Analysis of Various Degrees and Sizes in a Fuzzy Semigraph

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***Abstract*—In semigraphs, the role of a node in an edge is classified into two: end nodes and middle nodes. This classification of nodes in a fuzzy semigraph, along with the membership values, contributes to the categorization of the degrees of nodes according to their roles. Four types of degrees—degree, edge degree, adjacent degree, and consecutive adjacent degree of a node, corresponding regularities, and three types of sizes—size, crisp size, and pseudo size are introduced for a fuzzy semigraph. An application that models a sequence of activities and procedures to complete various tasks in a university as a fuzzy multi-semigraph is discussed to justify these new concepts.**

***Index Terms*—degrees, fuzzy multi semigraph, fuzzy semigraph, multi semigraph, order, regularities, sizes.**

I. Introduction and Preliminaries

The first to introduce fuzzy graphs [1] as an advancement of classical set theory was A. Rosenfeld. E. Sampathkumar developed the concept of semigraphs to overcome various flaws and broaden the scope of graph theory.

A. Semigraph

A semigraph [2] is a pair of sets $H^* = (N, E)$, where N is a non-empty set, called the node set and E is a set of r-tuples with distinct elements of N, for various $r \geq 2$, called the edge set satisfying the following:

(1) At most one node can be shared by any two elements in E,
(2) Any two edges $(n_1, n_2, ...n_r)$ and $(m_1, m_2, ...m_s)$ are equal if and only if $r = s$ and either $n_j = m_j$ or $n_j = m_{r-j+1}$ for $1 \leq j \leq r$.

If two edges in a semigraph share the same node, they are called adjacent edges. Comparable to this, two nodes in a semigraph are treated as adjacent if the nodes are positioned on the same edge of the semigraph. Likewise, two nodes are consecutively or sequentially adjacent if the semigraph contains an edge, a sequence of nodes in which the two nodes appear consecutively. In a semigraph, a partial edge is a sub-edge of an edge where the edge's consecutive nodes are also consecutive in the sub-edge. There are different types of semigraphs in which an r-uniform semigraph [3] is one where each edge has cardinality r. E. Sampathkumar in [2] has defined various degrees of a node n in a semigraph as follows:

(1) The number of edges with n as an end node is the degree of n, denoted by $deg(n)$.
(2) The number of edges where n is either an end node or a middle node is known as the edge degree of n, denoted by $deg_e(n)$.
(3) The number of nodes that are adjacent to the node n is the adjacent degree of n, denoted by $deg_a(n)$.
(4) The number of nodes that are consecutively adjacent to the node n is the consecutive adjacent degree of n, denoted by $deg_{ca}(n)$.

Each of these degrees defines different types of regular semigraphs [4]. A semigraph H^* is D_k-regular if every node has degree k. Similarly H^* is ED_k-regular, AD_k-regular, and CAD_k-regular if every node, respectively, has an edge degree, an adjacent degree, and a consecutive adjacent degree of k.

There are three different graphs associated to a semigraph H^*, each having the same node set N, are the end node graph H_e^*, the adjacency graph H_a^*, and the consecutive adjacency graph H_{ca}^*, in which the edges are defined as follows [2]:

(1) Two nodes in H_e^* are adjacent if, and only if, the nodes are the end nodes of an edge in H^*.
(2) Two nodes in H_a^* are adjacent if, and only if, the nodes are adjacent in H^*.
(3) Two nodes in H_{ca}^* are adjacent if, and only if, the nodes are consecutively adjacent in H^*.

B. Fuzzy Semigraph

Fuzzy semigraph [5] is a new concept that K. Radha and P. Renganathan proposed by connecting the ideas of fuzzy graphs and semigraphs. Let $H^* = (N, E)$ be a semigraph. A fuzzy semigraph H defined on H^* is a 4-tuple (N, ρ, ν, φ) in which $\rho : N \to [0, 1]$, $\nu : N \times N \to [0, 1]$, and $\varphi : E \to [0, 1]$ are functions satisfying the following conditions:

(1) $\nu(n_1, n_2) \leq \rho(n_1) \wedge \rho(n_2) \ \forall \ n_1, n_2 \in N$,
(2) $\varphi(e) = \nu(n_1, n_2) \wedge \nu(n_2, n_3) \wedge \cdots \wedge \nu(n_{r-1}, n_r) \leq \rho(n_1) \wedge \rho(n_r)$
if $e = (n_1, n_2, \ldots, n_r), r \geq 2$ is an edge in H^*.

If $e = (n_1, n_2, ..., n_r)$, $r \geq 2$ is an edge and n and m be any nodes in a semigraph with the properties,

$$\varphi(e) = \rho(n_1) \wedge \rho(n_r) \text{ and } \nu(n, m) = \rho(n) \wedge \rho(m)$$

then the fuzzy semigraph is referred to as an effective fuzzy semigraph [5]. As the nodes of ρ-membership value zero are considered not existing and also the function ν is defined from $N \times N$ is such that, a pair of nodes that are not consecutively adjacent in the underlying semigraph H^* must have the ν-membership value zero. Furthermore, in the second condition given by

$$\varphi(e) = \nu(n_1, n_2) \wedge \nu(n_2, n_3) \wedge \cdots \wedge \nu(n_{r-1}, n_r)$$

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$$\leq \rho(n_1) \wedge \rho(n_r),$$

the right side inequality is obvious.

Considering these requirements, the definition of a fuzzy semigraph established in [5] is modified in this paper. This definition suggests four kinds of degrees and three kinds of sizes. The studies of these degrees, sizes, and their relationships are included in this paper. For more information on concepts in fuzzy graph theory, see the references [1], [6]–[9], and refer [10]–[13] for fundamental notions in graphs and semigraphs.

C. Notations

H^*	semigraph
H	fuzzy semigraph
N	node set
E	edge set
deg	degree
deg_e	edge degree
deg_a	adjacent degree
deg_{ca}	consecutive adjacent degree
D_k	k-regular
ED_k	k-edge regular
AD_k	k-adjacent regular
CAD_k	k-consecutive adjacent regular
$\lvert X \rvert$	cardinality of the set X
$\lvert e \rvert$	cardinality/no.of nodes of the edge e
H_e^*	end node graph associated to the semigraph H^*
H_a^*	adjacency graph associated to the semigraph H^*
H_{ca}^*	consecutive adjacency graph associated to the semigraph H^*

II. Main Result

Definition 1: (Revised) Let $H^* = (N, E)$ be a semigraph. Then a fuzzy semigraph $H = (N, \rho, \nu, \varphi)$ defined on H^* consist of a node set N and functions $\rho : N \to [0, 1]$, $\nu : N \times N \to [0, 1]$, and $\varphi : E \to [0, 1]$ that satisfy the conditions for all n and m in N :

(1) $\rho(n) > 0$,
(2) $\nu(n, m) = 0$, if n and m are not consecutively adjacent,
(3) $\nu(n, m) \leq \rho(n) \wedge \rho(m)$,
(4) $\varphi(e) = \nu(n_1, n_2) \wedge \nu(n_2, n_3) \wedge \cdots \wedge \nu(n_{r-1}, n_r)$ if $e = (n_1, n_2, \ldots, n_r)$ is an edge in H^*, $r \geq 2$.

The semigraph $H^* = (N, E)$ is called the underlying semigraph of the fuzzy semigraph H.

The membership value of partial edges induced from that of an edge is assigned as follows:

Definition 2: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which $H^* = (N, E)$ be the underlying semigraph. Let Y be the set of all partial edges in H. Then the membership value of each partial edge in H is defined using the function $\kappa : Y \to [0, 1]$ such that

$$\kappa(n_i, n_{i+1}, \ldots, n_{j-1}, n_j) = \nu(n_i, n_{i+1}) \wedge \nu(n_{i+1}, n_{i+2}) \wedge \cdots \wedge \nu(n_{j-1}, n_j)$$

Where $(n_i, n_{i+1}, \ldots, n_{j-1}, n_j)$ be a partial edge of an edge in H.

Definition 3: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph. Suppose n is any node in H. Then various degrees of n are:

(1) Degree of n in H is $\sum \varphi(e)$ where the summation is taken over all the edges e with n as an end node, expressed as $d(n)$.
(2) Edge degree of n in H is $\sum \varphi(x)$ where the summation is taken over all the edges x with the node n, either as an end node or as a middle node, expressed as $d_e(n)$.
(3) Adjacent degree of n in H is $\sum \kappa(p)$ where the summation is taken over all the partial edges p with n as an end node, denoted as $d_a(n)$.
(4) Consecutive adjacent degree of n in H is $\sum \nu(n, m)$ where the summation is taken over all the nodes m which are consecutively adjacent to n in H, that is either (n, m) or (m, n) is a partial edge in H, denoted as $d_{ca}(n)$. Put differently

$$d_{ca}(n) = \sum_{\substack{n \neq m \\ n, m \in N}} \nu(n, m).$$

Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which $H^* = (N, E)$ be the underlying semigraph. Suppose the membership value of each edge in H is one, then the degree, edge degree, adjacent degree, and consecutive adjacent degree of any node n in H coincide with the degree, edge degree, adjacent degree, and consecutive adjacent degree of n in H^* respectively.

Also, these degrees are in a chain relation. For any node n in a fuzzy semigraph H,

$$d(n) \leq d_e(n) \leq d_{ca}(n) \leq d_a(n).$$

Theorem 4: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph whose underlying semigraph is $H^* = (N, E)$. Then

(1) $\displaystyle\sum_{n \in N} d(n) = 2 \sum_{e \in E} \varphi(e)$,
(2) $\displaystyle\sum_{n \in N} d_e(n) = \sum_{e \in E} |e| \varphi(e)$.

Proof: Each edge has two end nodes. So that (1) follows trivially.
Note that the edge degree of a node n in H gives the sum of membership values of the edges in which the node n is either an end node or a middle node. Since the edge degree of end nodes is counted in $\sum d(n)$,

$$\sum_{n \in N} d_e(n) = \sum_{n \in N} d(n) + \sum_{e \in E} (|e| - 2) \varphi(e),$$

where $|e| - 2$ gives the number of middle nodes lying on the edge e. Hence,

$$\sum_{n \in N} d_e(n) = \sum_{n \in N} d(n) + \sum_{e \in E} |e| \varphi(e) - 2 \sum_{e \in E} \varphi(e) = \sum_{e \in E} |e| \varphi(e).$$

■

Theorem 5: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph whose underlying semigraph is $H^* = (N, E)$. Let C be the set of all consecutively adjacent pairs of nodes in H^* and $\nu = c$ is a constant function on C. Then

(1) $\displaystyle\sum_{n \in N} d_a(n) + \sum_{n \in N} d_e(n) = \sum_{e \in E} |e|^2 \varphi(e)$,

(2) $\sum_{n\in N} d_{ca}(n) + \sum_{n\in N} d(n) = 2\sum_{n\in N} d_e(n)$.

Proof: Given $\nu(n,m) = c$, for any pair (n,m) in C, where c is a constant need not be an integer.
For each node n in an edge e, there are $|e|-1$ partial edges with n as an end node. Hence each edge e in H contributes $|e|(|e|-1)c$ to the sum of adjacent degrees of the nodes in H, thus

$$\sum_{n\in N} d_a(n) = \sum_{e\in E} |e|(|e|-1)c = \sum_{e\in E} |e|^2 c - \sum_{e\in E} |e|c.$$

That is,

$$\sum_{n\in N} d_a(n) + \sum_{e\in E} |e|c = \sum_{e\in E} |e|^2 c$$
$$\sum_{n\in N} d_a(n) + \sum_{e\in E} |e|\varphi(e) = \sum_{e\in E} |e|^2 \varphi(e).$$

Then by Theorem 4

$$\sum_{n\in N} d_a(n) + \sum_{n\in N} d_e(n) = \sum_{e\in E} |e|^2 \varphi(e),$$

which proves (1).
On every edge, there is only one node consecutively adjacent to each of the two end nodes and two nodes consecutively adjacent to each middle node, so that an edge e in H contributes $(2|e|-2)c$ to the sum of the consecutive adjacent degree of each node in H. That is

$$\begin{aligned}\sum_{n\in N} d_{ca}(n) &= \sum_{e\in E}(2|e|-2)c \\ &= 2\sum_{e\in E}|e|c - 2\sum_{e\in E} c \\ &= 2\sum_{e\in E}|e|\varphi(e) - 2\sum_{e\in E}\varphi(e)\end{aligned}$$

Then by Theorem 4

$$\sum_{n\in N} d_{ca}(n) = 2\sum_{n\in N} d_e(n) - \sum_{n\in N} d(n)$$
$$\sum_{n\in N} d_{ca}(n) + \sum_{n\in N} d(n) = 2\sum_{n\in N} d_e(n).$$

■

Notice that in counting consecutive adjacent degrees of all the nodes, we count each 2-partial edge exactly twice.

Theorem 6: Let $H = (N,\rho,\nu,\varphi)$ be a fuzzy semigraph. Then

$$\sum_{n\in N} d_{ca}(n) = 2 \times \sum_{\substack{n\neq m\\ n,m\in N}} \nu(n,m).$$

Corollary 7: Let $H = (N,\rho,\nu,\varphi)$ be a fuzzy semigraph whose underlying semigraph $H^* = (N,E)$ is an r-uniform. Let C be the set of all consecutively adjacent pairs of nodes in H^* and the function ν is constant on C with constant value c, need not be an integer. Then

(1) $\sum_{n\in N} d(n) = 2|E|c$,

(2) $\sum_{n\in N} d_e(n) = r|E|c$,

(3) $\sum_{n\in N} d_a(n) = r(r-1)|E|c$,

(4) $\sum_{n\in N} d_{ca}(n) = 2(r-1)|E|c$.

A fuzzy graph is a 2-uniform fuzzy semigraph, in such case Corollary 7 gives

$$\sum d(n) = \sum d_e(n) = \sum d_a(n) = \sum d_{ca}(n)$$

where the summation is taken over all the nodes n in H.

Following the definition of the order of a fuzzy graph, the order of a fuzzy semigraph can be defined as follows:

Definition 8: Let $H = (N,\rho,\nu,\varphi)$ be a fuzzy semigraph whose underlying semigraph H^* is (N,E). Then the order, expressed as $O(H)$ of the fuzzy semigraph H is

$$O(H) = \sum_{n\in N} \rho(n).$$

Also considering the variety of adjacencies, we can define three types of sizes on a fuzzy semigraph H-the crisp size $CS(H)$, the size $S(H)$ and the pseudo size $PS(H)$, as follows:

$$CS(H) = \sum_{e\in E} \varphi(e),$$
$$S(H) = \sum_{\substack{n\neq m\\ n,m\in N}} \nu(n,m) \text{ and}$$
$$PS(H) = \sum_{e\in E} |e|.$$

An immediate bound for $O(H)$ for a fuzzy semigraph $H = (N,\rho,\nu,\varphi)$,

$$|N|\left(\bigwedge_{n\in N}\rho(n)\right) \le O(H) \le |N|\left(\bigvee_{n\in N}\rho(n)\right).$$

Another lower bound for order of H in terms of edge membership value is given in Theorem 9, where the set E_n denotes the collection of all edges e in E where the node n lies on e.

Theorem 9: Let $H = (N,\rho,\nu,\varphi)$ be a fuzzy semigraph whose underlying semigraph is $H^* = (N,E)$. Then

$$O(H) \ge \sum_{n\in N}\bigvee_{e\in E_n}\varphi(e).$$

Proof: Let $e = (n_1, n_2, \ldots, n_r)$ be an edge in the fuzzy semigraph H. Since $\varphi(e) \le \rho(n_1)\wedge\rho(n_2)\wedge\cdots\wedge\rho(n_r)$. Thus $\rho(n_i) \ge \varphi(e)$, for $1 \le i \le r$. So that for any $n \in N$,

$$\rho(n) \ge \bigvee_{e\in E_n}\varphi(e).$$

Hence

$$O(H) = \sum_{n\in N}\rho(n) \ge \sum_{n\in N}\bigvee_{e\in E_n}\varphi(e).$$

■

The following bounds of crisp size are obvious for a fuzzy semigraph $H = (N,\rho,\nu,\varphi)$,

(1) $CS(H) \le S(H)$,

(2) $|E|\left(\bigwedge_{e\in E}\varphi(e)\right) \le CS(H) \le |E|\left(\bigvee_{e\in E}\varphi(e)\right)$.

Theorem 10: Let $H = (N,\rho,\nu,\varphi)$ be a fuzzy semigraph in which $H^* = (N,E)$ is the underlying semigraph. Then

$$CS(H) \le |E|\bigvee_{n\in N}\rho(n)$$

and

$$\bigwedge_{e\in E}\varphi(e)\left(\sum_{e\in E}(|e|-1)\right)\leq S(H)$$

$$\leq\bigvee_{n\in N}\rho(n)\left(\sum_{e\in E}(|e|-1)\right).$$

Proof: Each edge $e=(n_1,n_2,\ldots,n_r)$ in H satisfies the condition

$$\varphi(e)\leq\rho(n_1)\wedge\rho(n_2)\wedge\cdots\wedge\rho(n_r).$$

Thus

$$CS(H)\leq\sum_{e\in E}\bigvee_{n\in N}\rho(n)=|E|\bigvee_{n\in N}\rho(n).$$

Since $\nu(n_1,n_2)\leq\bigvee_{n\in N}\rho(n)$ for any $n_1,n_2\in N$ and there are $|e|-1$ consecutively adjacent pairs of nodes in an edge e, thus

$$S(H)\leq\sum_{e\in E}\bigvee_{n\in N}\rho(n)(|e|-1)$$

$$=\bigvee_{n\in N}\rho(n)\left(\sum_{e\in E}(|e|-1)\right).$$

Also for any nodes $n,m\in N$, $\nu(n,m)\geq\bigwedge_{e\in E}\varphi(e)$; so that

$$S(H)\geq\sum_{e\in E}\bigwedge_{e\in E}\varphi(e)(|e|-1)$$

$$=\bigwedge_{e\in E}\varphi(e)\left(\sum_{e\in E}(|e|-1)\right).$$

■

As a consequence of the above result,

$$CS(H)\leq\sum_{e\in E}\bigwedge_{n\in N_e}\rho(n)$$

where the set N_e is the collection of nodes n in the edge e.

Corollary 11: Let $H=(N,\rho,\nu,\varphi)$ be a fuzzy semigraph whose underlying semigraph is $H^*=(N,E)$. Then

$$\bigwedge_{e\in E}\varphi(e)\left(\sum_{n\in N}deg_e(n)-|E|\right)\leq S(H)$$

$$\leq\bigvee_{n\in N}\rho(n)\left(\sum_{n\in N}deg_e(n)-|E|\right).$$

Proof: The Theorem 10 gives

$$\bigwedge_{e\in E}\varphi(e)\sum_{e\in E}(|e|-1)\leq S(H)$$

$$\leq\bigvee_{n\in N}\rho(n)\sum_{e\in E}(|e|-1).$$

That is

$$\bigwedge_{e\in E}\varphi(e)\left(\sum_{e\in E}|e|-|E|\right)\leq S(H)$$

$$\leq\bigvee_{n\in N}\rho(n)\left(\sum_{e\in E}|e|-|E|\right).$$

The result follows because $\sum_{n\in N}deg_e(n)=\sum_{e\in E}|e|$. ■

Note that $PS(H)=\sum_{e\in E}|e|$. Thus the Corollary 11 can be restated as

$$\bigwedge_{e\in E}\varphi(e)\left(PS(H)-|E|\right)\leq S(H)$$

$$\leq\bigvee_{n\in N}\rho(n)\left(PS(H)-|E|\right),$$

or

$$\bigwedge_{e\in E}\varphi(e)PS(H)\leq S(H)\leq\bigvee_{n\in N}\rho(n)PS(H).$$

Many interesting relations between these parameters have been noticed.

Theorem 12: Let $H=(N,\rho,\nu,\varphi)$ be an effective fuzzy semigraph such that ρ is a constant function, where the underlying semigraph $H^*=(N,E)$. Then

(1) $CS(H)=\frac{|E|}{|N|}O(H)$,
(2) $S(H)=CS(H)\left(\frac{PS(H)}{|E|}-1\right)$,
(3) $S(H)=\frac{O(H)}{|N|}\left(PS(H)-CS(H)\right)$.

Proof: Given that the fuzzy semigraph H is effective and the function ρ is constant. Let $\rho(n)=c$ for each node n in H and c is a constant that need not be an integer. Then the order of H,

$$O(H)=\sum_{n\in N}\rho(n)=|N|c.$$

Then $CS(H)=\sum_{e\in E}\varphi(e)=|E|c=|E|\dfrac{O(H)}{|N|}$.

Now,

$$S(H)=\sum\nu(n,m),$$

where the summation is taken over all the nodes n and m such that $n\neq m$ in H. Thus

$$\begin{aligned}S(H)&=\sum_{e\in E}(|e|-1)c\\&=c\sum_{e\in E}|e|-c\sum_{e\in E}1=\frac{CS(H)}{|E|}\sum_{e\in E}|e|-c|E|\\&=\frac{CS(H)}{|E|}\sum_{e\in E}|e|-CS(H)\\&=\frac{CS(H)}{|E|}PS(H)-CS(H)\\&=CS(H)\left(\frac{PS(H)}{|E|}-1\right).\end{aligned}$$

So that (3) holds. ■

Let $H=(N,\rho,\nu,\varphi)$ be a fuzzy semigraph. Then $\delta(H)$ and $\Delta(H)$ are the minimum degree and maximum degree of nodes respectively in the fuzzy semigraph H. That is

$$\delta(H)=\wedge\{d(n):n\in N\}\text{ and}$$

$$\Delta(H)=\vee\{d(n):n\in N\}.$$

Similarly

$$\delta_{ca}(H)=\wedge\{d_{ca}(n):n\in N\}\text{ and}$$

$$\Delta_{ca}(H)=\vee\{d_{ca}(n):n\in N\}$$

are the minimum consecutive adjacent degree and maximum consecutive adjacent degree of nodes respectively in the fuzzy semigraph H.

Corollary 13: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph. If cardinality of N is p, then

$$0 \leq CS(H) \leq \frac{p(p-1)}{2}\Delta(H), \text{and}$$

$$0 \leq S(H) \leq \frac{p(p-1)}{2}\Delta_{ca}(H).$$

Theorem 14: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph such that $|N| = p$. Then

(1) $\frac{p\delta(H)}{2} \leq CS(H) \leq \frac{p\Delta(H)}{2}$,
(2) $\frac{p\delta_{ca}(H)}{2} \leq S(H) \leq \frac{p\Delta_{ca}(H)}{2}$.

Proof: Note that $CS(H) = \sum_{e \in E} \varphi(e)$.

Also $\delta(H) \leq d(n) \leq \Delta(H)$ for any node n in H. Thus

$$\sum_{n \in N} \delta(H) \leq \sum_{n \in N} d(n) \leq \sum_{n \in N} \Delta(H)$$

$$p\delta(H) \leq 2CS(H) \leq p\Delta(H)$$

$$\frac{p\delta(H)}{2} \leq CS(H) \leq \frac{p\Delta(H)}{2}.$$

Since $\delta_{ca}(H) \leq d_{ca}(n) \leq \Delta_{ca}(H)$ for any node n in H,

$$\sum_{n \in N} \delta_{ca}(H) \leq \sum_{n \in N} d_{ca}(n) \leq \sum_{n \in N} \Delta_{ca}(H)$$

$$p\delta_{ca}(H) \leq 2S(H) \leq p\Delta_{ca}(H)$$

$$\frac{p\delta_{ca}(H)}{2} \leq S(H) \leq \frac{p\Delta_{ca}(H)}{2}.$$

■

The following are some simple observations on the relations of various parameters associated with a fuzzy semigraph.

For a fuzzy semigraph $H = (N, \rho, \nu, \varphi)$

(1) $0 \leq PS(H) \leq |N|(|N| - 1)$.
(2) $\sum_{n \in N} deg_e(n) = PS(H)$.
(3) $PS(H) \geq 2$, if the underlying semigraph of H is not trivial.
(4) $PS(H) = r|E|$, if the underlying semigraph of H is r-uniform.
(5) $\sum_{n \in N} d_e(n) = cPS(H)$, if φ is the constant function c, where c need not be an integer.
(6) If k_1 and k_2 are the minimum and maximum cardinality of an edge in H then $|E|k_1 \leq PS(H) \leq |E|k_2$.
(7) By Theorem 4

$$\sum_{n \in N} d_e(n) \leq \frac{PS(H)}{2} \sum_{n \in N} d(n).$$

(8) By Theorem 5

$$\sum_{n \in N} d_a(n) + \sum_{n \in N} d_e(n) \leq \frac{PS(H)^2}{2} \sum_{n \in N} d(n).$$

A. Regular Fuzzy Semigraph

Various degrees associated with a node in a fuzzy semigraph excogitate various regularity concepts.

Definition 15: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph. Then for a real number k, H is said to be

(1) k-regular if each node has degree k.
(2) k-edge regular if each node has an edge degree k.
(3) k-adjacent regular if each node has adjacent degree k.
(4) k-consecutive adjacent regular if each node has consecutive adjacent degree equal to k.

The degrees and the ρ value of nodes together play important roles in many applications.

Definition 16: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph. The total degree of a node n is thus defined as follows:

(1) The total degree of n, $td(n) = d(n) + \rho(n)$.
(2) The total edge degree of n, $td_e(n) = d_e(n) + \rho(n)$.
(3) The total adjacent degree of n, $td_a(n) = d_a(n) + \rho(n)$.
(4) The total consecutive adjacent degree of n, denoted by $td_{ca}(n) = d_{ca}(n) + \rho(n)$.

These variant degrees naturally lead to other variant regularity concepts.

Definition 17: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph such that $td(n) = k$ for all the nodes n in H and for some constant k. Then H is called a total regular fuzzy semigraph of total degree k or k-totally regular fuzzy semigraph.
k-totally edge regular, k-totally adjacent regular, and k-totally consecutive adjacent regular fuzzy semigraphs are defined similarly.

The aforementioned definitions make it abundantly evident that the degree of regularity need not be an integer and that no general relationship exists between regular and totally regular fuzzy semigraphs of any kind.

The following theorems relate the concepts of various degrees in fuzzy semigraph with those in the underlying semigraph in certain situations.

Theorem 18: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph such that the function $\rho : N \to [0, 1]$ is constant. Then for each n in N

(1) $d(n) \leq \rho(n)deg(n)$
(2) $d_e(n) \leq \rho(n)deg_e(n)$
(3) $d_a(n) \leq \rho(n)deg_a(n)$
(4) $d_{ca}(n) \leq \rho(n)deg_{ca}(n)$

The inequality in the Theorem 18 becomes an equality if the fuzzy semigraph is effective.

Corollary 19: Let $H = (N, \rho, \nu, \varphi)$ be an effective fuzzy semigraph such that ρ is a constant function. Then for each n in N,

(1) $d(n) = \rho(n)deg(n)$
(2) $d_e(n) = \rho(n)deg_e(n)$
(3) $d_a(n) = \rho(n)deg_a(n)$
(4) $d_{ca}(n) = \rho(n)deg_{ca}(n)$

Corollary 20: Let $H = (N, \rho, \nu, \varphi)$ be an effective fuzzy semigraph.

(1) Suppose the underlying semigraph of H, in which ρ is constant, is D_k-regular, then H is $k\rho(n)$-regular fuzzy semigraph and $\rho(n)(k+1)$-totally regular fuzzy semigraph.
(2) Suppose the underlying semigraph of H, in which ρ is constant, is ED_k-regular, then H is $k\rho(n)$-edge regular

fuzzy semigraph and $\rho(n)(k+1)$-totally edge regular fuzzy semigraph.

(3) Suppose the underlying semigraph of H, in which ρ is constant, is AD_k-regular, then H is $k\rho(n)$-regular fuzzy semigraph and $\rho(n)(k+1)$-totally adjacent regular fuzzy semigraph.

(4) Suppose the underlying semigraph of H, in which ρ is constant, is CAD_k-regular, then H is $k\rho(n)$-consecutive adjacent regular fuzzy semigraph and $\rho(n)(k+1)$-totally consecutive adjacent regular fuzzy semigraph.

Theorem 21: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph. If ρ is a constant function then the following are equivalent:

(1) H is regular fuzzy semigraph,

(2) H is totally regular fuzzy semigraph.

Conversely if (1) and (2) are equivalent then ρ is a constant function.

Proof: Assume that the two criteria are equivalent and H is a k_1-regular and a k_2-totally regular fuzzy semigraph. If there exist at least one pair of nodes n and m in H such that $\rho(n) \neq \rho(m)$, then $d(n) = k_1 = d(m)$ and $td(n) = k_1 + \rho(n) = td(m) = k_1 + \rho(m)$. But then $\rho(n) = \rho(m)$, a contradiction.

Conversely, Assume that ρ is a constant function say, $\rho(n) = c$, where c need not be an integer, for each node n in H. Then $d(n) = k$ for all n in N if, and only if, $td(n) = k + c$ for all n in N, which proves the result. ■

A similar feature applies to fuzzy semigraphs which are edge, adjacent, or consecutive adjacent regular.

Theorem 22: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph. Then $\rho : N \to [0, 1]$ is a constant function if the fuzzy semigraph H is both regular and totally regular fuzzy semigraph.

Proof: Let H be a k_1-regular and k_2-totally regular fuzzy semigraph, for some constants k_1 and k_2. That is $d(n) = k_1$ and $td(n) = k_2$ for any nodes n in H. Then for each n in N,

$$d(n) + \rho(n) = k_2$$

$$k_1 + \rho(n) = k_2$$

$$\rho(n) = k_2 - k_1$$

This shows ρ is a constant function. ■

The fuzzy semigraphs which are edge regular, adjacent regular, or consecutive adjacent regular equally display the aforementioned feature.

The converse of the Theorem 22 might not be accurate. Take into consideration the fuzzy semigraph $H = (N, \rho, \nu, \varphi)$ shown in Fig. 1. Here $d(n_1) = 0.7$, $d(n_3) = 0.6$, $d_e(n_1) = 0.7$, $d_e(n_3) = 0.6$, $d_a(n_1) = 1$, $d_a(n_2) = 0.5$, $d_{ca}(n_1) = 0.8$, and $d_{ca}(n_2) = 0.5$. Note that ρ, the membership value of nodes is a constant function, $\rho(n_i) = 0.6$ for $1 \leq i \leq 4$, but H is neither regular nor totally regular fuzzy semigraph of any kind in this instance.

While considering cycles, when φ is a constant function we noticed some regularity among middle nodes and end nodes separately and is termed as me-regular.

Definition 23: A fuzzy semigraph H is said to be a me-regular fuzzy semigraph if all the middle nodes have the same degree and all the end nodes have the same degree.

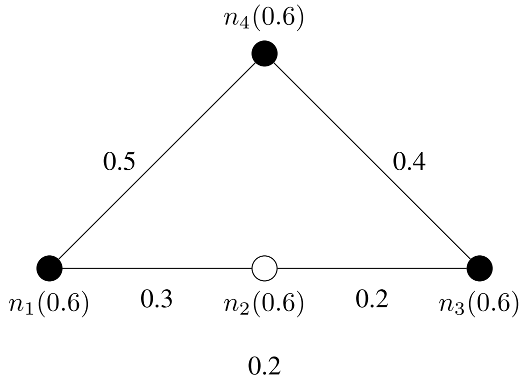

Fig. 1. counter example for the converse of Theorem 22.

If $d(n) = k_1$ for all middle nodes n and $d(n) = k_2$ for all end nodes n in H. Then H is referred to as a (k_1, k_2)-me-regular fuzzy semigraph.

Similarly one can define me-edge regular, me-adjacent regular, and me-consecutive adjacent regular fuzzy semigraphs.

Theorem 24: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which the underlying semigraph is an odd cycle. Then H is a me-regular fuzzy semigraph if, and only if, the function φ is constant.

Proof: Let H be a (l, m)-me-regular fuzzy semigraph, for some constants l and m. Suppose $e_1, e_2, \ldots, e_{2r+1}$ are the edges in the underlying semigraph in that order. Let $\varphi(e_1) = k$, for some constant k. Then $\varphi(e_2) = m - k$, $\varphi(e_3) = m - (m - k) = k$ and so on. There fore

$$\varphi(e_i) = \begin{cases} k, & \text{if } i \text{ is odd,} \\ m - k & \text{if } i \text{ is even.} \end{cases}$$

Hence $\varphi(e_1) = \varphi(e_{2r+1}) = k$. Suppose n be the node in which the edges e_1 and e_{2r+1} are incident. Then $d(n) = 2k$. which implies $m = 2k$ and $k = \frac{m}{2}$. Thus $\varphi(e_i) = \frac{m}{2}$ for all i. This shows φ is a constant function.

Conversely, suppose that φ be a constant function, say $\varphi(e) = c$ for all edges in H. Then

$$d(n) = \begin{cases} 0, & \text{if } n \text{ is not an end node of any edge in } H \\ 2c, & \text{if } n \text{ is an end node of any edge in } H. \end{cases}$$

Thus H is $(0, 2c)$-me-regular fuzzy semigraph. ■

Theorem 25: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which the underlying semigraph is an even cycle. Then H is a me-regular fuzzy semigraph if, and only if, either the function φ is constant or alternative edges have the same membership value.

Proof: Let H be a (l, m)-me-regular fuzzy semigraph, for some constants l and m. Let $e_1, e_2, \ldots, e_{2r}$ be the edges in the underlying semigraph in that order. Suppose $\varphi(e_1) = k$, then

$$\varphi(e_i) = \begin{cases} k, & \text{if } i \text{ is odd} \\ m - k, & \text{if } i \text{ is even.} \end{cases}$$

Suppose $k = m - k$ then φ is a constant function. Otherwise, the alternative edges have the same membership value.

Conversely, if φ is a constant function then from the proof of Theorem 24, H is a $(0, 2c)$-me-regular fuzzy semigraph. If each alternative edges have the same membership value,

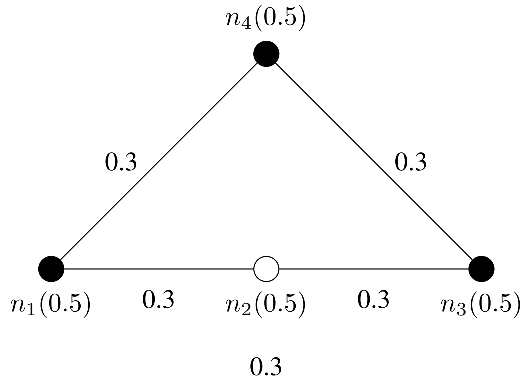

Fig. 2. neither adjacent regular nor me-adjacent regular fuzzy semigraph

say c and d respectively, then H is a $(0, c+d)$-me-regular fuzzy semigraph. ∎

Corollary 26: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which the underlying semigraph is a cycle. Suppose that H is a (m,e)-regular fuzzy semigraph. Then either the membership value of each edge is a constant or the alternative edges have the same membership value.

Note that for a fuzzy semigraph $H = (N, \rho, \nu, \varphi)$ the membership values of edges and partial edges of H are constant if the function ν is constant.

Theorem 27: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which the underlying semigraph is a cycle. Suppose ν is a constant function then H is a me-regular and a consecutive adjacent regular fuzzy semigraph.

The Theorem 27 does not necessarily produce a fuzzy semigraph which is adjacent regular or me-adjacent regular. Consider the fuzzy semigraph $H = (N, \rho, \nu, \varphi)$ given in the Fig. 2. Here $d_a(n_1) = 0.9$ and $d_a(n_4) = 0.6$. Thus H is neither adjacent regular nor me-adjacent regular fuzzy semigraph.

Theorem 28: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph where ν is a constant function and the underlying semigraph H^* is a uniform cycle. Then H is a me-adjacent regular fuzzy semigraph.

Proof: Assume that $\nu(n, m) = c$ for any consecutive adjacent nodes n and m in H and let H^* be an r-uniform semigraph. Then

$$d_a(m) = \begin{cases} c(r-1), & \text{if } m \text{ is a middle node,} \\ 2c(r-1), & \text{if } m \text{ is an end node.} \end{cases}$$

Thus the fuzzy semigraph H is a me-adjacent regular fuzzy semigraph. ∎

Theorem 29: Let $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which the underlying semigraph is a cycle with $|N|$ is odd. Then ν is a constant function if, and only if, H is consecutive adjacency regular fuzzy semigraph.

Theorem 30: Suppose that $H = (N, \rho, \nu, \varphi)$ be a fuzzy semigraph in which the underlying semigraph is a cycle with $|N|$ is even. Then H is consecutive adjacent regular fuzzy semigraph if, and only if, ν is either a constant function or the alternative edges have the same membership value.

Theorem 31: The crisp size of a k_1-regular fuzzy semigraph is $\frac{|N|k_1}{2}$ and the size of a k_2-consecutive adjacent regular fuzzy semigraph is $\frac{|N|k_2}{2}$.

Proof: Let $H = (N, \rho, \nu, \varphi)$ be a k_1-regular fuzzy semigraph with the underlying semigraph $H^* = (N, E)$. Then the crisp size of H is $CS(H) = \sum_{e \in E} \varphi(e)$. Then

$$\sum_{n \in N} d(n) = 2 \sum_{e \in E} \varphi(e) = 2CS(H)$$

Since H is a k_1-regular fuzzy semigraph,

$$\sum_{n \in N} k_1 = 2CS(H)$$

$$|N|k_1 = 2CS(H)$$

$$CS(H) = \frac{|N|k_1}{2}.$$

Similarly, $S(H) = \sum \nu(n, m)$ where the summation is taken over all the nodes n and m such that $n \neq m$ in H. Since H is k_2-consecutive adjacent regular fuzzy semigraph and $\sum_{n \in N} d_{ca}(n) = 2S(H)$,

$$\sum_{n \in N} k_2 = 2S(H)$$

$$|N|k_2 = 2S(H)$$

$$S(H) = \frac{|N|k_2}{2}.$$

∎

Corollary 32: The size of a (k_1, k_2)-me-regular fuzzy semigraph H is $\frac{|E_n|k_2 + |M_n|k_1}{2}$, where $|E_n|, |M_n|$ denote the cardinality of the collection of end nodes and middle only nodes in H respectively.

Corollary 33: The crisp size of a fuzzy semigraph H and the size of the associated end-node fuzzy graph H_e are the same. Similarly if H is an r-uniform fuzzy semigraph the size of an adjacency fuzzy graph H_a is at most $|E| \left(\frac{r(r-1)}{2}\right)$-times the crisp size of H, whereas the size of a consecutive adjacency fuzzy graph H_{ca} of such a uniform fuzzy semigraph H is at most $|E|(r-1)$ times the crisp size of H.

Theorem 34: Let $H = (N, \rho, \nu, \varphi)$ be a k_1-totally regular and k_2-totally consecutive adjacent regular fuzzy semigraph. Then $2CS(H) + O(H) = |N|k_1$ and $2S(H) + O(H) = |N|k_2$.

Proof: Since H is a k_1-totally regular fuzzy semigraph, $td(n) = k_1$ for all $n \in N$. Thus $d(n) + \rho(n) = k_1$ for all $n \in N$. So

$$\sum_{n \in N} d(n) + \sum_{n \in N} \rho(n) = \sum_{n \in N} k_1$$

$$2CS(H) + O(H) = |N|k_1.$$

Similarly, since H is a k_2-totally consecutive adjacent regular fuzzy semigraph, $td_{ca}(n) = k_2$ for all $n \in N$. Thus $d_{ca}(n) + \rho(n) = k_2$ for all $n \in N$. So

$$\sum_{n \in N} d_{ca}(n) + \sum_{n \in N} \rho(n) = \sum_{n \in N} k_2$$

$$2S(H) + O(H) = |N|k_2.$$

∎

The following results are immediate consequences of Theorem 34.

Corollary 35: Let $H = (N, \rho, \nu, \varphi)$ be a k_1-regular and k_2-totally regular fuzzy semigraph. Then

$$O(H) = |N|(k_2 - k_1).$$

Corollary 36: Let $H = (N, \rho, \nu, \varphi)$ be a k_1-regular and k_2-consecutive adjacent regular fuzzy semigraph with p number of nodes. Then the crisp size and size of H is given by $\frac{pk_1}{2}$ and $\frac{pk_2}{2}$ respectively.

B. Fuzzy Multi Semigraph

Further, the concept of a semigraph and hence that of a fuzzy semigraph is generalized because of the following observations:

Semigraph modeling is possible in any physical scenario where sequential activities describe the situation. But in all such situations, any two sequential activities may have more than one node in common. So we have proposed (communicated article titled: Planarity Index of Fuzzy Semigraphs [14]) a generalization of semigraph to multi-semigraph and accordingly fuzzy semigraph to fuzzy multi semigraph similar to graph versus multi graph; any two edges in a graph can have at most one node in common whereas in multi-graph two edges can have more than one node in common.

A multi semigraph is a pair $H^* = (N, E)$ where $N \neq \phi$ is the set of nodes and the edge set E is a collection of r - tuples, for various $r \geq 2$, of distinct elements of N such that an edge $(n_1, n_2, \ldots, n_r)$ and $(n_r, n_{r-1}, \ldots, n_2, n_1)$ are same.

Given this terminology, the semigraph defined by E. Sampathkumar in [2] can be regarded as a simple semigraph that requires an additional axiom namely, any two edges in E can have at most one node in common.

Since the concept of multi-edges can be effectively integrated with the theory of fuzziness, a fuzzy multi-semigraph is defined.

Consider the multi semigraph $H^* = (N, E)$. Then a fuzzy multi semigraph defined on H^* is defined as $H = (N, \rho, E_1, E_2)$ where $N \neq \phi$ and $\rho : N \to [0, 1]$ be a function and $E_1 = \{((n, m), \nu_1(n, m), \nu_2(n, m), \ldots, \nu_j(n, m)) \mid (n, m) \in N \times N\}$ be a fuzzy multi subset of $N \times N$ where $\nu_i : N \times N \to [0, 1]$ such that

$$\nu_i(n_1, n_2) \leq \rho(n_1) \wedge \rho(n_2) \quad \text{if} \quad (n_1, n_2) \in N \times N$$

in which $i = 1, 2, \ldots, j$, and $E_2 = \{(e, \varphi_1(e), \varphi_2(e), \ldots, \varphi_j(e)) | e \in E\}$ be a fuzzy multi subset of E where $\varphi_i : E \to [0, 1]$ which satisfies

$$\varphi_i(e) = \nu_i(n_1, n_2) \wedge \nu_i(n_2, n_3) \wedge \cdots \wedge \nu_i(n_{r-1}, n_r),$$

$$\forall\ i = 1, 2, \ldots, j$$

if the edge e is the r-tuple $(n_1, n_2, \ldots, n_{r-1}, n_r)$ and $j = max\{i \mid \varphi_i(e) \neq 0, e \in E\}$.

All the results examined in this article apply to fuzzy multi-semigraphs too.

C. Application

One can associate semigraph structure with any real-life situation where sequential activities are involved.

For example, consider the functioning of the University of Calicut. A student who enrolled at the university and completed his graduation successfully has to submit the application form along with the necessary documents to obtain the degree certificate. These documents will be received in a particular section, and one of the assistant officers of the section who is dealing with the certificate of that particular programme or center will put up the file. It will pass through a sequence of sections and officers in a particular order, related to the verification of documents, processing, and approval of the certificate.

The whole activity of the university thus generates a well-structured semigraph. For the sake of undirected edges, the sequence of activities can be considered in either direction. Also, it may not be a simple semigraph because each edge corresponds to a particular activity, so there can be more nodes (that is, sections) common to two edges.

The Fig. 3 depicts a part of this semigraph, where the loosely dashed line represents the activity of issuing a BSc Mathematics certificate, the solid line, the densely dashed line, the double line, and the double dotted line respectively represent the activity of issuing the certificate of the courses BSc Zoology, BCom, and BSc self-financing and the examination hall ticket of the course BCom.

The initial destination for the file is the Assistant Officer (AO) in charge of the student's respective degree programme. The AO checks the attached documents and verifies them with the help of the tabulation department. Thereafter, the file will be forwarded to higher officers and the student gets the degree certificate only after the verification and concurrence of the Section Officer (SO), Assistant Registrar (AR), Deputy Registrar (DR), Controller of Examination (CE), and Vice-Chancellor (VC) or the statutory boards like the Senate and Syndicate. Such a situation is explained in Fig. 3. Here, small circles with no filling are used for all middle nodes, whereas different edges are segregated using different patterns.

Now consider the various kinds of degrees that are defined in this paper. The membership values can be assigned by considering the situation or the problem to address. For example, membership values for nodes can be assigned so that the degree of each node gives a measure of the amount of work in all the tasks where the node (which represents an authority) is in the initial (that is, to initiate the process by verifying the documents) or in the concluding phase of the tasks. The edge degree of each node measures the minimum workload of all the tasks in which the node is involved. The consecutive adjacent degree of each node gives a measure of its works by combining the nodes just preceding the works and succeeding it taken by that authority (which was represented by that node) to verify or convince the task with a nearby authority.

III. Conclusion

Research on the idea of fuzzy graphs is explored in Mathematics as well as in their applications to other fields. This work defines various degrees and sizes of a fuzzy semigraph; to connect these notions, an analysis of these

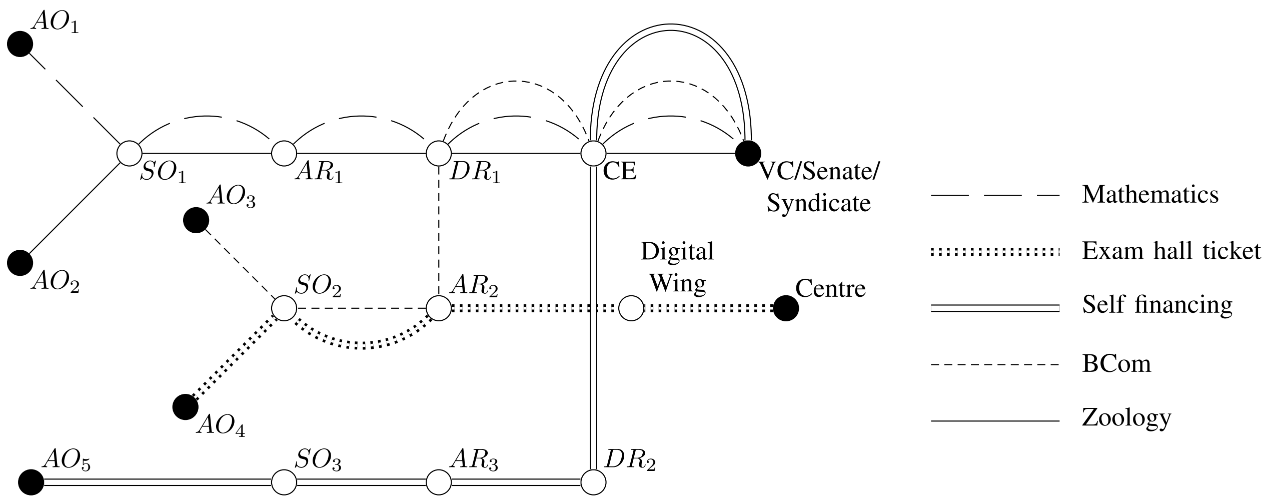

Fig. 3. A multi-semigraph representation of the activity described in the application.

parameters is made, which helps to determine the general properties of a fuzzy semigraph. Any real-life situation, where a sequence of activities constitutes a task can be depicted as a multi semigraph. By assigning appropriate membership values, the various degrees introduced in this paper have specific objectives. Further study of the work aimed at putting the concept of multi-set on the node set of a semigraph.

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