The Multipoint-based Hermite–Hadamard Inequalities for Fractional Integrals with Exponential Kernels

Zhengrong Yuan and Tingsong Du*

Abstract—**This paper concentrates on addressing fractional inequalities for exponential type convex functions. By means of exponential-type convexity, we firstly establish Hermite–Hadamard (HH) type inequalities for fractional integrals with exponential kernels. Secondly, based on the discovered fractional identity by separating $[a, b]$ to n equal subintervals, and the fact that the twice derivative in absolute value is exponential type convex, we present multipoint-based HH inequalities, which cover the trapezoid- and Bullen-type inequalities for $n = 1$ and 2, correspondingly. During the period, some numerical examples with graphs are provided to show the validity of the deduced inequalities.**

Index Terms—**Convex functions, fractional integrals, Hermite–Hadamard inequalities, multipoint-based inequalities**

I. Introduction

FRACTIONAL calculus, the branch of calculus studying integrals and derivatives of non integer order, has gained increasing popularity and interest in recent years. Because the classical calculus cannot model the entirety of real-world phenomena, researchers have investigated various types of fractional integrals. Riemann–Liouville (RL) fractional integrals [1], [2], [3], multiplicative-RL fractional integrals [4], k-RL fractional integrals [5], Hadamard-type k-fractional integrals [6], conformable fractional integrals [7], [8], and Katugampola fractional integrals [9] are just a few of the options available. These fractional operators are applied in different disciplines such as the medical sciences [10], physics [11], economy [12], engineering sciences [13], etc. For other applications associated with this topic, please see [14], [15], [16], [17] and the references cited therein.

Recently, Ahmad *et al.* [18] introduced a class of fractional integrals with exponential kernels for the first time. And they used these integrals to prove the fractional variants of Hermite–Hadamard (HH) type inequalities. Inspired by the ideas of this article, the scholars explored various of inequalities with the assistance of the fractional integrals. For instance, considering the first- and second-order differentiable functions, Wu *et al.* [19] studied the left and right side of the fractional HH-type inequalities. Further, Yuan *et al.* [20] deduced the parameterized fractional integral inequalities, which unified the midpoint-, Simpson-, Bullen-, and trapezoid-type inequalities. In Ref. [21], the authors gave some estimations of the upper bound of Ostrowski-type fractional inequalities through convex functions. And Rashid *et al.* [22] derived the fractional HH-, HH–Fejér- and Pachpatte-type inequalities for exponentially convex functions. Making use of the Mercer concept, Butt *et al.* [23] constructed fractional versions of HH-, HH–Fejér-, and Pachpatte-type inequalities for harmonically convex functions. In 2023, Botmart *et al.* [24] addressed the midpoint- and HH–Fejér-type inequalities via convexity and harmonically convexity, respectively. By introducing the interval-valued fractional integrals with exponential kernels, Zhou *et al.* [25] established the fractional integral inclusions. For more information about fractional integrals with exponential kernels, one can refer to [26], [27], [28], [29] and [30]. In this study, we focus on investigating the multipoint-based inequalities with relation to fractional integrals having exponential kernels.

In 2021, İşcan *et al.* [31] presented the general integer-order inequalities for functions whose first derivatives in absolute value are convex. And the established inequalities involve multiple points, which transform into the trapezoid- and Bullen-type inequalities for the parameter $n = 1$ and 2, separately. For simplicity, we call these kinds of inequalities as the multipoint-based HH inequalities. Based on the multipoint-based HH identity proved by İşcan *et al.* in Ref. [31], Yıldız *et al.* [32] further discussed integer-order integral inequalities for s-convex functions. And Erden *et al.* [33] constructed Newton's like inequalities of integer-order integrals including multiple points. Moreover, by means of RL fractional integrals, Ekinci and Özdemir [34] considered fractional variants of the multipoint-based HH inequalities for once-differentiable convex functions. In 2023, utilizing functions whose second derivatives absolute values are exponential type convex, Yıldız and Yergöz [35] obtained multipoint-based integer-order inequalities. However, there are comparatively few investigations regarding multipoint-based inequalities for the class of twice-differentiable functions, especially in the setting of the fractional integral operators. Taking inspiration from the foregoing work, it becomes obvious that exploiting fractional integral having exponential kernels in tandem with twice-differentiable exponential type convex functions can yield new multipoint-based inequalities.

The paper unfolds as follows. We recollect preliminary information for related concepts and fractional integrals in Sec. II. In Sec. III, by leveraging fractional integrals with exponential kernels, we prove the HH-type inequalities for exponential type convex functions. In Sec. IV, we derive a fractional integral identity by separating $[a, b]$ to n equal subintervals. Taking advantage of the identity, we establish multipoint-based fractional inequalities for twice-

Manuscript received December 31, 2023; revised April 2, 2024.

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differentiable exponential type convex functions. Meanwhile, we give examples to illustrate the obtained inequalities more intuitively.

II. Preliminaries

This section states some necessary definitions, theorems and related fractional integral results, which are used in this paper. Here and further, let $I \subseteq \mathbb{R}$ be a real interval and I° be the interior of I. The definition of convexity is represented as follows:

Definition 2.1: [36] A function $g : I \to \mathbb{R}$ is said to be convex, if

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$$

holds for every $x, y \in I$ and $t \in [0,1]$.

The HH inequality, which is famous in scientific literature, has a significant place in analysis mathematics. It is stated that if $g : I \to \mathbb{R}$ is a convex function on the interval I, then

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \leq \frac{g(a)+g(b)}{2}$$

holds for any $a, b \in I$ with $a < b$. This prominent inequality gives estimates for the mean value of a continuous convex function.

Bullen [37] proved the following inequality, which is known as the Bullen's inequality for convex functions,

$$\frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \leq \frac{1}{2}\left[g\left(\frac{a+b}{2}\right) + \frac{g(a)+g(b)}{2}\right].$$

In [38], Kadakal and İşcan proposed a class of exponential type convex functions.

Definition 2.2: [38] A nonnegative function $g : I \to \mathbb{R}$ is said to be exponential type convex, if the coming inequality

$$g(tx + (1-t)y) \leq \left(e^t - 1\right)g(x) + \left(e^{1-t} - 1\right)g(y)$$

holds for every $x, y \in I$ and $t \in [0,1]$.

In the same paper, they used this class of functions to prove a variant of HH inequality.

Theorem 2.1: [38] Let $g : [a,b] \to \mathbb{R}$ be an exponential type convex function. If the function $g \in L^1([a,b])$, then the following HH-type inequality is given

$$\begin{aligned}\frac{1}{2(e^{\frac{1}{2}}-1)}g\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \\ &\leq (e-2)(g(a)+g(b)). \qquad (1)\end{aligned}$$

Using the twice-differentiable functions, Alomari *et al.* [39] presented the following identity, which is related to the right part of HH inequality.

Lemma 2.1: [39] Let $g : I \to \mathbb{R}$ be a twice-differentiable function on I°, where $a, b \in I$ with $a < b$. If the function $g'' \in L^1([a,b])$, then the subsequent identity holds

$$\begin{aligned}&\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \\ &= \frac{(b-a)^2}{2}\int_0^1 t(1-t)g''(ta+(1-t)b)\mathrm{d}t.\end{aligned}$$

To establish multipoint-based integer-order HH inequalities, Yıldız and Yergöz [35] employed the equality below.

Lemma 2.2: [35] Assume that $g : I \to \mathbb{R}$ is a continuously differentiable function on I°, where $a, b \in I$ with $a < b$ and $n \in \mathbb{Z}^+$. If the function $g'' \in L^1([a,b])$, then the coming identity is valid

$$\begin{aligned}&\sum_{i=0}^{n-1}\frac{1}{2n}\left[\begin{array}{c} g\left(\frac{(n-i)a+ib}{n}\right) \\ +g\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\end{array}\right] - \frac{1}{b-a}\int_a^b g(\tau)\mathrm{d}\tau \\ &= \sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\int_0^1 t(1-t) \\ &\quad \times g''\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\mathrm{d}t. \quad (2)\end{aligned}$$

Ahmad *et al.* [18] proposed the following fractional integrals with exponential kernels.

Definition 2.3: [18] Let the function $g \in L^1([a,b])$ and the order $\alpha \in (0,1)$. The fractional integrals with exponential kernels, denoted by $\mathcal{I}_{a+}^\alpha g$ and $\mathcal{I}_{b-}^\alpha g$, respectively, are defined as the coming expressions:

$$\mathcal{I}_{a+}^\alpha g(x) = \frac{1}{\alpha}\int_a^x \exp\left(-\frac{1-\alpha}{\alpha}(x-\tau)\right)g(\tau)\mathrm{d}\tau,\ x > a,$$

and

$$\mathcal{I}_{b-}^\alpha g(x) = \frac{1}{\alpha}\int_x^b \exp\left(-\frac{1-\alpha}{\alpha}(\tau-x)\right)g(\tau)\mathrm{d}\tau,\ x < b.$$

From Definition 2.3, we can also readily observe that

$$\lim_{\alpha\to1}\mathcal{I}_{a+}^\alpha g(x) = \int_a^x g(\tau)\mathrm{d}\tau, \quad \lim_{\alpha\to1}\mathcal{I}_{b-}^\alpha g(x) = \int_x^b g(\tau)\mathrm{d}\tau.$$

Ahmad *et al.* [18] also established a fractional HH-type inequality below.

Theorem 2.2: [18] Assume that the function $g : [a,b] \to (0,+\infty)$ is convex. If the function $g \in L^1([a,b])$, then the subsequent inequality for fractional integrals with exponential kernels is obtained

$$\begin{aligned}g\left(\frac{a+b}{2}\right) &\leq \frac{1-\alpha}{2\left(1-e^{-\frac{1-\alpha}{\alpha}(b-a)}\right)}\left[\mathcal{I}_{a+}^\alpha g(b) + \mathcal{I}_{b-}^\alpha g(a)\right] \\ &\leq \frac{g(a)+g(b)}{2}.\end{aligned}$$

In the case of twice-differentiable convex functions, Wu *et al.* [19] proved the succeeding identity.

Lemma 2.3: [19] Let $g : [a,b] \to \mathbb{R}$ be a twice-differentiable function on (a,b). If the function $g'' \in L^1([a,b])$, then the following fractional integral equality can be derived

$$\begin{aligned}&\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a+}^\alpha g(b) + \mathcal{I}_{b-}^\alpha g(a)\right] - \frac{g(a)+g(b)}{2} \\ &= \frac{(b-a)^2}{2\rho(1-e^{-\rho})} \\ &\quad \times \int_0^1 \left(e^{-\rho t} + e^{-\rho(1-t)} - 1 - e^{-\rho}\right)g''(ta+(1-t)b)\mathrm{d}t,\end{aligned}$$

where

$$\rho = \frac{1-\alpha}{\alpha}(b-a).$$

In 2021, Kadakal *et al.* [40] provided the improved power-mean integral inequality.

Theorem 2.3: [40] Suppose that f and g are both real functions defined on $[a,b]$. If $q \geq 1$, and $|f|, |f||g|^q \in L^1([a,b])$, then the following inequality holds

$$\int_a^b |f(\tau)g(\tau)|\mathrm{d}\tau \leq \frac{1}{b-a}\left[\begin{array}{l} \left(\int_a^b (b-\tau)|f(\tau)|\mathrm{d}\tau\right)^{1-\frac{1}{q}} \\ \times\left(\int_a^b (b-\tau)|f(\tau)||g(\tau)|^q\mathrm{d}\tau\right)^{\frac{1}{q}} \\ +\left(\int_a^b (\tau-a)|f(\tau)|\mathrm{d}\tau\right)^{1-\frac{1}{q}} \\ \times\left(\int_a^b (\tau-a)|f(\tau)||g(\tau)|^q\mathrm{d}\tau\right)^{\frac{1}{q}} \end{array}\right].$$

We conclude this section by reviewing the beta function, which is defined by

$$\beta(\mu,\nu)=\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\mathrm{d}t, \mu,\nu>0.$$

III. Fractional HH-type inequalities

With the assistance of fractional integrals with exponential kernels, this section establishes the HH-type inequalities for exponential type convex functions.

Theorem 3.1: Let $g:[a,b]\to[0,+\infty)$ be an exponential type convex function. For $\alpha\in(0,1)\setminus\{\frac{b-a}{1+b-a}\}$, if the function $g\in L^1([a,b])$, then one can acquire the following fractional integral inequality

$$\begin{aligned} &\frac{1}{2\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right) \\ &\leq \frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right] \\ &\leq \Phi_1(\rho)\frac{g(a)+g(b)}{2}, \end{aligned} \tag{3}$$

where

$$\Phi_1(\rho)=\frac{1}{(\rho^2-1)(1-e^{-\rho})} \times\left[\begin{array}{l} (\rho^2+\rho)\left(e^{-\rho}-e^{1-\rho}\right)-2e^{-\rho} \\ +\left(\rho^2-\rho\right)(e-1)+2 \end{array}\right],\ \rho\neq 0,1.$$

Proof: Using the property of the exponential type convex function g, we have that

$$g\left(\frac{x+y}{2}\right)\leq\left(e^{\frac{1}{2}}-1\right)g(x)+\left(e^{\frac{1}{2}}-1\right)g(y).$$

Taking advantage of the change of variables $x=ta+(1-t)b$ and $y=tb+(1-t)a$, we get that

$$\begin{aligned} g\left(\frac{a+b}{2}\right)\leq&\left(e^{\frac{1}{2}}-1\right)g(ta+(1-t)b) \\ &+\left(e^{\frac{1}{2}}-1\right)g(tb+(1-t)a). \end{aligned} \tag{4}$$

Multiplying both sides of the inequality (4) by $e^{-\rho t}$, and integrating the resulting inequality with respect to t over $[0,1]$, we obtain that

$$\begin{aligned} &\frac{1-e^{-\rho}}{\rho\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right) \\ &\leq\int_0^1 e^{-\rho t}g(ta+(1-t)b)\mathrm{d}t \\ &\quad+\int_0^1 e^{-\rho t}g(tb+(1-t)a)\mathrm{d}t \\ &=\frac{1}{b-a}\left[\begin{array}{l}\int_a^b e^{-\frac{1-\alpha}{\alpha}(b-u)}g(u)\mathrm{d}u \\ +\int_a^b e^{-\frac{1-\alpha}{\alpha}(u-a)}g(u)\mathrm{d}u\end{array}\right] \\ &=\frac{\alpha}{b-a}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right], \end{aligned}$$

which yields that

$$\frac{1}{2\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right)\leq\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right].$$

This finishes the proof of the first inequality in (3).

For the proof of the second inequality in (3), we note that if the function g is exponential type convex, then for every $t\in[0,1]$, we have that

$$g(ta+(1-t)b)\leq\left(e^t-1\right)g(a)+\left(e^{1-t}-1\right)g(b),$$

and

$$g(tb+(1-t)a)\leq\left(e^t-1\right)g(b)+\left(e^{1-t}-1\right)g(a).$$

By adding the above two inequalities, we get that

$$\begin{aligned} &g(ta+(1-t)b))+g(tb+(1-t)a) \\ &\leq\left(e^t+e^{1-t}-2\right)(g(a)+g(b)). \end{aligned} \tag{5}$$

Multiplying both sides of the inequality (5) by $e^{-\rho t}$, and integrating the resulting inequality regarding t over $[0,1]$, we obtain that

$$\begin{aligned} &\int_0^1 e^{-\rho t}\left[g(ta+(1-t)b))+g(tb+(1-t)a)\right]\mathrm{d}t \\ &\leq(g(a)+g(b))\int_0^1 e^{-\rho t}\left(e^t+e^{1-t}-2\right)\mathrm{d}t, \end{aligned}$$

that is

$$\begin{aligned} &\frac{\alpha}{b-a}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right] \\ &\leq\frac{(\rho^2+\rho)\left(e^{-\rho}-e^{1-\rho}\right)-2e^{-\rho}+\left(\rho^2-\rho\right)(e-1)+2}{\rho^3-\rho} \\ &\quad\times(g(a)+g(b)). \end{aligned} \tag{6}$$

Both sides of the inequality (6) are multiplied by $\frac{\rho}{2(1-e^{-\rho})}$ simultaneously, and we acquire that

$$\begin{aligned} &\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}_{a^+}^{\alpha}g(b)+\mathcal{I}_{b^-}^{\alpha}g(a)\right] \\ &\leq\frac{(\rho^2+\rho)\left(e^{-\rho}-e^{1-\rho}\right)-2e^{-\rho}+\left(\rho^2-\rho\right)(e-1)+2}{2(\rho^2-1)(1-e^{-\rho})} \\ &\quad\times(g(a)+g(b)). \end{aligned}$$

Thus, the proof of Theorem 3.1 is completed. ■

Remark 3.1: In Theorem 3.1, if we take $\alpha\to 1$, i.e., $\rho=\frac{1-\alpha}{\alpha}(b-a)\to 0$, then we have that

$$\lim_{\alpha\to 1}\frac{1-\alpha}{2(1-e^{-\rho})}=\frac{1}{2(b-a)},$$

and

$$\lim_{\rho\to 0}\Phi_1(\rho)=2(e-2).$$

Thus, the inequality (3) is transformed to the inequality (1).

The fractional HH-type inequality involving midpoint can be represented as follows.

Theorem 3.2: Let $g:[a,b]\to[0,+\infty)$ be an exponential type convex function. For $\alpha\in(0,1)\setminus\{\frac{b-a}{1+b-a}\}$, if the function $g\in L^1([a,b])$, then the following fractional integral inequality is given

$$\begin{aligned}&\frac{1}{2\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right)\\&\le\frac{1-\alpha}{2\left(1-e^{-\frac{\rho}{2}}\right)}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\&\le\Phi_2(\rho)\frac{g(a)+g(b)}{2},\end{aligned}\tag{7}$$

where

$$\begin{aligned}\Phi_2(\rho)=&\frac{1}{(\rho^2-1)\left(1-e^{-\frac{\rho}{2}}\right)}\\&\times\left[\begin{array}{l}2\left(\rho^2-1\right)e^{-\frac{\rho}{2}}-2\rho^2e^{\frac{1-\rho}{2}}\\+\left(\rho^2-\rho\right)(e-1)+2\end{array}\right],\ \rho\neq 0,1.\end{aligned}$$

Proof: Since the function g is exponential type convex on $[a,b]$, for $x,y\in[a,b]$, we have that

$$g\left(\frac{x+y}{2}\right)\le\left(e^{\frac{1}{2}}-1\right)g(x)+\left(e^{\frac{1}{2}}-1\right)g(y).$$

For $x=\frac{t}{2}a+\frac{2-t}{2}b$ and $y=\frac{2-t}{2}a+\frac{t}{2}b$, we obtain that

$$\begin{aligned}&\frac{1}{\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right)\\&\le g\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+g\left(\frac{2-t}{2}a+\frac{t}{2}b\right).\end{aligned}\tag{8}$$

Multiplying both sides of the inequality (8) by $e^{-\frac{\rho}{2}t}$, then integrating the obtained result concerning t over $[0,1]$, we get that

$$\begin{aligned}&\frac{2\left(1-e^{-\frac{\rho}{2}}\right)}{\rho\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right)\\&\le\int_0^1e^{-\frac{\rho}{2}t}g\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\mathrm{d}t\\&\quad+\int_0^1e^{-\frac{\rho}{2}t}g\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\mathrm{d}t\\&=\frac{2}{b-a}\left[\begin{array}{l}\int_{\frac{a+b}{2}}^{b}e^{-\frac{1-\alpha}{\alpha}(b-u)}g(u)\mathrm{d}u\\+\int_a^{\frac{a+b}{2}}e^{-\frac{1-\alpha}{\alpha}(u-a)}g(u)\mathrm{d}u\end{array}\right]\\&=\frac{2\alpha}{b-a}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right],\end{aligned}$$

which deduces that

$$\begin{aligned}&\frac{1}{2\left(e^{\frac{1}{2}}-1\right)}g\left(\frac{a+b}{2}\right)\\&\le\frac{1-\alpha}{2\left(1-e^{-\frac{\rho}{2}}\right)}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right].\end{aligned}$$

This completes the proof of the first part of inequality (7).

To prove the second part of the inequality (7), employing the exponential-type convexity of the function g, for every $t\in[0,1]$, it follows that

$$g\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\le\left(e^{\frac{t}{2}}-1\right)g(a)+\left(e^{\frac{2-t}{2}}-1\right)g(b),$$

and

$$g\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\le\left(e^{\frac{2-t}{2}}-1\right)g(a)+\left(e^{\frac{t}{2}}-1\right)g(b).$$

By adding the above two inequalities, we get that

$$\begin{aligned}&g\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+g\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\\&\le\left(e^{\frac{t}{2}}+e^{\frac{2-t}{2}}-2\right)(g(a)+g(b)).\end{aligned}\tag{9}$$

Multiplying both sides of the inequality (9) by $e^{-\frac{\rho}{2}t}$, and integrating the resultant inequality regarding t over $[0,1]$, we acquire that

$$\begin{aligned}&\int_0^1e^{-\frac{\rho}{2}t}\left[g\left(\frac{t}{2}a+\frac{2-t}{2}b\right)+g\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\right]\mathrm{d}t\\&\le(g(a)+g(b))\int_0^1e^{-\frac{\rho}{2}t}\left(e^{\frac{t}{2}}+e^{\frac{2-t}{2}}-2\right)\mathrm{d}t,\end{aligned}$$

that is

$$\begin{aligned}&\frac{2\alpha}{b-a}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\&\le\frac{4\left(\rho^2-1\right)e^{-\frac{\rho}{2}}-4\rho^2e^{\frac{1-\rho}{2}}+2\left(\rho^2-\rho\right)(e-1)+4}{\rho^3-\rho}\\&\quad\times(g(a)+g(b)).\end{aligned}\tag{10}$$

Both sides of the inequality (10) are multiplied by $\frac{\rho}{4\left(1-e^{-\frac{\rho}{2}}\right)}$ at the same time, and we have that

$$\begin{aligned}&\frac{1-\alpha}{2\left(1-e^{-\frac{\rho}{2}}\right)}\left[\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)\right]\\&\le\frac{2\left(\rho^2-1\right)e^{-\frac{\rho}{2}}-2\rho^2e^{\frac{1-\rho}{2}}+\left(\rho^2-\rho\right)(e-1)+2}{2\left(\rho^2-1\right)\left(1-e^{-\frac{\rho}{2}}\right)}\\&\quad\times(g(a)+g(b)).\end{aligned}$$

Thus, the proof of Theorem 3.2 is finished. ∎

Remark 3.2: In Theorem 3.2, if we take $\alpha\to 1$, i.e., $\rho=\frac{1-\alpha}{\alpha}(b-a)\to 0$, then we have that

$$\lim_{\alpha\to 1}\frac{1-\alpha}{2\left(1-e^{-\frac{\rho}{2}}\right)}=\frac{1}{b-a},$$

and

$$\lim_{\rho\to 0}\Phi_2(\rho)=2(e-2).$$

Thus, the inequality (7) changes into the inequality (1).

IV. MULTIPOINT-BASED FRACTIONAL INEQUALITIES

For twice-differentiable functions, this section firstly formulates a fractional integral identity by separating $[a,b]$ to n equal subintervals. On the basis of the integral identity, and the fact that the twice derivative in absolute value is exponential type convex, we address multipoint-based HH inequalities. In the meantime, numerical examples are provided to show the validity of the deduced inequalities as well. Before giving the subsequent results, we introduce the following notation:

$$\varrho=\frac{\rho}{n}=\frac{(b-a)(1-\alpha)}{n\alpha},\ \alpha\in(0,1),\ n\in\mathbb{Z}^{+},\ a<b.$$

We need the following lemma.

Lemma 4.1: Let $g:[a,b]\to\mathbb{R}$ be a twice-differentiable function on (a,b), $n\in\mathbb{Z}^{+}$ and $i\in\mathbb{N}$. If the function $g''\in L^{1}([a,b])$, then the following identity for fractional integrals with exponential kernels holds

$$\begin{aligned}&\sum_{i=0}^{n-1}\frac{1}{2n}\left[g(\omega(i))+g(\omega(i+1))-\frac{1-\alpha}{1-e^{-\varrho}}\Psi_{g}(\mathcal{I};\alpha,i)\right]\\&=\sum_{i=0}^{n-1}\frac{(b-a)^{2}}{2n^{3}\varrho(1-e^{-\varrho})}\\&\quad\times\int_{0}^{1}\left[\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t,\end{aligned}\tag{11}$$

where

$$\omega(i)=a+\frac{i}{n}(b-a),$$

and

$$\Psi_{g}(\mathcal{I};\alpha,i)=\mathcal{I}^{\alpha}_{\omega(i)^{+}}g(\omega(i+1))+\mathcal{I}^{\alpha}_{\omega(i+1)^{-}}g(\omega(i)).$$

Proof: Considering the right side of the identity (11), we can write that

$$\begin{aligned}&\int_{0}^{1}\left[\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t\\&=-\frac{n}{b-a}\left[\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g'\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\Bigg|_{0}^{1}\\&\quad+\frac{n\varrho}{b-a}\int_{0}^{1}\left[\begin{array}{l}\left(e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g'\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t\\&=\frac{n^{2}\varrho}{(b-a)^{2}}\left(1-e^{-\varrho}\right)\left[g(\omega(i))+g(\omega(i+1))\right]\\&\quad-\frac{n^{2}\varrho^{2}}{(b-a)^{2}}\\&\quad\times\int_{0}^{1}\left[\begin{array}{l}\left(e^{-\varrho t}+e^{-\varrho(1-t)}\right)\\\times g\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t.\end{aligned}\tag{12}$$

Making use of the substitution $x=t\omega(i)+(1-t)\omega(i+1)$ for the integral above, we get that

$$\begin{aligned}&\int_{0}^{1}\left(e^{-\varrho t}+e^{-\varrho(1-t)}\right)g\big(t\omega(i)+(1-t)\omega(i+1)\big)\mathrm{d}t\\&=\frac{n}{b-a}\int_{\omega(i)}^{\omega(i+1)}\left(\begin{array}{l}e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\\+e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))}\end{array}\right)g(x)\mathrm{d}x\\&=\frac{n\alpha}{b-a}\left[\mathcal{I}^{\alpha}_{\omega(i)^{+}}g(\omega(i+1))+\mathcal{I}^{\alpha}_{\omega(i+1)^{-}}g(\omega(i))\right].\end{aligned}\tag{13}$$

Putting the identity (13) into the identity (12), we have that

$$\begin{aligned}&\int_{0}^{1}\left[\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t\\&=\frac{n^{2}\varrho\left(1-e^{-\varrho}\right)}{(b-a)^{2}}\\&\quad\times\left\{\begin{array}{l}g(\omega(i))+g(\omega(i+1))\\-\frac{1-\alpha}{1-e^{-\varrho}}\left[\begin{array}{l}\mathcal{I}^{\alpha}_{\omega(i)^{+}}g(\omega(i+1))\\+\mathcal{I}^{\alpha}_{\omega(i+1)^{-}}g(\omega(i))\end{array}\right]\end{array}\right\}.\end{aligned}\tag{14}$$

Multiplying both sides of the identity (14) by $\frac{(b-a)^{2}}{2n^{3}\varrho(1-e^{-\varrho})}$, we obtain that

$$\begin{aligned}&\frac{1}{2n}\left\{\begin{array}{l}g(\omega(i))+g(\omega(i+1))\\-\frac{1-\alpha}{1-e^{-\varrho}}\left[\begin{array}{l}\mathcal{I}^{\alpha}_{\omega(i)^{+}}g(\omega(i+1))\\+\mathcal{I}^{\alpha}_{\omega(i+1)^{-}}g(\omega(i))\end{array}\right]\end{array}\right\}\\&=\frac{(b-a)^{2}}{2n^{3}\varrho(1-e^{-\varrho})}\\&\quad\times\int_{0}^{1}\left[\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t.\end{aligned}$$

Consequently, we have that

$$\begin{aligned}&\sum_{i=0}^{n-1}\frac{1}{2n}\left\{\begin{array}{l}g(\omega(i))+g(\omega(i+1))\\-\frac{1-\alpha}{1-e^{-\varrho}}\left[\begin{array}{l}\mathcal{I}^{\alpha}_{\omega(i)^{+}}g(\omega(i+1))\\+\mathcal{I}^{\alpha}_{\omega(i+1)^{-}}g(\omega(i))\end{array}\right]\end{array}\right\}\\&=\sum_{i=0}^{n-1}\frac{(b-a)^{2}}{2n^{3}\varrho(1-e^{-\varrho})}\\&\quad\times\int_{0}^{1}\left[\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\end{array}\right]\mathrm{d}t.\end{aligned}$$

This completes the proof. ■

Remark 4.1: In Lemma 4.1, if we take $\alpha\to 1$, i.e., $\varrho=\frac{(b-a)(1-\alpha)}{n\alpha}\to 0$, then we have that

$$\lim_{\alpha\to 1}\frac{1-\alpha}{1-e^{-\varrho}}=\frac{n}{b-a},\tag{15}$$

and

$$\lim_{\varrho\to 0}\frac{1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}}{\varrho(1-e^{-\varrho})}=-t^{2}+t.\tag{16}$$

Thus, the equality (11) reduces to the equality (2).

Corollary 4.1: Consider Lemma 4.1, we can get the subsequent results.

(1) For $n=1$, we have Lemma 2.3.

(2) For $n=2$, we have the coming Bullen-type equality for fractional integrals with exponential kernels:

$$\begin{aligned}&\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}\right]\\&-\frac{1-\alpha}{4(1-e^{-\frac{\varrho}{2}})}\left[\begin{array}{l}\mathcal{I}^{\alpha}_{a^{+}}g\left(\frac{a+b}{2}\right)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}g(a)\\+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}g(b)+\mathcal{I}^{\alpha}_{b^{-}}g\left(\frac{a+b}{2}\right)\end{array}\right]\\&=\frac{(b-a)^{2}}{8\rho(1-e^{-\frac{\varrho}{2}})}\\&\quad\times\int_{0}^{1}\left\{\begin{array}{l}\left(1+e^{-\frac{\varrho}{2}}-e^{-\frac{\varrho}{2}t}-e^{-\frac{\varrho}{2}(1-t)}\right)\\\times\left[\begin{array}{l}g''\left(ta+(1-t)\frac{a+b}{2}\right)\\+g''\left(t\frac{a+b}{2}+(1-t)b\right)\end{array}\right]\end{array}\right\}\mathrm{d}t,\end{aligned}$$

which is a new Bullen-type identity.

For brevity, we will use the subsequent notation in the sequel:

$$\begin{aligned}&\Xi(g;[a,b],n)\\&:=\sum_{i=0}^{n-1}\frac{1}{2n}\left[g(\omega(i))+g(\omega(i+1))-\frac{1-\alpha}{1-e^{-\varrho}}\Psi_{g}(\mathcal{I};\alpha,i)\right].\end{aligned}$$

In particular, for $n=1$, we have that

$$\begin{aligned}&\Xi(g;[a,b],1)\\&:=\frac{g(a)+g(b)}{2}-\frac{1-\alpha}{2(1-e^{-\rho})}\left[\mathcal{I}^{\alpha}_{a^{+}}g(b)+\mathcal{I}^{\alpha}_{b^{-}}g(a)\right],\end{aligned}$$

and for $n = 2$, we have that

$$\begin{aligned}&\Xi(g;[a,b],2)\\&:=\frac{1}{2}\left[g\left(\frac{a+b}{2}\right)+\frac{g(a)+g(b)}{2}\right]\\&\quad-\frac{1-\alpha}{4\left(1-e^{-\frac{\varrho}{2}}\right)}\left[\begin{array}{l}\mathcal{I}^{\alpha}_{a^+}g\left(\frac{a+b}{2}\right)+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(a)\\+\mathcal{I}^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(b)+\mathcal{I}^{\alpha}_{b^-}g\left(\frac{a+b}{2}\right)\end{array}\right].\end{aligned}$$

By means of Lemma 4.1, we derive the following multipoint-based fractional integral inequality.

Theorem 4.1: Assume that all conditions in Lemma 4.1 are satisfied. For $\alpha \in (0,1) \setminus \{\frac{b-a}{n+b-a}\}$, if the function $|g''|$ is exponential type convex on $[a,b]$, then the following inequality is valid

$$\begin{aligned}&|\Xi(g;[a,b],n)|\\&\leq\sum_{i=0}^{n-1}\frac{(b-a)^2\Delta_1(\varrho)}{2n^3\varrho(1-e^{-\varrho})}\left[|g''(\omega(i))|+|g''(\omega(i+1))|\right],\end{aligned} \quad (17)$$

where

$$\begin{aligned}\Delta_1(\varrho)&=\frac{1}{\varrho\left(1-\varrho^2\right)}\\&\quad\times\left[\begin{array}{l}\left(2\varrho^3+\varrho^2-\varrho-2\right)e^{-\varrho}\\-\left(\varrho^3+\varrho^2\right)e^{1-\varrho}-\left(\varrho^3-\varrho^2\right)e\\+2\varrho^3-\varrho^2-\varrho+2\end{array}\right],\varrho\neq 0,1.\end{aligned}$$

Proof: From Lemma 4.1, we deduce that

$$\begin{aligned}&|\Xi(g;[a,b],n)|\\&\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\\&\quad\times\int_0^1\left[\begin{array}{l}\left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right|\\\times\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|\end{array}\right]\mathrm{d}t.\end{aligned} \quad (18)$$

Since $1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)} \geq 0$ for any $t \in [0,1]$, and the function $|g''|$ is exponential type convex on $[a,b]$, we acquire that

$$\begin{aligned}&\int_0^1\left[\begin{array}{l}\left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right|\\\times\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|\end{array}\right]\mathrm{d}t\\&\leq\int_0^1\left\{\begin{array}{l}\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\\\times\left[\begin{array}{l}\left(e^t-1\right)|g''(\omega(i))|\\+\left(e^{1-t}-1\right)|g''(\omega(i+1))|\end{array}\right]\end{array}\right\}\mathrm{d}t.\end{aligned} \quad (19)$$

Direct computation deduces that

$$\begin{aligned}&\int_0^1\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\left(e^t-1\right)\mathrm{d}t\\&=\int_0^1\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\left(e^{1-t}-1\right)\mathrm{d}t\\&=\frac{1}{\varrho\left(1-\varrho^2\right)}\\&\quad\times\left[\begin{array}{l}\left(2\varrho^3+\varrho^2-\varrho-2\right)e^{-\varrho}-\left(\varrho^3+\varrho^2\right)e^{1-\varrho}\\-\left(\varrho^3-\varrho^2\right)e+2\varrho^3-\varrho^2-\varrho+2\end{array}\right].\end{aligned} \quad (20)$$

A combination of the inequalities (18), (19) and identity (20) yields the desired result. Thus, the proof is done here. ■

Remark 4.2: In Theorem 4.1, if we take $\alpha \to 1$, i.e., $\varrho = \frac{(b-a)(1-\alpha)}{n\alpha} \to 0$, then we have that

$$\lim_{\varrho\to 0}\frac{\Delta_1(\varrho)}{\varrho(1-e^{-\varrho})}=\frac{17}{6}-e. \quad (21)$$

Using the results (15) and (21), the inequality (17) is transformed to

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\frac{1}{2n}\left[g(\omega(i))+g(\omega(i+1))\right]-\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right|\\&\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3}\left(\frac{17}{6}-e\right)\left[|g''(\omega(i))|+|g''(\omega(i+1))|\right],\end{aligned}$$

which is recorded by [35, Theorem 3.1].

Corollary 4.2: Consider Theorem 4.1, we can get the following results.
(1) For $n = 1$, we have the subsequent fractional trapezoid-type inequality:

$$|\Xi(g;[a,b],1)|\leq\frac{(b-a)^2\Delta_1(\rho)}{2\rho(1-e^{-\rho})}\left[|g''(a)|+|g''(b)|\right].$$

(2) For $n = 2$, we have the coming fractional Bullen-type inequality:

$$\begin{aligned}&|\Xi(g;[a,b],2)|\\&\leq\frac{(b-a)^2\Delta_1\left(\frac{\rho}{2}\right)}{8\rho\left(1-e^{-\frac{\rho}{2}}\right)}\left[|g''(a)|+2\left|g''\left(\frac{a+b}{2}\right)\right|+|g''(b)|\right],\end{aligned}$$

where $\Delta_1(\cdot)$ is defined in Theorem 4.1.

To illustrate the result of Theorem 4.1 more intuitively, we provide an example with graphs.

Example 4.1: Consider the function $g(x) = x^4 + x^2$ on the interval $(-\infty,+\infty)$, the corresponding $|g''(x)| = 12x^2 + 2$ is nonnegative convex. Due to the fact that every nonnegative convex function is exponential type convex function, see [38], we know that the function $|g''(x)|$ is also exponential type convex. If we take $a = 0, b = 1, n = 1$, $\alpha \in (0,1) \setminus \{0.5\}$, then all hypotheses in Theorem 4.1 are met. Obviously, we have $\varrho = \frac{(b-a)(1-\alpha)}{n\alpha} = \frac{1-\alpha}{\alpha}$. The left part of the inequality (17) can be recorded as

$$\begin{aligned}|L_1(\alpha)|&=\left|\begin{array}{l}1-\frac{1-\alpha}{2\alpha\left(1-e^{-\frac{1-\alpha}{\alpha}}\right)}\\\times\int_0^1\left(e^{-\frac{1-\alpha}{\alpha}(1-x)}+e^{-\frac{1-\alpha}{\alpha}x}\right)\left(x^4+x^2\right)\mathrm{d}x\end{array}\right|\\&=\left|\begin{array}{l}\frac{1}{(1-\alpha)^4\left(1-e^{-\frac{1-\alpha}{\alpha}}\right)}\\\times\left[\begin{array}{l}\left(17\alpha^4+5\alpha^3-\alpha^2+3\alpha\right)e^{-\frac{1-\alpha}{\alpha}}\\-47\alpha^4+37\alpha^3-17\alpha^2+3\alpha\end{array}\right]\end{array}\right|.\end{aligned}$$

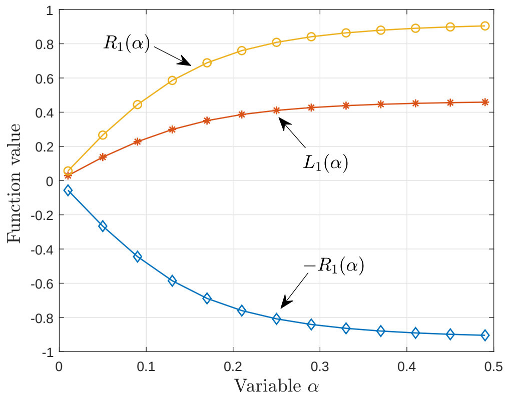
The right part of the inequality (17) can be written as

$$R_1(\alpha)=\frac{8\alpha}{(1-\alpha)\left(1-e^{-\frac{1-\alpha}{\alpha}}\right)}\Delta_1\left(\frac{1-\alpha}{\alpha}\right).$$

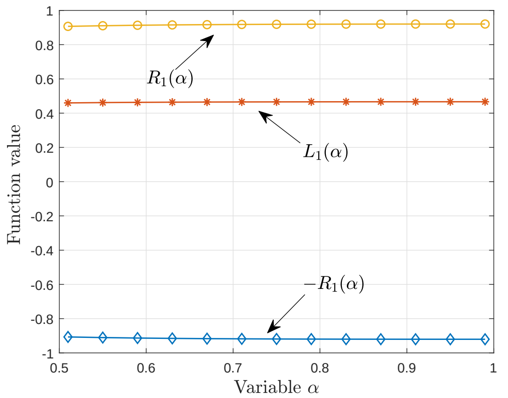
For variable $\alpha \in (0,0.5) \cup (0.5,1)$, we plot the graphical depiction of the functions $L_1(\alpha)$, $R_1(\alpha)$ and $-R_1(\alpha)$ in Fig. 1, respectively. It is not laborious to observe that $-R_1(\alpha) < L_1(\alpha) < R_1(\alpha)$, which agrees with the outcome stated in Theorem 4.1.

With the aid of the Hölder's integral inequality, we propose the succeeding theorem.

Theorem 4.2: Assume that all conditions in Lemma 4.1 are satisfied. For $p,q \in (1,\infty)$ with $p^{-1}+q^{-1}=1$, if the function $|g''|^q$ is exponential type convex on $[a,b]$, then one



(a)



(b)

Fig. 1: An example to the inequality (17) in Theorem 4.1

can attain the following inequality

$$\begin{aligned}&|\Xi(g;[a,b],n)|\\ &\leq \sum_{i=0}^{n-1}\frac{(b-a)^2(e-2)^{\frac{1}{q}}}{2n^3}\left(\Delta(\varrho,p)\right)^{\frac{1}{p}}\\ &\quad\times\left[|g''(\omega(i))|^q+|g''(\omega(i+1))|^q\right]^{\frac{1}{q}}, \qquad (22)\end{aligned}$$

where

$$\Delta(\varrho,p)=\int_0^1\left(\frac{1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}}{\varrho\left(1-e^{-\varrho}\right)}\right)^p\mathrm{d}t,\ \ \varrho\neq 0.$$

Proof: From Lemma 4.1 and the Hölder's integral inequality, we deduce that

$$\begin{aligned}&|\Xi(g;[a,b],n)|\\ &\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\\ &\quad\times\int_0^1\left[\begin{array}{l}\left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right|\\ \times\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|\end{array}\right]\mathrm{d}t\\ &\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\\ &\quad\times\left(\int_0^1\left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^1\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}. \qquad (23)\end{aligned}$$

In accordance with the exponential-type convexity of $|g''|^q$, we get that

$$\begin{aligned}&\int_0^1\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|^q\mathrm{d}t\\ &\leq|g''(\omega(i))|^q\int_0^1\left(e^t-1\right)\mathrm{d}t\\ &\quad+|g''(\omega(i+1))|^q\int_0^1\left(e^{1-t}-1\right)\mathrm{d}t\\ &=(e-2)\left[|g''(\omega(i))|^q+|g''(\omega(i+1))|^q\right]. \qquad (24)\end{aligned}$$

Noticing that $1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\geq 0$ for any $t\in[0,1]$, and applying the inequality (24) to the inequality (23), we obtain the desired outcome. Thus, the proof is completed. ■

Remark 4.3: In Theorem 4.2, if we take $\alpha\to 1$, i.e., $\varrho=\frac{(b-a)(1-\alpha)}{n\alpha}\to 0$, using the results (15) and (16), then the inequality (22) turns into

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\frac{1}{2n}\left[g(\omega(i))+g(\omega(i+1))\right]-\frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right|\\ &\leq\sum_{i=0}^{n-1}\frac{(b-a)^2(e-2)^{\frac{1}{q}}}{2n^3}\beta^{\frac{1}{p}}(p+1,p+1)\\ &\quad\times\left[|g''(\omega(i))|^q+|g''(\omega(i+1))|^q\right]^{\frac{1}{q}},\end{aligned}$$

which is given by [35, Theorem 3.4].

We present the succedent theorem, which is related to the power-mean integral inequality.

Theorem 4.3: Assume that all conditions in Lemma 4.1 are satisfied. For $\alpha\in(0,1)\setminus\{\frac{b-a}{n+b-a}\}$, if the function $|g''|^q$ is exponential type convex with $q\in(1,+\infty)$, then the following inequality is given

$$\begin{aligned}&|\Xi(g;[a,b],n)|\\ &\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\left(\Delta_2(\varrho)\right)^{1-\frac{1}{q}}\left(\Delta_1(\varrho)\right)^{\frac{1}{q}}\\ &\quad\times\left[|g''(\omega(i))|^q+|g''(\omega(i+1))|^q\right]^{\frac{1}{q}}, \qquad (25)\end{aligned}$$

where

$$\Delta_2(\varrho)=\frac{\varrho+\varrho e^{-\varrho}+2e^{-\varrho}-2}{\varrho},\ \ \varrho\neq 0,$$

and $\Delta_1(\varrho)$ is the same as in Theorem 4.1.

Proof: Utilizing Lemma 4.1, the power-mean integral inequality, and the exponential-type convexity of $|g''|^q$, we have that

$$\begin{aligned}&|\Xi(g;[a,b],n)|\\ &\leq\sum_{i=0}^{n-1}\frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\\ &\quad\times\int_0^1\left(\begin{array}{l}\left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right|\\ \times\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|\end{array}\right)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})} \\
&\quad \times \left(\int_0^1 \left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right| \mathrm{d}t\right)^{1-\frac{1}{q}} \\
&\quad \times \left[\int_0^1 \left(\begin{array}{l} \left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right| \\ \times\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right|^q \end{array}\right) \mathrm{d}t\right]^{\frac{1}{q}} \\
&\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})} \\
&\quad \times \left\{\int_0^1 \left[\begin{array}{l} \left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \\ \times \left(\begin{array}{l} \left(e^{1-t}-1\right)|g''(\omega(i+1))|^q \\ +\left(e^t-1\right)|g''(\omega(i))|^q \end{array}\right) \end{array}\right] \mathrm{d}t\right\}^{\frac{1}{q}} \\
&\quad \times \left[\int_0^1 \left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \mathrm{d}t\right]^{1-\frac{1}{q}}. \qquad (26)
\end{aligned}
$$

Direct computation yields that

$$
\begin{aligned}
&\int_0^1 \left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \mathrm{d}t \\
&= \frac{\varrho+\varrho e^{-\varrho}+2e^{-\varrho}-2}{\varrho}. \qquad (27)
\end{aligned}
$$

Making use of the identities (20) and (27) in the inequality (26), we obtain the desired outcome. Thus, the proof is accomplished. ■

Remark 4.4: In Theorem 4.3, if we take $\alpha \to 1$, i.e., $\varrho = \frac{(b-a)(1-\alpha)}{n\alpha} \to 0$, then we have that

$$
\lim_{\varrho \to 0} \frac{\Delta_2(\varrho)}{\varrho(1-e^{-\varrho})} = \frac{1}{6}. \qquad (28)
$$

Using the results (15), (21) and (28), the inequality (25) is transformed to

$$
\begin{aligned}
&\left|\sum_{i=0}^{n-1} \frac{1}{2n}\big[g(\omega(i))+g(\omega(i+1))\big] - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x\right| \\
&\leq \sum_{i=0}^{n-1} \frac{6^{\frac{1}{q}-1}(b-a)^2}{2n^3}\left(\frac{17}{6}-e\right)^{\frac{1}{q}} \\
&\quad \times \left(|g''(\omega(i))|^q+|g''(\omega(i+1))|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

which is presented by [35, Theorem 3.7].

Corollary 4.3: Consider Theorem 4.3, we can get the following results.

(1) For $n=1$, we have the succeeding fractional trapezoid-type inequality:

$$
\begin{aligned}
&|\Xi(g;[a,b],1)| \\
&\leq \frac{(b-a)^2}{2\rho(1-e^{-\rho})}\left(\Delta_2(\rho)\right)^{1-\frac{1}{q}}\left(\Delta_1(\rho)\right)^{\frac{1}{q}} \\
&\quad \times \left[|g''(a)|^q+|g''(b)|^q\right]^{\frac{1}{q}}.
\end{aligned}
$$

(2) For $n=2$, we have the subsequent fractional Bullen-type inequality:

$$
\begin{aligned}
&|\Xi(g;[a,b],2)| \\
&\leq \frac{(b-a)^2}{8\rho\left(1-e^{-\frac{\rho}{2}}\right)}\left(\Delta_2\left(\frac{\rho}{2}\right)\right)^{1-\frac{1}{q}}\left(\Delta_1\left(\frac{\rho}{2}\right)\right)^{\frac{1}{q}} \\
&\quad \times \left[\begin{array}{l} \left(|g''(a)|^q+\left|g''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\ +\left(\left|g''\left(\frac{a+b}{2}\right)\right|^q+|g''(b)|^q\right)^{\frac{1}{q}} \end{array}\right],
\end{aligned}
$$

where $\Delta_1(\cdot)$ and $\Delta_2(\cdot)$ are the same as in Theorems 4.1 and 4.3, respectively.

With the help of the improved power-mean integral inequality, the following theorem is given.

Theorem 4.4: Assume that all conditions in Lemma 4.1 are satisfied. For $\alpha \in (0,1)\backslash\{\frac{b-a}{n+b-a}\}$, if $|g''|^q$ is exponential type convex with $q>1$, then the following inequality holds

$$
\begin{aligned}
&|\Xi(g;[a,b],n)| \\
&\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\left(\frac{\Delta_2(\varrho)}{2}\right)^{1-\frac{1}{q}} \\
&\quad \times \left\{\begin{array}{l} \left[\begin{array}{l} \Delta_3(\varrho)|g''(\omega(i))|^q \\ +\Delta_4(\varrho)|g''(\omega(i+1))|^q \end{array}\right]^{\frac{1}{q}} \\ +\left[\begin{array}{l} \Delta_4(\varrho)|g''(\omega(i))|^q \\ +\Delta_3(\varrho)|g''(\omega(i+1))|^q \end{array}\right]^{\frac{1}{q}} \end{array}\right\}, \qquad (29)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Delta_3(\varrho) \\
&= \frac{1}{\varrho(\varrho-1)^2(\varrho+1)^2} \\
&\quad \times \left[\begin{array}{l} \left(-\frac{5}{2}\varrho^5+5\varrho^3-\varrho^2-\frac{1}{2}\varrho-1\right)e^{-\varrho} \\ +\left(\varrho^5-3\varrho^3-2\varrho^2\right)e^{1-\varrho} \\ +\left(\varrho^5-3\varrho^3+2\varrho^2\right)e \\ -\frac{5}{2}\varrho^5+5\varrho^3+\varrho^2-\frac{1}{2}\varrho+1 \end{array}\right], \varrho \neq 0,1,
\end{aligned}
$$

$$
\begin{aligned}
&\Delta_4(\varrho) \\
&= \frac{1}{\varrho(\varrho-1)^2(\varrho+1)^2} \\
&\quad \times \left[\begin{array}{l} \left(\frac{1}{2}\varrho^5-\varrho^4-2\varrho^3+4\varrho^2-\frac{1}{2}\varrho-1\right)e^{-\varrho} \\ +\left(\varrho^4+2\varrho^3+\varrho^2\right)e^{1-\varrho} \\ -\left(\varrho^4-2\varrho^3+\varrho^2\right)e \\ +\frac{1}{2}\varrho^5+\varrho^4-2\varrho^3-4\varrho^2-\frac{1}{2}\varrho+1 \end{array}\right], \varrho \neq 0,1,
\end{aligned}
$$

and $\Delta_2(\varrho)$ is identical to that in Theorem 4.3.

Proof: In the light of Lemma 4.1, the improved power-mean integral inequality, and the exponential-type convexity of $|g''|^q$, we have that

$$
\begin{aligned}
&|\Xi(g;[a,b],n)| \\
&\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})} \\
&\quad \times \int_0^1 \left[\begin{array}{l} \left|1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right| \\ \times\left|g''\big(t\omega(i)+(1-t)\omega(i+1)\big)\right| \end{array}\right] \mathrm{d}t
\end{aligned}
$$

$$\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}$$
$$\times \left\{ \left[\int_0^1 (1-t)\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\mathrm{d}t \right]^{1-\frac{1}{q}} \right.$$
$$\times \left[\int_0^1 \begin{pmatrix} (1-t)\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \\ \times\left|g''\left(t\omega(i)+(1-t)\omega(i+1)\right)\right|^q \end{pmatrix} \mathrm{d}t \right]^{\frac{1}{q}}$$
$$+ \left[\int_0^1 t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\mathrm{d}t \right]^{1-\frac{1}{q}}$$
$$\left. \times \left[\int_0^1 \begin{pmatrix} t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \\ \times\left|g''\left(t\omega(i)+(1-t)\omega(i+1)\right)\right|^q \end{pmatrix} \mathrm{d}t \right]^{\frac{1}{q}} \right\}$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}$$
$$\times \left\{ \left[\int_0^1 (1-t)\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\mathrm{d}t \right]^{1-\frac{1}{q}} \right.$$
$$\times \left[\int_0^1 \begin{pmatrix} (1-t) \\ \times\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \\ \times \begin{pmatrix} \left(e^{1-t}-1\right) \\ \times|g''(\omega(i+1))|^q \\ +\left(e^t-1\right)|g''(\omega(i))|^q \end{pmatrix} \end{pmatrix} \mathrm{d}t \right]^{\frac{1}{q}}$$
$$+ \left[\int_0^1 \begin{pmatrix} t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right) \\ \times \begin{pmatrix} \left(e^{1-t}-1\right) \\ \times|g''(\omega(i+1))|^q \\ +\left(e^t-1\right)|g''(\omega(i))|^q \end{pmatrix} \end{pmatrix} \mathrm{d}t \right]^{\frac{1}{q}}$$
$$\left. \times \left[\int_0^1 t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\mathrm{d}t \right]^{1-\frac{1}{q}} \right\}. \quad (30)$$

Direct computations yield that

$$\int_0^1 (1-t)\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\mathrm{d}t$$
$$= \int_0^1 t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\mathrm{d}t$$
$$= \frac{\varrho+\varrho e^{-\varrho}+2e^{-\varrho}-2}{2\varrho}, \quad (31)$$

$$\int_0^1 (1-t)\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\left(e^t-1\right)\mathrm{d}t$$
$$= \int_0^1 t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\left(e^{1-t}-1\right)\mathrm{d}t$$
$$= \frac{1}{\varrho(\varrho-1)^2(\varrho+1)^2}$$
$$\times \begin{bmatrix} \left(-\frac{5}{2}\varrho^5+5\varrho^3-\varrho^2-\frac{1}{2}\varrho-1\right)e^{-\varrho} \\ +\left(\varrho^5-3\varrho^3-2\varrho^2\right)e^{1-\varrho} \\ +\left(\varrho^5-3\varrho^3+2\varrho^2\right)e \\ -\frac{5}{2}\varrho^5+5\varrho^3+\varrho^2-\frac{1}{2}\varrho+1 \end{bmatrix}, \quad (32)$$

and

$$\int_0^1 (1-t)\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\left(e^{1-t}-1\right)\mathrm{d}t$$
$$= \int_0^1 t\left(1+e^{-\varrho}-e^{-\varrho t}-e^{-\varrho(1-t)}\right)\left(e^t-1\right)\mathrm{d}t$$
$$= \frac{1}{\varrho(\varrho-1)^2(\varrho+1)^2}$$
$$\times \begin{bmatrix} \left(\frac{1}{2}\varrho^5-\varrho^4-2\varrho^3+4\varrho^2-\frac{1}{2}\varrho-1\right)e^{-\varrho} \\ +\left(\varrho^4+2\varrho^3+\varrho^2\right)e^{1-\varrho} \\ -\left(\varrho^4-2\varrho^3+\varrho^2\right)e \\ +\frac{1}{2}\varrho^5+\varrho^4-2\varrho^3-4\varrho^2-\frac{1}{2}\varrho+1 \end{bmatrix}. \quad (33)$$

Submitting identities (31), (32) and (33) to the inequality (30), we obtain the desired result. Thus, the proof is accomplished. ■

Remark 4.5: In Theorem 4.4, if we take $\alpha \to 1$, i.e., $\varrho = \frac{(b-a)(1-\alpha)}{n\alpha} \to 0$, then we have that

$$\lim_{\alpha\to 1} \frac{\Delta_3(\varrho)}{\varrho(1-e^{-\varrho})} = \frac{131}{12} - 4e, \quad (34)$$

and

$$\lim_{\alpha\to 1} \frac{\Delta_4(\varrho)}{\varrho(1-e^{-\varrho})} = 3e - \frac{97}{12}. \quad (35)$$

Using the results (15), (28), (34) and (35), the inequality (29) changes into

$$\left| \sum_{i=0}^{n-1} \frac{1}{2n}\left[g(\omega(i))+g(\omega(i+1))\right] - \frac{1}{b-a}\int_a^b g(x)\mathrm{d}x \right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3}\left(\frac{1}{12}\right)^{1-\frac{1}{q}}$$
$$\times \left\{ \begin{array}{c} \begin{bmatrix} \left(\frac{131}{12}-4e\right)|g''(\omega(i))|^q \\ +\left(3e-\frac{97}{12}\right)|g''(\omega(i+1))|^q \end{bmatrix}^{\frac{1}{q}} \\ +\begin{bmatrix} \left(3e-\frac{97}{12}\right)|g''(\omega(i))|^q \\ +\left(\frac{131}{12}-4e\right)|g''(\omega(i+1))|^q \end{bmatrix}^{\frac{1}{q}} \end{array} \right\},$$

which is provided by [35, Theorem 3.14].

Remark 4.6: The upper bound of the inequality (29) is superior to that of the inequality (25). In fact, by using concavity of the function $h:[0,+\infty)\to\mathbb{R}, h(x)=x^s, 0<s\leq 1$, we can write the right hand-side of the inequality (29) as follows:

$$\sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\left(\frac{\Delta_2(\varrho)}{2}\right)^{1-\frac{1}{q}}$$
$$\times \left\{ \begin{array}{c} \begin{bmatrix} \Delta_3(\varrho)|g''(\omega(i))|^q \\ +\Delta_4(\varrho)|g''(\omega(i+1))|^q \end{bmatrix}^{\frac{1}{q}} \\ +\begin{bmatrix} \Delta_4(\varrho)|g''(\omega(i))|^q \\ +\Delta_3(\varrho)|g''(\omega(i+1))|^q \end{bmatrix}^{\frac{1}{q}} \end{array} \right\}$$
$$\leq 2\sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\left(\frac{\Delta_2(\varrho)}{2}\right)^{1-\frac{1}{q}}$$
$$\times \begin{bmatrix} \frac{\Delta_3(\varrho)+\Delta_4(\varrho)}{2}|g''(\omega(i))|^q \\ +\frac{\Delta_3(\varrho)+\Delta_4(\varrho)}{2}|g''(\omega(i+1))|^q \end{bmatrix}^{\frac{1}{q}}$$
$$= \sum_{i=0}^{n-1} \frac{(b-a)^2}{2n^3\varrho(1-e^{-\varrho})}\left(\Delta_2(\varrho)\right)^{1-\frac{1}{q}}\left(\Delta_1(\varrho)\right)^{\frac{1}{q}}$$
$$\times \left[|g''(\omega(i))|^q+|g''(\omega(i+1))|^q\right]^{\frac{1}{q}}, \quad (36)$$

where we use the following fact that

$$\Delta_3(\varrho)+\Delta_4(\varrho)=\Delta_1(\varrho).$$

For a more intuitive comparison of the obtained bounds in Theorems 4.3 and 4.4, we supply an example with tables here.

Example 4.2: Consider the function $g(x) = q^2 e^{\frac{x}{q}}$ on $(-\infty, +\infty)$ for $q \in (1, +\infty)$, the corresponding $|g''(x)|^q = e^x$ is nonnegative convex, and the function $|g''(x)|^q$ is also exponential type convex. If we take $a = 0, b = 1, n = 1, q = 2$ and $\alpha \in (0,1)\backslash\{0.5\}$, then all hypotheses in Theorems 4.3 and 4.4 are met. Obviously, we have $\varrho = \frac{(b-a)(1-\alpha)}{n\alpha} = \frac{1-\alpha}{\alpha}$. Owing to the fact that the left-hand side of the inequality (25) is the same as that of the inequality (29), it can be transformed into

$$\begin{aligned}
&|L_2(\alpha)| \\
&= \Bigg|2 + 2e^{\frac{1}{2}} - \frac{2(1-\alpha)}{\alpha\left(1 - e^{-\frac{1-\alpha}{\alpha}}\right)} \\
&\quad \times \int_0^1 \left(e^{-\frac{1-\alpha}{\alpha}(1-x)} + e^{-\frac{1-\alpha}{\alpha}x}\right) e^{\frac{x}{2}} \mathrm{d}x\Bigg| \\
&= \Bigg|2 + 2e^{\frac{1}{2}} - \frac{4(1-\alpha)}{(3\alpha-2)(2-\alpha)\left(1 - e^{-\frac{1-\alpha}{\alpha}}\right)} \\
&\quad \times \left[(2-\alpha)\left(e^{\frac{3\alpha-2}{2\alpha}} - 1\right) + (3\alpha-2)\left(e^{\frac{1}{2}} - e^{-\frac{1-\alpha}{\alpha}}\right)\right]\Bigg|,
\end{aligned}$$

where $\alpha \neq \frac{2}{3}$. The right-hand side of the inequality (25) in Theorem 4.3 can be recorded as

$$\begin{aligned}
R_2(\alpha) = &\frac{(1+e)^{\frac{1}{2}}\alpha}{2(1-\alpha)\left(1 - e^{-\frac{1-\alpha}{\alpha}}\right)} \\
&\times \left[\Delta_2\left(\frac{1-\alpha}{\alpha}\right)\right]^{\frac{1}{2}} \left[\Delta_1\left(\frac{1-\alpha}{\alpha}\right)\right]^{\frac{1}{2}}.
\end{aligned}$$

The right-hand side of the inequality (29) in Theorem 4.4 can be written as

$$\begin{aligned}
R_3(\alpha) = &\frac{\alpha}{2(1-\alpha)\left(1 - e^{-\frac{1-\alpha}{\alpha}}\right)} \left(\frac{\Delta_2\left(\frac{1-\alpha}{\alpha}\right)}{2}\right)^{\frac{1}{2}} \\
&\times \left\{ \begin{array}{c} \left[\Delta_3\left(\frac{1-\alpha}{\alpha}\right) + e\Delta_4\left(\frac{1-\alpha}{\alpha}\right)\right]^{\frac{1}{2}} \\ + \left[\Delta_4\left(\frac{1-\alpha}{\alpha}\right) + e\Delta_3\left(\frac{1-\alpha}{\alpha}\right)\right]^{\frac{1}{2}} \end{array} \right\}.
\end{aligned}$$

If we take $\alpha = \frac{1}{4}$, then we have that $\left|L_2\left(\frac{1}{4}\right)\right| \approx 0.094371$, $\left|R_2\left(\frac{1}{4}\right)\right| \approx 0.117126$, and $\left|R_3\left(\frac{1}{4}\right)\right| \approx 0.116926$. Clearly, $0.094371 < 0.116926 < 0.117126$, which shows that the bound provided by the inequality (29) is better than that given by the inequality (25).

When the parameter α takes different values, we have the following numerical results.

TABLE I: Numerical comparison of Theorem 4.3 and Theorem 4.4 for $\alpha \in (0, 0.5)$

values of α	values of $\lvert L_2(\alpha)\rvert$	values of $R_2(\alpha)$ in Theorem 4.3	values of $R_3(\alpha)$ in Theorem 4.4
0.1	0.055920	0.069685	0.069527
0.2	0.086816	0.107816	0.107623
0.3	0.099032	0.122868	0.122664
0.4	0.103846	0.128797	0.128589

From Tables I and II, we can clearly see that the values on the left are less than the values on the right, which corresponds to the theoretical results given in Theorems 4.3 and 4.4. In particular, we observe that the values of $R_3(\alpha)$ are smaller than the values of $R_2(\alpha)$, indicating that the inequality (29) in Theorem 4.4 gives a better approximate estimate than the inequality (25) in Theorem 4.3.

TABLE II: Numerical comparison of Theorem 4.3 and Theorem 4.4 for $\alpha \in (0.5, 1)$

values of α	values of $\lvert L_2(\alpha)\rvert$	values of $R_2(\alpha)$ in Theorem 4.3	values of $R_3(\alpha)$ in Theorem 4.4
0.6	0.106885	0.132539	0.132328
0.7	0.107345	0.133106	0.132895
0.8	0.107561	0.133372	0.133160
0.9	0.107650	0.133482	0.133271

For functions whose second derivatives are bounded, we obtain the another estimative result below.

Theorem 4.5: Assuming that every condition involved in Lemma 4.1 is satisfied. If the function g'' is bounded, i.e., $m \leq g''(u) \leq M$ for all $u \in [a,b]$ with $m, M \in \mathbb{R}$, then one can receive the following inequality

$$\begin{aligned}
\frac{m(b-a)^2}{2n^2\varrho(1-e^{-\varrho})}\Delta_2(\varrho) &\leq \Xi(g;[a,b],n) \\
&\leq \frac{M(b-a)^2}{2n^2\varrho(1-e^{-\varrho})}\Delta_2(\varrho), \qquad (37)
\end{aligned}$$

where $\Delta_2(\varrho)$ is the same as in Theorem 4.3.

Proof: By using the change of variables, we have that

$$\begin{aligned}
&\frac{1-\alpha}{1-e^{-\varrho}}\left[\mathcal{I}^{\alpha}_{\omega(i)^+}g(\omega(i+1)) + \mathcal{I}^{\alpha}_{\omega(i+1)^-}g(\omega(i))\right] \\
&= \frac{1-\alpha}{\alpha\left(1-e^{-\varrho}\right)} \\
&\quad \times \int_{\omega(i)}^{\omega(i+1)} \left(e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)} + e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))}\right) g(x)\mathrm{d}x \\
&= \frac{1-\alpha}{\alpha\left(1-e^{-\varrho}\right)} \\
&\quad \times \int_{\omega(i)}^{\omega(i+1)} \left[\begin{array}{l} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\ \times g\big(\omega(i) + \omega(i+1) - x\big) \end{array}\right] \mathrm{d}x \\
&= \frac{1-\alpha}{2\alpha\left(1-e^{-\varrho}\right)} \\
&\quad \times \int_{\omega(i)}^{\omega(i+1)} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\
&\quad \times \left[g(x) + g\big(\omega(i) + \omega(i+1) - x\big)\right]\mathrm{d}x. \qquad (38)
\end{aligned}$$

Employing the identity transformation, we obtain that

$$\begin{aligned}
&g(\omega(i)) + g(\omega(i+1)) \\
&\quad - \frac{1-\alpha}{1-e^{-\varrho}}\left[\mathcal{I}^{\alpha}_{\omega(i)^+}g(\omega(i+1)) + \mathcal{I}^{\alpha}_{\omega(i+1)^-}g(\omega(i))\right] \\
&= g(\omega(i)) + g(\omega(i+1)) - \frac{1-\alpha}{2\alpha\left(1-e^{-\varrho}\right)} \\
&\quad \times \int_{\omega(i)}^{\omega(i+1)} \left[\begin{array}{l} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\ \times \left[g(x) + g\big(\omega(i) + \omega(i+1) - x\big)\right] \end{array}\right] \mathrm{d}x \\
&= -\frac{1-\alpha}{2\alpha\left(1-e^{-\varrho}\right)} \\
&\quad \times \int_{\omega(i)}^{\omega(i+1)} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\
&\quad \times \left[\begin{array}{l} g(x) + g\big(\omega(i) + \omega(i+1) - x\big) \\ -g(\omega(i)) - g(\omega(i+1)) \end{array}\right] \mathrm{d}x, \qquad (39)
\end{aligned}$$

where we utilize the following fact that

$$\int_{\omega(i)}^{\omega(i+1)} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \mathrm{d}x = \frac{2\alpha\left(1-e^{-\varrho}\right)}{1-\alpha}.$$

Since $m \leq g''(u) \leq M, u \in [a,b]$, for $\tau \in [\omega(i), \omega(i+1)]$, we obtain that

$$\int_{\tau}^{\omega(i)+\omega(i+1)-\tau} m\mathrm{d}u \leq \int_{\tau}^{\omega(i)+\omega(i+1)-\tau} g''(u)\mathrm{d}u \leq \int_{\tau}^{\omega(i)+\omega(i+1)-\tau} M\mathrm{d}u,$$

which yields that

$$\begin{aligned} &m\big(\omega(i)+\omega(i+1)-2\tau\big) \\ &\leq g'\big(\omega(i)+\omega(i+1)-\tau\big) - g'(\tau) \\ &\leq M\big(\omega(i)+\omega(i+1)-2\tau\big). \end{aligned} \tag{40}$$

Integrating the inequality (40) with respect to τ on $[\omega(i), x]$, we obtain that

$$\begin{aligned} &\int_{\omega(i)}^{x} m\big(\omega(i)+\omega(i+1)-2\tau\big)\mathrm{d}\tau \\ &\leq \int_{\omega(i)}^{x} \left[g'\big(\omega(i)+\omega(i+1)-\tau\big) - g'(\tau)\right]\mathrm{d}\tau \\ &\leq \int_{\omega(i)}^{x} M\big(\omega(i)+\omega(i+1)-2\tau\big)\mathrm{d}\tau, \end{aligned}$$

that is,

$$\begin{aligned} &M\big(\omega(i)-x\big)\big(\omega(i+1)-x\big) \\ &\leq g(x) + g\big(\omega(i)+\omega(i+1)-x\big) - g(\omega(i)) - g(\omega(i+1)) \\ &\leq m\big(\omega(i)-x\big)\big(\omega(i+1)-x\big). \end{aligned} \tag{41}$$

Multiplying the inequality (41) by $-\frac{1-\alpha}{2\alpha(1-e^{-\varrho})}\left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right)$ and integrating the resultant inequality with respect to x on $[\omega(i), \omega(i+1)]$, we have that

$$\begin{aligned} &-\frac{m(1-\alpha)}{2\alpha\left(1-e^{-\varrho}\right)} \\ &\times \int_{\omega(i)}^{\omega(i+1)} \left[\begin{array}{c} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\ \times\big(\omega(i)-x\big)\big(\omega(i+1)-x\big) \end{array}\right]\mathrm{d}x \\ &\leq -\frac{1-\alpha}{2\alpha\left(1-e^{-\varrho}\right)} \\ &\times \int_{\omega(i)}^{\omega(i+1)} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\ &\times \left[\begin{array}{c} g(x) + g\big(\omega(i)+\omega(i+1)-x\big) \\ -g(\omega(i)) - g(\omega(i+1)) \end{array}\right]\mathrm{d}x \\ &\leq -\frac{M(1-\alpha)}{2\alpha\left(1-e^{-\varrho}\right)} \\ &\times \int_{\omega(i)}^{\omega(i+1)} \left[\begin{array}{c} \left(\begin{array}{c} e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} \\ +e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)} \end{array}\right) \\ \times\big(\omega(i)-x\big)\big(\omega(i+1)-x\big) \end{array}\right]\mathrm{d}x. \end{aligned} \tag{42}$$

Direct complication yields that,

$$\begin{aligned} &\int_{\omega(i)}^{\omega(i+1)} \left[\begin{array}{c} \left(e^{-\frac{1-\alpha}{\alpha}(x-\omega(i))} + e^{-\frac{1-\alpha}{\alpha}(\omega(i+1)-x)}\right) \\ \times\big(\omega(i)-x\big)\big(\omega(i+1)-x\big) \end{array}\right]\mathrm{d}x \\ &= -\frac{2\alpha^2}{(1-\alpha)^2} \\ &\quad\times \left[\frac{b-a}{n}\left(1+e^{-\varrho}\right) + \frac{2\alpha}{1-\alpha}\left(e^{-\varrho}-1\right)\right]. \end{aligned} \tag{43}$$

Employing the identities (39) and (43) into the inequality (42), we obtain that

$$\begin{aligned} &\frac{m(b-a)^2}{n^2} \cdot \frac{\varrho + \varrho e^{-\varrho} + 2e^{-\varrho} - 2}{\varrho^2(1-e^{-\varrho})} \\ &\leq g(\omega(i)) + g(\omega(i+1)) \\ &\quad - \frac{1-\alpha}{1-e^{-\varrho}}\left[\mathcal{I}^{\alpha}_{\omega(i)^+} g(\omega(i+1)) + \mathcal{I}^{\alpha}_{\omega(i+1)^-} g(\omega(i))\right] \\ &\leq \frac{M(b-a)^2}{n^2} \cdot \frac{\varrho + \varrho e^{-\varrho} + 2e^{-\varrho} - 2}{\varrho^2(1-e^{-\varrho})}. \end{aligned}$$

Consequently, we have that

$$\begin{aligned} &\frac{m(b-a)^2}{2n^2} \cdot \frac{\varrho + \varrho e^{-\varrho} + 2e^{-\varrho} - 2}{\varrho^2(1-e^{-\varrho})} \\ &\leq \sum_{i=0}^{n-1} \frac{1}{2n}\left[g(\omega(i)) + g(\omega(i+1)) - \frac{1-\alpha}{1-e^{-\varrho}}\Psi_g(\mathcal{I};\alpha,i)\right] \\ &\leq \frac{M(b-a)^2}{2n^2} \cdot \frac{\varrho + \varrho e^{-\varrho} + 2e^{-\varrho} - 2}{\varrho^2(1-e^{-\varrho})}. \end{aligned}$$

This completes the proof. ■

For displaying the result of Theorem 4.5 more intuitively, we offer an example here.

Example 4.3: Let us consider the function $g : [a,b] = [1,2] \to \mathbb{R}$ defined by $g(x) = \frac{1}{(\ln 2)^2}\left(\frac{1}{2}\right)^x$. Then, the function $g''(x) = \left(\frac{1}{2}\right)^x$ is bounded on $[1,2]$ with $m = \frac{1}{4}$ and $M = \frac{1}{2}$. In Theorem 4.5, for $n = 1$ and $\alpha \in (0,1)$, we have $\varrho = \frac{(b-a)(1-\alpha)}{n\alpha} = \frac{1-\alpha}{\alpha}$. The left side of the double inequality (37) can be written as

$$\begin{aligned} \Omega_1(\alpha) &= \frac{m(b-a)^2}{2n^2} \cdot \frac{\varrho + \varrho e^{-\varrho} + 2e^{-\varrho} - 2}{\varrho^2(1-e^{-\varrho})} \\ &= \frac{\left(\alpha^2+\alpha\right)e^{-\frac{1-\alpha}{\alpha}} - 3\alpha^2 + \alpha}{8(1-\alpha)^2\left(1-e^{-\frac{1-\alpha}{\alpha}}\right)}. \end{aligned}$$

The right side of the double inequality (37) is as follows:

$$\begin{aligned} \Omega_2(\alpha) &= \frac{M(b-a)^2}{2n^2} \cdot \frac{\varrho + \varrho e^{-\varrho} + 2e^{-\varrho} - 2}{\varrho^2(1-e^{-\varrho})} \\ &= \frac{\left(\alpha^2+\alpha\right)e^{-\frac{1-\alpha}{\alpha}} - 3\alpha^2 + \alpha}{4(1-\alpha)^2\left(1-e^{-\frac{1-\alpha}{\alpha}}\right)}. \end{aligned}$$

The middle part of the inequality (37) can be recorded as

$$\begin{aligned} \Omega_3(\alpha) &= \frac{3}{8(\ln 2)^2} - \frac{1-\alpha}{2(\ln 2)^2\alpha\left(1-e^{-\frac{1-\alpha}{\alpha}}\right)} \\ &\quad\times \int_1^2 \left(e^{-\frac{1-\alpha}{\alpha}(2-x)} + e^{-\frac{1-\alpha}{\alpha}(x-1)}\right)\left(\frac{1}{2}\right)^x \mathrm{d}x. \end{aligned}$$

In Fig. 2, we draw functions $\Omega_1(\alpha)$, $\Omega_2(\alpha)$ and $\Omega_3(\alpha)$ for $\alpha \in (0,1)$, respectively. It is obvious that $\Omega_1(\alpha) < \Omega_3(\alpha) < \Omega_2(\alpha)$, which is consistent with the result given in Theorem 4.5.

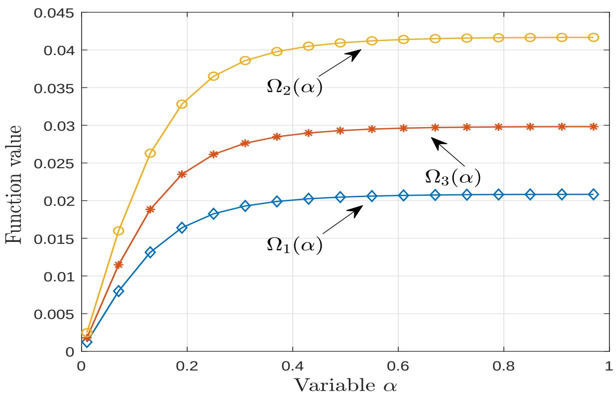


Fig. 2: An example to the inequality (37) in Theorem 4.5

V. Conclusions

In the setting of fractional integrals with exponential kernels, the HH-type inequalities for exponential type convex functions are presented here. And then, a fractional integral identity by separating $[a, b]$ to n equal subintervals is proved, from which we construct multipoint-based HH inequalities in the case of twice-differentiable functions.

Following the ideas and methods presented in this paper, the researchers can address similar inequalities using other types of fractional integrals, like the k-fractional integrals with exponential kernels [41], and interval-valued fractional double integrals with exponential kernels [42], which is a new direction in future researches.

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