ln I_1(t) + \ln I_2(t)$. Using Itô's formula, the equations (10) and (13),

$$\begin{aligned}
&dU = \beta S(t) - \xi I_1(t) - (\mu+\epsilon_1+\epsilon_2) - \frac{1}{2}\sigma_3^2\\
&+\alpha S(t) + \xi I_2(t) - (\mu+\delta_1+\delta_2) - \frac{1}{2}\sigma_2^2\\
&+\sigma_2 dB_2(t) + \sigma_3 dB_3(t).
\end{aligned}$$

Then by taking integral of the above equation from 0 to t and dividing to t,

$$\begin{aligned}
&\frac{U(t)-U(0)}{t} =\\
&\beta\langle S(t)\rangle - \xi\langle I_1(t)\rangle - (\mu+\epsilon_1+\epsilon_2) - \frac{1}{2}\sigma_3^2\\
&+\alpha\langle S(t)\rangle + \xi\langle I_2(t)\rangle - (\mu+\delta_1+\delta_2) - \frac{1}{2}\sigma_2^2\\
&+\sigma_2\frac{1}{t}\int_0^t dB_2(v) + \sigma_3\frac{1}{t}\int_0^t dB_3(v)
\end{aligned} \tag{14}$$

Then by putting the value of (3) in (14) we obtained

$$\begin{aligned}
&\frac{U(t)-U(0)}{t} = \frac{\alpha\Lambda}{\mu} + \xi\langle I_2(t)\rangle - (\mu+\delta_1+\delta_2) - \frac{1}{2}\sigma_2^2\\
&-\frac{\alpha(\mu+\delta_1+\delta_2)}{\mu}\langle I_1(t)\rangle - \frac{\alpha(\mu+\epsilon_1+\epsilon_2)}{\mu}\langle I_2(t)\rangle\\
&+\frac{\beta\Lambda}{\mu} - (\mu+\epsilon_1+\epsilon_2) - \frac{1}{2}\sigma_3^2 - \frac{\beta(\mu+\delta_1+\delta_2)}{\mu}\langle I_1(t)\rangle\\
&-\frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu}\langle I_2(t)\rangle - \xi\langle I_1(t)\rangle + (\alpha+\beta)(-\Psi(t))\\
&+(\alpha+\beta)\frac{M(t)}{\mu t} + \sigma_2\frac{1}{t}\int_0^t dB_2(v) + \sigma_3\frac{1}{t}\int_0^t dB_3(v).
\end{aligned}$$

By the strong law of the large numbers for the continuous local martingale and (6),

$$\begin{aligned}
&\frac{U(t)-U(0)}{t} \ge \left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)(\mathcal{R}_0^*-1)\\
&+\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_2^2\right)(\mathcal{R}_s^*-1)\\
&-(\Lambda_1\vee\Lambda_2)[\langle I_1(t)\rangle + \langle I_2(t)\rangle],
\end{aligned}$$

where

$$\begin{aligned}
\Lambda_1 =& \frac{\alpha(\mu+\delta_1+\delta_2)}{\mu} + \frac{\beta(\mu+\delta_1+\delta_2)}{\mu} + \xi,\\
\Lambda_2 =& \frac{\alpha(\mu+\epsilon_1+\epsilon_2)}{\mu} + \frac{\beta(\mu+\epsilon_1+\epsilon_2)}{\mu} - \xi.
\end{aligned}$$

Under the condition $\alpha(\mu+\epsilon_1+\epsilon_2) > \mu\xi$, we can obtain $\Lambda_2 > 0$.

Let $t \to +\infty$,

$$\begin{aligned}
&\liminf_{t\to+\infty}[\langle I_1(t)\rangle + \langle I_2(t)\rangle] \ge\\
&\frac{\left(\mu+\delta_1+\delta_2+\frac{1}{2}\sigma_2^2\right)(\mathcal{R}_0^*-1)}{(\Lambda_1\vee\Lambda_2)}\\
&\frac{+\left(\mu+\epsilon_1+\epsilon_2+\frac{1}{2}\sigma_2^2\right)(\mathcal{R}_s^*-1)}{(\Lambda_1\vee\Lambda_2)}.
\end{aligned}$$

■

Remark 4. Under some conditions, obviously, both of WOM and IWOM can coexist, which is conductive to the dissemination of product information, but also the most beneficial to consumers.

IV. Conclusion

In this paper, we propose and investigated a stochastic model considering Internet word of mouth (IWOM). The existence and uniqueness of the positive solution to model (1) is obtained by constructing suitable Lyapunov function. Then we establish the sufficient conditions for extinction and persistence in mean of the infected 1 and the infected 2, respectively. Stochastic dynamic analysis shows that if the intensity of environmental noise σ_2 and σ_3 are large, it will reduce the number of the infected who spread the product information. This implies that external factors such as market policies and product information changes will bring risks to its marketing. In addition, we also found that when the transport rate ξ is small, it can ensure the coexistence of the two types of information transmission groups, which can affect the consumption decisions of more people.

In future research, we will consider whether there is a time lag in the propagation of product information. If so, what impact will it have on the dynamic behaviors of information dissemination?

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