Connectivity Remainder in Bipolar Fuzzy Graphs and Some Related Problems

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***Abstract*—The fuzzy membership function has non-collapsability and can be used to characterize uncertainties that cannot be explored by sudden events. As the membership function does not satisfy the complementarity law, a single membership function cannot simultaneously characterize the positive and negative aspects of things. This article considers structured fuzzy data with negative uncertainty information, which is modeled using bipolar fuzzy graphs. On this basis, we consider the connectivity remainder of the model and obtain the corresponding theoretical results. In addition, we have discussed some related fractional factor issues.**

***Index Terms*—bipolar fuzzy graph, connectivity remainder, fractional factor, embedding**

I. INTRODUCTION

PROBABILITY theory and fuzzy theory are two effective tools to deal with uncertain information. Their differences lie in: (1) probability function has collapsibility, but membership function (MF) has no collapsibility; (2) The probability function satisfies the complementary law, while the membership function does not satisfy the complementary law. The second feature requires that when considering practical problems, at least two MFs are required to depict the positive and negative effects of some uncertainty. For structured data with uncertainty, the bipolar fuzzy graph (BFG) is used to describe the positive and negative uncertainties of elements, as well as the positive and negative uncertainties associated relationship with elements. For recent advances in fuzzy graph and bipolar fuzzy graph, refer to Muhiuddin et al. [1], Li et al. [2] and [3], Gao et al. [4], Perumal [5], Ullah et al. [6], Gayathri et al. [7], Nie et al. [8], Jiang et al. [9] and Josy et al. [10].

Specifically, when considering fuzzy graphs which represent human relationship networks, where each vertex denotes a person and an edge between two people standard the "like" relation of each other. For instance, let v_1, v_2 and v_3 represents three persons who know each other (hence, there are edges between any two of them), and the values of membership function of their edges are stated by $(v_1v_2, 0.3)$, $(v_2v_3, 0.6)$ and $(v_3v_1, 0.9)$. On the other hand, if we define the edges in the fuzzy graph as "dislike" of each other, then whether the value of membership function of edges can be stated by $(v_1v_2, 0.7)$, $(v_2v_3, 0.4)$ and $(v_3v_1, 0.1)$? The answer is "No" since the membership function doesn't satisfy complementary low. Hence, in order to represent the "dislike" of edges between vertices, another membership function is necessary to be defined. As well as the edge set, the membership functions in vertex set should be introduced to express the positive and negative uncertainty of vertices. This is the main motivation we study the BFG instead of fuzzy graph (FG).

In recent years, the study of fuzzy graph theory has become a hotspot in graph theory and computer science, and many theoretical results have been applied to the field of chemistry. Islam and Pal [11] introduced edge *F*-index on FGs and application in molecular chemistry. Ganesan et al. [12] studied the strong domination integrity in fuzzy graphs. Khan et al. [13] modelled Cayley picture fuzzy graphs as interconnected networks. Das et al. [14] researched Picture fuzzy threshold graphs and applied them to medicine replenishment. Lu et al. [15] defined the cyclic connectivity index in bipolar fuzzy incidence graph and applied it in the study of anti-aging drugs.

Although there has been gratifying progress in BFGs, the characteristics of BFGs in most mathematical settings need to be further studied. Since connectivity is the basis of graph topology which determines the topological properties of the whole graph and the local properties between vertices, the study of connectivity enables us to further understand the construction of bipolar fuzzy graphs from a global perspective. This work aims to introduce connectivity remainder in bipolar fuzzy graph in terms of bipolar connectivity index, and study the characteristics of the new topological index from a theoretical prospect.

The subsequent of this paper is organized as follows. The setting of bipolar fuzzy graph is presented in the next section. The main contributions include new concepts and theorems are manifested in Section 3. Some related problems in fractional factors are showcased and we give the affirmative answer.

Manuscript received December 14, 2023; revised April 15, 2024. This work was supported in part by National Science Foundation of China (No. 12161094).

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II. Preliminary

This section aims to present the notations and terminologies in a BFG setting. Let $G=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ be a bipolar fuzzy graph with $(\mu_A^P,\mu_A^N):V\to[0,1]\times[-1,0]$, and $(\mu_B^P,\mu_B^N):V\times V\to$ $[0,1]\times[-1,0]$, where μ_B^P and μ_B^N are symmetry functions if *G* is an undirected FG. Moreover, $\mu_B^P(x_1x_2)\le\mu_A^P(x_1)\wedge\mu_A^P(x_2)$ and $\mu_B^N(x_1x_2)\ge\mu_A^N(x_1)$ $\vee\mu_A^N(x_2)$ for any $x_1,x_2\in V$. Set μ_A^* and μ_B^* as the vertex set and edge set of *G*, respectively. If x_1x_2 is not an edge in the crisp graph, then $\mu_B^P(x_1x_2)=\mu_B^N(x_1x_2)=0$. A bipolar fuzzy graph $G=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ is complete if $\mu_B^P(x_1x_2)=\mu_A^P(x_1)\wedge\mu_A^P(x_2)$ and $\mu_B^N(x_1x_2)=\mu_A^N(x_1)\vee\mu_A^N(x_2)$ for any $x_1,x_2\in V$. A bipolar fuzzy graph $H=(\nu_A^P,\nu_A^N,$ $\nu_B^P,\nu_B^N)$ is called a partial bipolar fuzzy subgraph (PBFS) of *G*, if $\nu_A^P(x)\le\mu_A^P(x)$ and $\nu_A^N(x)\ge\mu_A^N(x)$ for any $x\in V$, and $\nu_B^P(x_1x_2)\le\mu_B^P(x_1x_2)$ and $\nu_B^N(x_1x_2)\ge\mu_B^N(x_1x_2)$ for any $x_1x_2\in V\times V$. A PBFS with $\nu_A^P(x)\le\mu_A^P(x)$ and $\nu_A^N(x)\ge\mu_A^N(x)$ for any $x\in V$, and $\nu_B^P(x_1x_2)=\mu_B^P(x_1x_2)$ and $\nu_B^N(x_1x_2)=\mu_B^N(x_1x_2)$ for any $x_1x_2\in V\times V$ is called a bipolar fuzzy subgraph of *G*. A path *P* of length *n* in bipolar fuzzy graph is a sequence of different vertices $x_0,x_1,\cdots,x_n$ with $\nu_B^P(x_{i-1}x_i)>0$ and $\nu_B^N(x_{i-1}x_i)<0$ for any $i\in\{1,\cdots,n\}$. A path is called a cycle if $x_0=x_n$. The positive strength (PS) and negative strength (NS) of *P* are denoted by

$$S^P(P)=\wedge_{i\in\{1,\cdots,n\}}\{\mu_B^P(x_{i-1}x_i)\},$$

$$S^N(P)=\vee_{i\in\{1,\cdots,n\}}\{\mu_B^N(x_{i-1}x_i)\}.$$

The PS and negative strength of connectedness between vertices *x* and *y* are formulated by

$$CONN_G^P(x,y)=\vee\{S^P(P):P=x\cdots y\},$$

$$CONN_G^N(x,y)=\wedge\{S^N(P):P=x\cdots y\}.$$

If the PS (resp. NS) of *P* from *x* to *y* is exactly equal to $CONN_G^P(x,y)$ (res. $CONN_G^N(x,y)$), then we call *P* as a positive strongest (resp. negative strongest) $x-y$ path. An edge xy is

- positive α-strong if $\mu_B^P(xy)>CONN_{G-xy}^P(x,y)$;
- negative α-strong if $\mu_B^N(xy)<CONN_{G-xy}^N(x,y)$;
- bipolar α-strong if $\mu_B^P(xy)>CONN_{G-xy}^P(x,y)$ and $\mu_B^N(xy)<CONN_{G-xy}^N(x,y)$;
- positive β-strong if $\mu_B^P(xy)=CONN_{G-xy}^P(x,y)$;
- negative β-strong if $\mu_B^N(xy)=CONN_{G-xy}^N(x,y)$;
- bipolar β-strong if $\mu_B^P(xy)=CONN_{G-xy}^P(x,y)$ and $\mu_B^N(xy)=CONN_{G-xy}^N(x,y)$;
- positive δ-edge if $\mu_B^P(xy)<CONN_{G-xy}^P(x,y)$;
- negative δ-edge if $\mu_B^N(xy)>CONN_{G-xy}^N(x,y)$;
- bipolar δ-edge if $\mu_B^P(xy)<CONN_{G-xy}^P(x,y)$ and $\mu_B^N(xy)>CONN_{G-xy}^N(x,y)$.

An edge xy is called a positive fuzzy bridge (PFB) (resp. negative fuzzy bridge, NFB) if deleting xy from *G* reduces (resp. increases) the PS (resp. NS) of connectedness between a certain pair of vertices in *G*. An edge xy is called a bipolar fuzzy bridge if it is both a PFB and a NFB. A vertex $x\in\mu_A^*$ is called a positive fuzzy cut-vertex (PFCV) (resp. negative fuzzy cut-vertex, NFCV) of *G* if removing *x* will decrease (resp. increase) the positive strength (resp. NS) of connectedness between a certain pair of vertices. A vertex $x\in\mu_A^*$ is called a bipolar fuzzy cut-vertex if it is both PFVC and NFCV. A connected BFG $G=(\mu_A^P,\mu_A^N,$ $\mu_B^P,\mu_B^N)$ is a bipolar fuzzy tree (BFT) if it has a bipolar fuzzy spanning subgraph $F=(\mu_A^P,\mu_A^N,\nu_B^P,\nu_B^N)$ which is a tree, and for all edges *xy* not in *F*, there exists a path between *x* and *y* in *F*, whose PS is more than $\mu_B^P(xy)$ and negative strength is less than $\mu_B^N(xy)$.

A BFG $G=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ is called

- positive α-saturated if every vertex $x\in\mu_A^*$ incident at least one positive α-strong edge;
- negative α-saturated if every vertex $x\in\mu_A^*$ incident at least one negative α-strong edge;
- bipolar α-saturated if every vertex $x\in\mu_A^*$ incident at least one bipolar α-strong edge;
- positive β-saturated if every vertex $x\in\mu_A^*$ incident at least one positive β-strong edge;
- negative β-saturated if every vertex $x\in\mu_A^*$ incident at least one negative β-strong edge;
- bipolar β-saturated if every vertex $x\in\mu_A^*$ incident at least one bipolar β-strong edge;
- positive saturated if it is both positive α-saturated and positive β-saturated;
- negative saturated if it is both negative α-saturated and negative β-saturated;
- bipolar saturated if it is both bipolar α-saturated and bipolar β-saturated.

III. Bipolar connectivity remainder and main results

As illustrated above, $CONN_G^P(x,y)$ and $CONN_G^N(x,y)$ represent the maximum positive feasible resource and minimum negative feasible resource transformed between vertices *x* and *y* in a transform network (it transforms data, service, energy or other kinds of resources) which is modelled by a bipolar fuzzy graph *G*. However, in many real-world applications, the network produces some degree of congestion due to the conflicts between the small capacity of channels and the large amount of resources that need to be transformed. On the other hand, an overly amount of resources leads to the crash of the whole network because of the limited storage of each vertex (measured by μ_A^P and μ_A^N). Thus, it is necessary to characterize the amount of overload from both positive and negative aspects, so as to serve as a reference to improve the performance of the entire network. For this purpose, we extend the connectivity remainder [16] to a BFG setting.

Definition 1. Let $G=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ be a BFG. The bipolar connectivity remainder (BCR) of *G* is defined by $CR(G)=(CR^P(G),CR^N(G))$, where

$$CR^P(G)=\sum_{x,y\in\mu_A^*}(\wedge\{\mu_A^P(x),\mu_A^P(y)\}-CONN_G^P(x,y))$$

and

$$CR^N(G)=\sum_{x,y\in\mu_A^*}(\vee\{\mu_A^N(x),\mu_A^N(y)\}-CONN_G^N(x,y)),$$

denote the positive connectivity remainder and negative connectivity remainder, respectively.

According to its definition, we directly get that if $H=(\mu_A^P,\mu_A^N,\nu_B^P,\nu_B^N)$ is a PBFS of $G=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ with the same vertex set, then $CR^P(G)\le CR^P(H)$ and $CR^N(G)\ge CR^N(H)$. Furthermore, the vertices in *G* are arranged as follows according to the value of the positive membership function: $x_1,\cdots,x_n$ such that $0<\mu_A^P(x_1)\le\cdots\le\mu_A^P(x_n)$, then

$$0\le CR^P(G)\le\sum_{i=1}^{n-1}(n-i)\mu_A^P(x_i).$$

More exactly, by simply computing, we have

$$CR^P(G)=\sum_{i=1}^{n-1}(n-i)\mu_A^P(x_i)-\sum_{x,y\in\mu_A^*}CONN_G^P(x,y). \quad (1)$$

On the other hand, re-arrange the vertices by $y_1,\cdots,y_n$ such that $0>\mu_A^N(y_1)\ge\cdots\ge\mu_A^N(y_n)$, then

$$0\ge CR^N(G)\ge\sum_{i=1}^{n-1}(n-i)\mu_A^N(y_i).$$

The following equation is obtained using the similar fashion as (1):

$$CR^N(G)=\sum_{i=1}^{n-1}(n-i)\mu_A^N(y_i) \quad (2)$$
$$-\sum_{x,y\in\mu_A^*}CONN_G^N(x,y).$$

The equations (1) and (2) can be employed in special settings of positive membership function values or negative membership function values. For instance, if $\mu_A^P(x_i)=\mu_A^P(x_{i-1})+\varepsilon^P$ for vertex order $x_1,\cdots,x_n$, where $i\in\{2,\cdots,n\}$ and $\varepsilon^P\in(0,\frac{1}{n-1})$, then in view of (1), we deduce

$$CR^P(G)=\frac{n(n-1)(3\mu_A^P(x_1)+\varepsilon^P n)}{6}-\sum_{x,y\in\mu_A^*}CONN_G^P(x,y).$$

If $\mu_A^N(y_i)=\mu_A^N(y_{i-1})+\varepsilon^N$ for vertex order $y_1,\cdots,y_n$, where $i\in\{2,\cdots,n\}$ and $\varepsilon^N\in(-\frac{1}{n-1},0)$, then in terms of (2), we get

$$CR^N(G)=\frac{n(n-1)(3\mu_A^N(y_1)+\varepsilon^N n)}{6}-\sum_{x,y\in\mu_A^*}CONN_G^N(x,y).$$

An edge $e\in\mu_B^*$ is

- positive CR-reducing if $CR^P(G-e)<CR^P(G)$;
- positive CR-increasing if $CR^P(G-e)>CR^P(G)$;
- positive CR-neutral if $CR^P(G-e)=CR^P(G)$;
- negative CR-reducing if $CR^N(G-e)<CR^N(G)$;
- negative CR-increasing if $CR^N(G-e)>CR^N(G)$;
- negative CR-neutral if $CR^N(G-e)=CR^N(G)$.

In light of the definitions stated above, we immediately obtain the following truths: an edge $e\in\mu_B^*$ of BFG *G* is

- positive CR-increasing $\Leftrightarrow$ *e* is positive α-strong;
- negative CR-decreasing $\Leftrightarrow$ *e* is negative α-strong;
- positive CR-neutral $\Leftrightarrow$ *e* is positive β-strong or positive δ-edge;
- negative CR-neutral $\Leftrightarrow$ *e* is negative β-strong or negative δ-edge.

Moreover, we confirm that the crisp graph of BFG *G* is a tree $\Leftrightarrow$ no edge of *G* is positive CR-neutral or negative CR-neutral.

For vertex $x\in\mu_A^*$, we call it a

- positive remainder reducing vertex if $$CR^P(G-x)<CR^P(G);$$
- positive remainder enhancing vertex if $$CR^P(G-x)>CR^P(G);$$
- positive remainder inactive vertex if $$CR^P(G-x)=CR^P(G);$$
- negative remainder reducing vertex if

$$CR^N(G-x) < CR^N(G);$$

- negative remainder enhancing vertex if
$$CR^N(G-x) > CR^N(G);$$
- negative remainder inactive vertex if
$$CR^N(G-x) = CR^N(G).$$

In terms of the definition of isomorphic of bipolar fuzzy graphs, we know that if two bipolar fuzzy graphs G_1 and G_2 satisfying $G_1 \cong G_2$, then $CR^P(G_1) = CR^P(G_2)$ and $CR^N(G_1) = CR^N(G_2)$. Set $0 < w_1 \le w_2 \le \cdots \le w_m \le 1$, then

$$CR^P(G^{w_1}) \le CR^P(G^{w_2}) \le \cdots \le CR^P(G^{w_m}),$$
$$CR^N(G^{w_1}) \ge CR^N(G^{w_2}) \ge \cdots \ge CR^N(G^{w_m}).$$

The following theorem characterizes the BCR of the BFG by deleting a special vertex, which extends Theorem 4.1 in [16].

Theorem 1. If for any two vertices x and y, there is a positive strongest path (resp. negative strongest path) connecting them with z as a non-internal vertex, then $CR^P(G-z) \le CR^P(G)$ (resp. $CR^N(G-z) \ge CR^N(G)$).

Proof of Theorem 1. We only check the negative arguement using the same trick manifested in [16]. For any $y \in \mu_A^*$, we infer $\sum_{x\in\mu_A^*} (\vee\{\mu_A^N(x), \mu_A^N(z)\} - CONN_G^N(x,y)) \le 0$.

According to the assumption, z is not internal in any negative strongest path, we acquire

$$\begin{aligned} & CR^N(G-\{z\}) \\ &= \sum_{x,y\in\mu_A^*-\{z\}} (\vee\{\mu_A^N(x), \mu_A^N(y)\} - CONN_{G-\{z\}}^N(x,y)) \\ &= \sum_{x,y\in\mu_A^*-\{z\}} (\vee\{\mu_A^N(x), \mu_A^N(y)\} - CONN_G^N(x,y)). \end{aligned}$$

Furthermore, we get

$$\begin{aligned} & CR^N(G) \\ &= \sum_{x,y\in\mu_A^*-\{z\}} (\vee\{\mu_A^N(x), \mu_A^N(y)\} - CONN_G^N(x,y)) \\ &\quad + \sum_{x\in\mu_A^*} (\vee\{\mu_A^N(x), \mu_A^N(z)\} - CONN_G^N(x,z)) \\ &= CR^N(G-\{z\}) + \sum_{x\in\mu_A^*} (\vee\{\mu_A^N(x), \mu_A^N(z)\} \\ &\quad - CONN_G^N(x,z)) \\ &\le CR^N(G-\{z\}). \end{aligned}$$

□

Unfortunately, the reverse of Theorem 1 is not established in general, which can refer to Figure 6 in [16] for more details. Moreover, the following statements are deduced from Theorem 1.

- If x is a terminal vertex (with degree 1 in its crisp graph), then $CR^P(G-x) \le CR^P(G)$ and $CR^N(G-z) \ge CR^N(G)$.
- If G is a bipolar saturated fuzzy cycle, then we have $CR^P(G-x) \le CR^P(G)$ and $CR^N(G-z) \ge CR^N(G)$ for any $x \in \mu_A^*$.

Theorem 2. Let $G = (\mu_A^P, \mu_A^N, \mu_B^P, \mu_B^N)$ be a BFG with $n = |\mu_A^*| \ge 3$. Assume $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$ for any $x \in \mu_A^*$. For a given vertex $z \in \mu_A^*$, set

$$\Delta_z^P = \sum_{x,y\in\mu_A^*} CONN_G^P(x,y) - \sum_{x,y\in\mu_A^*-\{z\}} CONN_{G-\{z\}}^P(x,y),$$
$$\Delta_z^N = \sum_{x,y\in\mu_A^*} CONN_G^N(x,y) - \sum_{x,y\in\mu_A^*-\{z\}} CONN_{G-\{z\}}^N(x,y).$$

Then we have the following statements:

- z is a positive remainder reducing vertex $\Leftrightarrow \Delta_z^P < n-1$;
- z is a positive remainder enhancing vertex $\Leftrightarrow \Delta_z^P > n-1$;
- z is a positive remainder inactive vertex $\Leftrightarrow \Delta_z^P = n-1$;
- z is a negative remainder reducing vertex $\Leftrightarrow \Delta_z^N < -n+1$;
- z is a negative remainder enhancing vertex $\Leftrightarrow \Delta_z^N > -n+1$;
- z is a negative remainder inactive vertex $\Leftrightarrow \Delta_z^N = -n+1$.

Proof of Theorem 2. We only check the negative remainder reducing vertex part, and the rest parts can be verified in view of the same fashion.

Let z be a negative remainder reducing vertex. Then, we yield

$$\begin{aligned} & CR^N(G-z) < CR^N(G) \\ \Leftrightarrow & \sum_{x,y\in\mu_A^*-\{z\}} (\vee\{\mu_A^N(x), \mu_A^N(y)\} - CONN_{G-\{z\}}^N(x,y)) \\ & \quad < \sum_{x,y\in\mu_A^*} (\vee\{\mu_A^N(x), \mu_A^N(y)\} - CONN_G^N(x,y)) \\ \Leftrightarrow & \sum_{x,y\in\mu_A^*-\{z\}} (-1 - CONN_{G-\{z\}}^N(x,y)) \\ & \quad < \sum_{x,y\in\mu_A^*} (-1 - CONN_G^N(x,y)) \\ \Leftrightarrow & -\binom{n-1}{2} - \sum_{x,y\in\mu_A^*-\{z\}} (CONN_{G-\{z\}}^N(x,y)) \\ & \quad < -\binom{n}{2} - \sum_{x,y\in\mu_A^*} (CONN_G^N(x,y)) \\ \Leftrightarrow & \sum_{x,y\in\mu_A^*} (CONN_G^N(x,y)) - \sum_{x,y\in\mu_A^*-\{z\}} (CONN_{G-\{z\}}^N(x,y)) \\ & \quad < -n+1 \\ \Leftrightarrow & \Delta_z^N < -n+1. \end{aligned}$$

□

Using the principle of Theorem 2, it is not hard to check the following statements (assume $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$ for any $x \in \mu_A^*$).

- All vertices in a complete BFG are both positive remainder inactive and negative remainder inactive.
- No vertex in G is a positive remainder enhancing vertex (resp. negative remainder enhancing vertex);
- If there is a positive remainder inactive vertex (resp. negative remainder inactive vertex) in G, then all the resting

vertices are positive remainder inactive vertices (resp. negative remainder inactive vertices).

For the bipolar fuzzy cycle (BFC), we have the following properties.

Theorem 3. Let $C_n=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ be a bipolar fuzzy cycle with $\mu_A^P(x)=1$ and $\mu_A^N(x)=-1$ for any $x\in\mu_A^*$. We have the following statements:

(i)If C_n is positive β -saturated (resp. negative β -saturated) and $\mu_B^P(e)=\beta^P$ (resp. $\mu_B^N(e)=\beta^N$) for any $e\in\mu_B^*$, then $CR^P(C_n)=\binom{n}{2}(1-\beta^P)$ (resp. $CR^N(C_n)=\binom{n}{2}(-1-\beta^N)$).

(ii) If C_n is positive saturated (resp. negative saturated) with $\alpha_1^P,\alpha_2^P,\cdots,\alpha_{n/2}^P$ (resp. $\alpha_1^N,\alpha_2^N,\cdots,\alpha_{n/2}^N$) as strengths of positive α -strong (resp. negative α -strong) edges, and all positive β -strong (resp. negative β -strong) edges share the same strength β^P (resp. β^N), then we have $CR^P(C_n)=\frac{n}{2}[n(1-\beta^P)+2\beta^P-1]-\sum_{i=1}^{n/2}\alpha_i^P$ (resp. $CR^N(C_n)=\frac{n}{2}[n(-1-\beta^N)+2\beta^N+1]-\sum_{i=1}^{n/2}\alpha_i^N$). Furthermore, if $\alpha_1^P=\alpha_2^P=\cdots=\alpha_{n/2}^P=\alpha^P$ (resp. $\alpha_1^N=\alpha_2^N=\cdots=\alpha_{n/2}^N=\alpha^N$), then $CR^P(C_n)=\frac{n}{2}[n(1-\alpha^P-\beta^P)+2\beta^P-1]$ (resp. $CR^N(C_n)=\frac{n}{2}[n(1-\alpha^N-\beta^N)+2\beta^N-1]$).

(iii) If C_n contains *m* positive α -strong (resp. negative α -strong) edges with strengths $\alpha_1^P,\alpha_2^P,\cdots,\alpha_m^P$ (resp. $\alpha_1^N,\alpha_2^N,\cdots,\alpha_m^N$), and all positive β -strong (resp. negative β -strong) edges share the common strength β^P (resp. β^N), then $CR^P(C_n)\le\binom{n}{2}(1-\beta^P)+m\beta^P-\sum_{i=1}^{m}\alpha_m^P$ (resp. $CR^N(C_n)\ge\binom{n}{2}(-1-\beta^N)+m\beta^N-\sum_{i=1}^{m}\alpha_m^N$).

(iv) $CR^P(C_n)=CR^P(C_n-xy)$ (resp. $CR^N(C_n)=CR^N(C_n-xy)$) for each $xy\in\mu_B^*$ $\Leftrightarrow$ C_n is positive β -saturated (resp. negative β -saturated) and μ_B^P (resp. μ_B^N) is a constant function.

(v) For a positive real number r^P (resp. negative real number r^N), there exists a BFC C_n such that $CR^P(C_n)=r^P$ (resp. $CR^N(C_n)=r^N$).

Proof of Theorem 3. Since most of the results can be directly verified, we only check some negative parts of them.

To prove the negative part of (iii), denote E^N as the set of negative α -strong edges in C_n . We get $\sum_{xy\in E^N}CONN_G^N(x,y)=\sum_{i=1}^{m}\alpha_i^N$, $CONN_G^N(x,y)\le\beta^N$ where $x,y\in\mu_A^*$ and $xy\notin E^N$, and hence

$$\begin{aligned}CR^N(C_n)&=-\binom{n}{2}-(\sum_{x,y\in\mu_A^*,xy\notin E^N}CONN_G^N(x,y)\\&\quad+\sum_{xy\in E^N}CONN_G^N(x,y))\\&\ge\binom{n}{2}(-1-\beta^N)+m\beta^N-\sum_{i=1}^{m}\alpha_m^N .\end{aligned}$$

To prove the negative part of (v), let r^N be any negative real number. Select an integer *n* with $-\binom{n}{2}<2r^N$, the bipolar fuzzy cycle C_n can be constructed as follows: $\mu_A^P(x)=1$ and $\mu_A^N(x)=-1$ for any $x\in\mu_A^*$. $\mu_B^P(xy)=1-\frac{2r^P}{n(n-1)}$ and $\mu_B^N(xy)=-1-\frac{2r^N}{n(n-1)}$ for any $xy\in\mu_B^*$. Hence, $CONN_G^P(x,y)=1-\frac{2r^P}{n(n-1)}$ and $CONN_G^N(x,y)=-1-\frac{2r^N}{n(n-1)}$ for any $x,y\in\mu_A^*$. The negative connectivity remainder is calculated by

$$\begin{aligned}&CR^N(C_n)\\&=\sum_{x,y\in\mu_A^*}(\vee\{\mu_A^N(x),\mu_A^N(y)\}-CONN_G^N(x,y))\\&=-\binom{n}{2}-\sum_{x,y\in\mu_A^*}(CONN_G^N(x,y))\\&=-\binom{n}{2}-\binom{n}{2}(-1-\frac{2r^N}{n(n-1)})=r^N .\end{aligned}$$ □

The last result in this section is related to the BFT.

Theorem 4. Let $T=(\mu_A^P,\mu_A^N,\mu_B^P,\mu_B^N)$ be a bipolar fuzzy tree with $\mu_A^P(x)=1$ and $\mu_A^N(x)=-1$ for any $x\in\mu_A^*$. Let $\alpha_{\min}^P$, $\alpha_{\max}^P$, $\alpha_{\min}^N$, $\alpha_{\max}^N$ be the minimum positive strength of all positive α -strong edges in *T*, maximum positive strength of all positive α -strong edges in *T*, minimum negative strength of all negative α -strong edges in *T*, maximum negative strength of all negative α -strong edges in *T*, respectively. Then, we have

$$\binom{n}{2}(1-\alpha^P_{\max}) \le CR^P(C_n) \le \binom{n}{2}(1-\alpha^P_{\min}),$$
$$\binom{n}{2}(-1-\alpha^N_{\max}) \le CR^N(C_n) \le \binom{n}{2}(-1-\alpha^N_{\min}).$$

In what follows, we say a path is positive (negative) strong if it has only positive strong edges (PSEs) (negative strong edges NSEs). $V' \subset V$ is called a positive vertex cut (PVC) (resp. negative vertex cut, NVC) if either $G-V'$ is trivial or $CONN^P_{G-V'}(v,v') < CONN^P_G(v,v')$ (resp. $CONN^N_{G-V'}(v,v') < CONN^N_G(v,v')$) for some *v* and *v'*. $E' \subset E$ is called a positive edge cut (PEC) (resp. negative edge cut, NEC) if either $G-E'$ is disconnected or $CONN^P_{G-E'}(v,v') < CONN^P_G(v,v')$ (resp. $CONN^N_{G-E'}(v,v') < CONN^N_G(v,v')$) for some $v, v' \in V$. For a fuzzy vertex cut *V'* of *G*, the positive (resp. negative) sum $s^P(V') = \sum_{x\in V'} \min\{\mu^P_B(x,y) : xy$ is a PSE} (resp. $s^N(V') = \sum_{x\in V'} \max\{\mu^N_B(x,y) : xy$ is a NSE)), and if E' is a PEC (resp. NEC) of *G*, then its positive strong weight (PSW) (resp. negative strong weight, NSW) is denoted by $s^P(E') = \sum_{e\in E'} \mu^P_B(e)$ (resp. $s^N(E') = \sum_{e\in E'} \mu^N_B(e)$). The minimum of the strong weights of PVCs and PECs are the positive vertex connectivity and positive edge connectivity of *G* respectively, and the maximum of the negative weights of NVCs and NECs are the negative vertex connectivity and negative edge connectivity of *G* respectively. If a cycle C has more than one positive (resp. negative) weak edge, then we call it a positive (resp. negative) cycle.

Positive (resp. negative) cycle connectivity (CC) of a BFG *G* is the maximum (minimum) of the strengths of all positive (resp. negative) strong cycles in *G*. We call a vertex as positive (resp. negative) cyclic cutvertex if it's removing will reduce (increase) the positive (resp. negative) cycle connectivity of *G*, and an edge is referred as a positive (resp. negative) cyclic bridge if deleting it will decrease (increase) the positive (resp. negative) cycle connectivity. A bipolar fuzzy graph doesn't admit positive (resp. negative) cyclic cutvertices and positive (resp. negative) cyclic bridges is a positive (resp. negative) cyclically balanced graph. $V' \subseteq V$ is a positive (resp. negative) cyclic vertex cut if $CC^P(G-V') < CC^P(G)$ (resp. $CC^N(G-V') > CC^N(G)$) and $E' \subseteq E$ is a (positive) cyclic edge cut of *G* if $CC^P(G-E') < CC^P(G)$ (resp. $CC^N(G-E') > CC^N(G)$). A cycle is positive (resp. negative) strong if all its edges are positive (resp. negative) strong, and the positive (resp. negative) strength of a cycle is the value of its minimum (maximum) edge.

For two vertices *x* and *x'* in BFG *G*, the positive θ^P -evaluation and negative θ^N -evaluation of *x* and *x'* are denoted by

$$\theta^P(x,x') = \{\alpha^P : \alpha^P \text{ is the PS of a cycle containing } x \text{ and } x'\},$$
$$\theta^N(x,x') = \{\alpha^N : \alpha^N \text{ is the NS of a cycle containing } x \text{ and } x'\}.$$

If there is no cycle containing *x* and *x'*, then $\theta^P(x,x') = \theta^N(x,x') = \varnothing$ (and hence $C^P_G(x,x') = C^N_G(x,x') = 0$).

Generalized positive (resp. negative) cycle connectivity (GPCC, and resp. GNCC) between *x* and *x'* in a BFG *G* is given by

$$C^P_G(x,x') = \max\{\alpha^P : \alpha^P \in \theta^P(x,x'), x, x' \in V\},$$
$$C^N_G(x,x') = \min\{\alpha^N : \alpha^N \in \theta^N(x,x'), x, x' \in V\}.$$

The GPCC and GNCC of a BFG *G* are formulated by

$$C^P(G) = \max\{C^P_G(x,x') : x, x' \in V\},$$
$$C^N(G) = \min\{C^N_G(x,x') : x, x' \in V\}.$$

In the crisp graph, $C^P(G) = 1$ and $C^N(G) = -1$ if *G* is cyclic and $C^P(G) = C^N(G) = 0$ if *G* is a tree. For a BFG $G = (\mu^P_A, \mu^N_A, \mu^P_B, \mu^N_B)$, we have $CC^P(G) \le C^P(G)$ and $CC^N(G) \ge C^N(G)$. Let *H* be a PBFS of *G*, then $C^P(H) \le C^P(G)$ and $C^N(H) \ge C^N(G)$. If *G* is complete, then all cycles are positive and negative strong, hence $CC^P(G) = C^P(G) = m^P_{n-2}$ and $CC^N(G) = C^N(G) = m^N_3$ (assume $|V| = n$, we rank the value of $\mu^P_A(v)$ and $\mu^N_A(v)$ in increasing order by $\{m^P_1, \cdots, m^P_n\}$ and $\{m^N_1, \cdots, m^N_n\}$).

If *G* is a bipolar fuzzy tree. Then

$$C^P(G) = \begin{cases} 0, & D \text{ is a tree} \\ w^P_\delta, & \text{otherwise} \end{cases},$$
$$C^N(G) = \begin{cases} 0, & D \text{ is a tree} \\ w^N_\delta, & \text{otherwise} \end{cases}.$$

where w^P_δ (resp. w^N_δ) is the maximum (resp. minimum) MF value of positive (resp. negative) δ - edges in G. Since bipolar fuzzy cycle has no δ -edges, thus GPCC (resp. GNCC) and positive (resp. negative) cycle connectivity of a BFC are equal to the PS (resp. NS) of *G*. If two BFGs G_1 and G_2 are isomorphic, then $C^P(G_1) = C^P(G_2)$ and $C^N(G_1) = C^N(G_2)$. Suppose G' is a BFG obtained by deleting an edge from *G*, then *G* contains a unique cycle with largest (resp. smallest) positive (negative) strength if $C^P(G') < C^P(G)$ (resp. $C^N(G') > C^N(G)$).

A vertex *x* in BFG is called a positive (resp. negative) g -cyclic cutvertex (PgCC and NgCC) of *G* if $C^P(G-\{x\}) < C^P(G)$ (resp. $C^N(G-\{x\}) > C^N(G)$). A positive (resp. negative) g -cyclic bridge (PgCB and NgCB) of *G* is an edge *e* such that satisfying $C^P(G-e)<C^P(G)$ (resp. $C^N(G-e)>C^N(G)$). Note that a PgCC (resp. NgCC) may not be a positive (resp. negative) cyclic cutvertex and a PgCB (resp. NgCB) may not be a positive (resp. negative) cyclic bridge. Obviously, if $e=xx'$ is a PgCB (resp. NgCB) of a BFG *G*, then both *x* and x' are PgCC (resp. NgCC) of *G*. Moreover, all vertices and edges in a BFG are PgCC and NgCC, and PgCB and NgCB if crisp graph is a cycle. In fact, a vertex *x* in a BFG is a PgCC (resp. NgCC) if and only if *x* contains in all cycles with maximum PS (minimum NS).

Note that in complete bipolar fuzzy graph, all cycles are strong, the condition for the existence of PgCC (resp. NgCC) (or PgCB (resp. NgCB)) (that is, $m^P_{n-3}<m^P_{n-2}$ for positive and $m^N_4<m^N_3$) is the same as that of positive (resp. negative) CCs (or positive (resp. negative) CBs). Therefore, the corresponding PgCC (NgCC) (or PgCB (NGCB)) set is the same as the positive (resp. negative) CCs (or positive (resp. negative) CBs) set. That is, for complete BFG, the concept of positive cycle connectivity (PCC) (negative cycle connectivity, NCC) and positive (negative) g -cycle connectivity coincides.

If a BFG *G* contains exactly one cycle *C* of maximum PS (minimum negative strength). Let P be the set of positive (resp. negative) cyclic cutvertices and Q be the set of PgCC (NgCC) of *G*. Then $P=Q$ if and only if *C* is a positive (negative) strong cycle.

A BFG *G* is a positive cyclically stable (PCS) (resp. negative cyclically stable, NCS) if it lacks both PgCC (resp. NgCC) and PgCB (NgCB). Suppose *G* is not a tree, $V'\subseteq V$ is positive (resp. negative) g -cyclic vertex cut if $C^P(G-V')<C^P(G)$ (resp. $C^N(G-V')>C^N(G)$) (in this case, the positive and negative weight of V' are $S^P(V')=\sum_{x\in V'}\min\{\mu^P_B(xx'),\quad xx'\in E(G)\}$ and $S^N(V')=\sum_{x\in V'}\max\{\mu^N_B(xx'),xx'\in E(G)\}$). $E'\subseteq E$ is a positive (resp. negative) g -cyclic edge cut of *G* if $C^P(G-E')<C^P(G)$ (resp. $C^N(G-E')>C^N(G)$) (in this case, the PSW and NSW of E' are denoted by $\overline{S^P}(E')=\sum_{e_j\in E'}\mu^P_B(e_j)$ (e_j is a positive strong edge in E') and $\overline{S^N}(E')=\sum_{e_j\in E'}\mu^N_B(e_j)$ (e_j is a negative strong edge in E')).

The positive (resp. negative) g -cyclic vertex connectivity of *G* is denoted by $\kappa^P(G)$ (resp. $\kappa^N(G)$) is defined as the minimum (resp. maximum) of positive (resp. negative) strong weights of all positive (resp. negative) g -cyclic vertex cuts of *G*. The positive (resp. negative) g -cyclic edge connectivity $\overline{\kappa^P}(G)$ (resp. $\overline{\kappa^N}(G)$) is denoted as the minimum (resp. maximum) of the non-zero PSW (resp. NSW) of all positive (resp. negative) g -cyclic edge cuts of *G*. For a PBFS *H* of *G*, we have $\kappa^P(H)\le\kappa^P(G)$ and $\kappa^N(H)\ge\kappa^N(G)$.

Suppose, *G* is a complete BFG, we have $\kappa^P(G)=m^P_1$, $\kappa^N(G)=m^N_n$, $\kappa^P(G)\le\overline{\kappa^P}(G)\le\delta^P_s(G)$ and $\kappa^N(G)\ge\overline{\kappa^N}(G)\ge\delta^N_s(G)$. Actually, there always exist positive and negative g -cyclic end vertices in complete BFG. Furthermore, *G* is PCS (resp. NCS) if and only if $m^P_{n-3}=m^P_{n-2}$ (resp. $m^N_4=m^N_3$). Hence, a complete BFG *G* with at least four vertices is PCS (resp. NCS) if there exist a subgraph K_4 of *G*, where each cycle has the equal maximal PS (minimum NS).

When *G* is a BFT which is cyclic, let C be the only cycle of maximum PS (resp. minimum NS) in *G*, then

$$\kappa^P(G)=\overline{\kappa^P}(G)=\wedge\{\mu^P_B(xx'):xx' \text{ is a positive strong edge in } C\},$$

$$\kappa^N(G)=\overline{\kappa^N}(G)=\vee\{\mu^N_B(xx'):xx' \text{ is a negative strong edge in } C\}.$$

For a BFT *G* that is not cyclic, we have $\kappa^P(G)=\kappa^N(G)=0$.

IV. Related Problems

In this section, we discuss some related problems and give a theoretical analysis.

A. *Fractional Factors of Graphs on Surfaces*

Kawarabayashi and Ozeki [17] studied the existence of 2- and 3-factor in 4- and 5-connected graphs embedded in a surface with a small face-width (denoted by $fw(G)$). With the help of this pioneering contribution, Matsubara et al. [18] proved some results for a graph embedded in a surface to admit an [*a*,*b*]-factor (where *a*<*b*), and they also presented that parts of parameter bounds are sharp by counterexamples. Due to the equivalence of necessary and sufficient conditions for the existence of [*a*,*b*]-factor and fractional [*a*,*b*]-factor (FabF) in *a*<*b* setting (compared by Lovasz [19] and Zhang and Liu [20]), the theoretical conclusions presented in Matsubara et al. [18] also established for FabF if *a*<*b*. However, set *a*=*b* in [*a*,*b*]-factor and FabF, it is noteworthy that *k*-factor and fractional *k*-factor (F*k*F) have completely different characterizations. In what follows, we always assume that $a\le g(x)<f(x)\le b$ for all $x\in V(G)$ in (fractional) (*g*,*f*)-factor setting.

Motivated by such deficiency, we study the conditions for a graph embedded in a surface to have a fractional *k*-factor.

Theorem 5. Let $k\ge 3$ be an integer and *G* be a graph embedded in a surface of Euler genus (EG) *g*. If $\delta(G)\ge k+2$ and $fw(G)\ge\frac{(k+1)(2g-4)-2}{k-2}$, then *G* admits a F*k*F.

Proof. Assume that G meets all the conditions of Theorem 5 but has no fractional k-factor. There exist two disjoint subsets $S,T\subseteq V(G)$ which satisfy

$$k|S|+d_{G-S}(T)-k|T|\le -1 \quad (3)$$

Select S and T with minimum $|T|$. Thus, $T\ne\varnothing$ and $d_{G-S}(x)\le k-1$ for any $x\in T$.

Construct a bipartite graph H from G by setting $V(H)=S\cup T$ and $E(H)=E_G(S,T)$. Then $|V(H)|=|S|+|T|$ and $|E(H)|=e_G(S,T)$. In terms of Lemma 4 in [17] or Lemma 2.2 in [18], we infer

$$e_G(S,T)\le\begin{cases}2|S|+2|T|-2, & \text{if } H \text{ is flat,}\\ 2|S|+2|T|+2g-4, & \text{otherwise.}\end{cases} \quad (4)$$

Since $\delta(G)\ge k+2$, we determine

$$d_{G-S}(x)\ge (k+2)|T|-e_G(S,T). \quad (5)$$

In light of (3) and (5), we derive

$$e_G(S,T)\ge k|S|+2|T|+1.$$

Using (4), we get

$$(k-2)|S|\le\begin{cases}-3, & \text{if } H \text{ is flat,}\\ 2g-5, & \text{otherwise.}\end{cases} \quad (6)$$

If H is flat or $g\le 2$, then it leads to a contradiction. Thus, H is not flat and $g\ge 3$. By (5) and $d_{G-S}(x)\le k-1$ for any $x\in T$, we get $(k-1)|T|\ge d_{G-S}(T)\ge (k+2)|T|-e_G(S,T)$ and $e_G(S,T)\ge 3|T|$. Substituting this inequality into (4) yields $|T|\le 2|S|+2g-4$. Thus, by (6) and $k\ge 3$,

$$|S|+|T|\le 3|S|+2g-4\le\frac{(k+1)(2g-4)-3}{k-2}. \quad (7)$$

Since H is not flat, there is a non-contractible cycle C in H (in G), and a non-contractible curve γ with $\gamma\cap G\subseteq V(C)\subseteq V(H)=S\cup T$. By virtue of $S\cap T=\varnothing$ and (7), we have

$$|\gamma\cap G|\le|S\cup T|=|S|+|T|\le\frac{(k+1)(2g-4)-3}{k-2}.$$

It contradicts $fw(G)\ge\frac{(k+1)(2g-4)-2}{k-2}$. □

Using the same tricks, we get the following conclusion on the existence of fractional (g,f) -factor (FgfF) on surfaces.

Theorem 6. Let a,b be positive integers with $3\le a\le b$, and G be a graph embedded in a surface of EG g. If $\delta(G)\ge b+2$ and $fw(G)\ge\frac{(a+1)(2g-4)-2}{a-2}$, then G admits a FgfF.

Since the FgfF and (g,f)-factor have the same necessary and sufficient condition when $g<f$, we have the following result on the existence of (g,f)-factors on surfaces.

Theorem 7. Let a,b be positive integers with $3\le a<b$, and G be a graph embedded in a surface of EG g. If $\delta(G)\ge b+2$ and $fw(G)\ge\frac{(a+1)(2g-4)-2}{a-2}$, then G admits a (g,f)-factor.

B. *Difference Strategy of Choosing v' in I_2 in Gao et al. [21]*

This subsection aims to explain why the traditional trick can't be used to select I_2 in Gao et al. [21].

Suppose we select v' using the following principle:

$$v'=\arg\max_{v\in I_2}|N_{G-S}(v)|.$$

Let $j^*=d_{G-S}(v')$. We acquire

$$|N_{G-S}(I_2\setminus\{v'\})|\le\sum_{j=1}^{j^*-1}ji_j+j^*(i_{j^*}-1)$$
$$=\sum_{j=1}^{j^*}ji_j-j^*=\sum_{j=1}^{k-2}ji_j-j^*. \quad (8)$$

If $j^*\le k-2$, and

$$|N_{G-S}(I_2\setminus\{v'\})|\le\sum_{j=1}^{k-2}ji_j+(k-2)(i_{k-1}-1)$$
$$=\sum_{j=1}^{k-1}ji_j-i_{k-1}-k+2$$

if $j^*=k-1$ (i.e., $i_{k-1}\ge 1$).

In these circumstances, the proof of $i_{k-1}=0$ part when $|I_1|=0$ and $|T_0|+l\ge 1$ will become the following procedure:

If $i_{k-1}=0$, then in terms of (8), we verify

$$|U|\le|S|+|N_{G-S}(I_2\setminus\{v'\})|$$
$$\le|T_0|+l+\frac{\sum_{j=1}^{k-2}(j+1)(k-j)i_j}{k}+n-\frac{1}{k}+(\sum_{j=1}^{k-2}ji_j-j^*)$$
$$=|T_0|+l+n-\frac{1}{k}+\sum_{j=1}^{k-2}(-\frac{j^2}{k}+(2-\frac{1}{k})j+1)i_j-j^*.$$

Considering quadratic equation $\Theta(j)=-\frac{j^2}{k}+(2-\frac{1}{k})j+1$. We have $\max\{\Theta(j)\}=\Theta(j^*)$, and hence set

$$\Phi(j^*)=(-\frac{(j^*)^2}{k}+(2-\frac{1}{k})j^*+1)|I_2|-j^*$$
$$=-\frac{|I_2|}{k}(j^*)^2+((2-\frac{1}{k})|I_2|-1)j^*+|I_2|.$$

Symmetry axis of quadratic function $\Phi(j^*)$ is $j^*=\frac{(2k-1)|I_2|-k}{2|I_2|}$ which depends on the correlation between k and $|I_2|$. Since $j^*\in\{1,\cdots,k-2\}$, we infer

$$\max\{\Phi(j^*)\}=\begin{cases}\Phi(1), & \text{if } \frac{(2k-1)|I_2|-k}{2|I_2|}<1\\ \Phi(k-2), & \text{if } \frac{(2k-1)|I_2|-k}{2|I_2|}>k-2,\\ \Phi(\frac{(2k-1)|I_2|-k}{2|I_2|}), & \text{otherwise.}\end{cases}$$

If $\frac{(2k-1)|I_2|-k}{2|I_2|}<1$, then $|I_2|\le 0$, a contradiction. If $\frac{(2k-1)|I_2|-k}{2|I_2|}>k-2$, then

$$|U|\le|T_0|+l+n-\frac{1}{k}+(-\frac{1}{k}(k-2)^2$$
$$+(2-\frac{1}{k})(k-2)+1)|I_2|-(k-2)$$
$$\le|T_0|+l+1-\frac{1}{k}+(k-\frac{2}{k})|I_2|$$

and hence

$$t(G)\le\frac{|U|}{\omega(G-U)}$$
$$\le\frac{|T_0|+l+1-\frac{1}{k}+(k-\frac{2}{k})|I_2|}{|I_2|+|T_0|+l}$$
$$=1+\frac{1-\frac{1}{k}+(k-1-\frac{2}{k})|I_2|}{|I_2|+|T_0|+l}$$
$$\le 1+\frac{1-\frac{1}{k}+(k-1-\frac{2}{k})|I_2|}{|I_2|+1}$$
$$=1+(k-1-\frac{2}{k})-\frac{k-2-\frac{1}{k}}{|I_2|+1}$$
$$<k-\frac{2}{k},$$

which contradicts to $t(G)\ge k-\frac{1}{k}$. If $1\le\frac{(2k-1)|I_2|-k}{2|I_2|}\le k-2$, then we check that

$$\Phi(k-2)=(k-\frac{2}{k})|I_2|-k+2,$$
$$\Phi(k-3)=(k-\frac{6}{k})|I_2|-k+3,$$
$$\cdots$$
$$\Phi(j)=(-\frac{1}{k}(j)^2+(2-\frac{1}{k})j+1)|I_2|-j,$$
$$\Phi(j-1)=(-\frac{1}{k}(j-1)^2+(2-\frac{1}{k})(j-1)+1)|I_2|-j+1,$$
$$\cdots$$
$$\Phi(1)=(3-\frac{2}{k})|I_2|-1.$$

Therefore, we conclude that $\Phi(j)-\Phi(j-1)=(2-\frac{2j}{k})|I_2|-1$. Set $j_{\rm opt}\in\{1,2,\cdots,k-2\}$ which satisfies $\frac{k}{2k-2(j_{\rm opt}+1)}>|I_2|\ge\frac{k}{2k-2j_{\rm opt}}$. Hence, $\{\Phi(k-2),\Phi(k-3),\cdots,\Phi(j_{\rm opt}+1)\}$ is a strictly monotonically increasing sequence in terms of $\frac{k}{2k-2(k-2)}>\cdots>\frac{k}{2k-2(j_{\rm opt}+1)}>|I_2|$, and $\{\Phi(j_{\rm opt}),\cdots,\Phi(1)\}$ is a strictly monotonically decreasing sequence in terms of $|I_2|\ge\frac{k}{2k-2j_{\rm opt}}>\cdots>\frac{k}{2k-2}$. Hence, $\max\Phi(j^*)=\Phi(j_{\rm opt})$ and

$$|U|\le|T_0|+l+n-\frac{1}{k}+(-\frac{1}{k}j_{\rm opt}^2+$$
$$(2-\frac{1}{k})j_{\rm opt}+1)|I_2|-j_{\rm opt}$$
$$\le|T_0|+l+k-1-\frac{1}{k}+(-\frac{1}{k}j_{\rm opt}^2$$
$$+(2-\frac{1}{k})j_{\rm opt}+1)|I_2|-j_{\rm opt}$$
$$=|T_0|+l+k-1-j_{\rm opt}-\frac{1}{k}+(2j_{\rm opt}+1$$
$$-\frac{j_{\rm opt}(1+j_{\rm opt})}{k})|I_2|.$$

Therefore,

$$t(G)\le\frac{|U|}{\omega(G-U)}$$
$$\le\frac{|T_0|+l+k-1-j_{\rm opt}-\frac{1}{k}+(2j_{\rm opt}+1-\frac{j_{\rm opt}(1+j_{\rm opt})}{k})|I_2|}{|I_2|+|T_0|+l}$$
$$=1+\frac{k-1-j_{\rm opt}-\frac{1}{k}+(2j_{\rm opt}-\frac{j_{\rm opt}(1+j_{\rm opt})}{k})|I_2|}{|I_2|+|T_0|+l}$$
$$\le 1+\frac{k-1-j_{\rm opt}-\frac{1}{k}+(2j_{\rm opt}-\frac{j_{\rm opt}(1+j_{\rm opt})}{k})|I_2|}{|I_2|+1}$$
$$=1+(2j_{\rm opt}-\frac{j_{\rm opt}(1+j_{\rm opt})}{k})$$

$$+\frac{k-1-\frac{1}{k}-3j_{\text{opt}}+\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}}{|I_2|+1}.$$

If $k-1-\frac{1}{k}-3j_{\text{opt}}+\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}\le 0$, then

$$t(G)\le 1+2j_{\text{opt}}-\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}$$
$$=-\frac{j_{\text{opt}}^2}{k}+(2-\frac{1}{k})j_{\text{opt}}+1$$
$$\le -\frac{(k-2)^2}{k}+(2-\frac{1}{k})(k-2)+1$$
$$=k-\frac{2}{k}$$

which contradicts to $t(G)\ge k-\frac{1}{k}$. Thus, $k-1-\frac{1}{k}-3j_{\text{opt}}+\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}>0$ and

$$t(G)\le 1+(2j_{\text{opt}}-\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k})+\frac{k-1-\frac{1}{k}-3j_{\text{opt}}+\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}}{\frac{k}{2k-2j_{\text{opt}}}+1}.$$

To prove $t(G)<k-\frac{1}{k}$, it is necessary to prove

$$(2j_{\text{opt}}-\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k})+\frac{k-1-\frac{1}{k}-3j_{\text{opt}}+\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}}{\frac{k}{2k-2j_{\text{opt}}}+1}<k-1-\frac{1}{k}.$$

That is,

$$k-1-\frac{1}{k}<(k-1-\frac{1}{k})\frac{3k-2j_{\text{opt}}}{2k-2j_{\text{opt}}}-(2j_{\text{opt}}-\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k})\frac{3k-2j_{\text{opt}}}{2k-2j_{\text{opt}}}+3j_{\text{opt}}-\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k}.$$

which equals to

$$(2j_{\text{opt}}-\frac{j_{\text{opt}}(1+j_{\text{opt}})}{k})\frac{k}{2k-2j_{\text{opt}}}<(k-1-\frac{1}{k})\frac{k}{k-2j_{\text{opt}}}+j_{\text{opt}},$$

and it can be easily confirmed due to $j_{\text{opt}}\in\{1,\cdots,k-2\}$.

However, this trick is not available when $|I_1|=0$ and $|T_0|+l=0$. This is why the selecting policy of v' will be revised if $i_{k-1}=0$.

C. Remark on Difference of Isolated Bound for FabF and Fractional (a,b,n)-Critical Graph (FabnCG)

Recently, there have been two papers on the relationship between isolated toughness and FabF (resp. FabnCG). The main contributions are stated as follows.

Theorem 8. (Gao et al. [22]) Let G be a graph, a, b be positive integers with $2\le a\le b$. If $\delta(G)\ge a$ and $I(G)\ge a-1+\frac{a}{b}$, then G admits a FabF.

Theorem 9. (Gao et al. [23]) Let G be a graph, and a, b, n be positive integers with $2\le a\le b$ and $(n_{a,b}-1)a\le b\le n_{a,b}a-1$, where $i\ge 2$ is an integer. If $\delta(G)\ge a+n$ and $I(G)>a-1+\frac{n+1}{n_{a,b}}$, then G is a FabnCG.

Compared to Theorem 8 and Theorem 9, it is essential to ask some questions:

(1) Why isolated toughness bound in Theorem 8 is denoted by "$\ge$", while isolated toughness bound in Theorem 9 is denoted by "$>$"?

(2) Why the tight isolated toughness bound in Theorem 8 is a function with regard to a and b, while the sharp $I(G)$ bound in Theorem 9 is a function with regard to a and n?

(3) Why the best isolated toughness bound in Theorem 9 is a piecewise function, and the $I(G)$ bound for a graph admits FabF only a simple expression?

In fact, the answer to the aforementioned three questions are involved in the counterexample analysis of [22] and [23]. Let $a(|T_0|+l)-1=mb+c$, where $m\in\mathbb{N}\cup\{0\}$ and $c\in\{0,\cdots,b-1\}$. In terms of $\frac{1}{b}\le\frac{c+1}{b}\le 1$, we have

$$\max\{\frac{\left\lfloor(a-1+\frac{a}{b})(|T_0|+l)+n-\frac{1}{b}\right\rfloor}{|T_0|+l}\}$$
$$=a-1+\frac{a}{b}+\max\{\frac{n-\frac{c+1}{b}}{|T_0|+l}\}.$$

If $n=0$, then the last term $\max\{\frac{n-\frac{c+1}{b}}{|T_0|+l}\}$ is negative, which implies that the objective function reaches the maximum value when $|T_0|+l$ tends to be infinite. Since graph G is finite, which leads to that the extreme graph can't be reached and hence the isolated toughness bound in Theorem 8 takes "$\ge$". If $n\in\mathbb{N}$, then the last term is positive, which implies that the objective function reaches the maximum value when $|T_0|+l$ takes the minimum value, which leads to that the extreme graph can be reached and hence the isolated toughness bound in Theorem 9 takes ">"'.

In addition, when $n=0$, the last term $\max\{\frac{n-\frac{c+1}{b}}{|T_0|+l}\}$ tends to ∞ and the value focuses on $a-1+\frac{a}{b}$. This explains the isolated toughness bound in Theorem 8 with regard to a and b and only one expression of the final form. When it comes to $n\in\mathbb{N}$,

$$\frac{\left\lfloor (a-1+\frac{a}{b})(|T_0|+l)+n-\frac{1}{b}\right\rfloor}{|T_0|+l}$$

$$\le\frac{\left\lfloor n_{a,b}(a-1+\frac{a}{b})+n-\frac{1}{b}\right\rfloor}{n_{a,b}}$$

$$=a-1+\frac{\left\lfloor \frac{n_{a,b}a-1}{b}\right\rfloor+n}{n_{a,b}}.$$

Since the value of $\left\lfloor \frac{n_{a,b}a-1}{b}\right\rfloor$ independent of b, the sharp $I(G)$ bound in Theorem 9 is independent of the value of b. And it is the piecewise function because whether $\left\lfloor \frac{n_{a,b}a-1}{b}\right\rfloor$ takes depends on the relationship between a and b. It answers both the second and the third questions.

V. Conclusion

Bipolar fuzzy graphs have a wide range of applications in structured fuzzy data, and almost all network problems can have corresponding models and theories. This article provides the theoretical analysis of connectivity remainder in bipolar fuzzy graph setting, and discusses some related topics on fraction factors by obtaining relevant remarks.

The following questions can serve as future research directions.

1. The important difference between fuzzy set theory and general set theory can be summarized that fuzzy set operations do not satisfy the complementarity law. Based on this fact, the BFG is a compensation for the complementarity law, using the negative MF to characterize the negative uncertainty in the graph. Similar settings include intuitionistic fuzzy graph and Pythagorean fuzzy graph, and hence the connectivity remainder under these fuzzy graph assumptions needs further study.

2. With the diversification of structural data, in many applications, it is necessary to embed graph structures into manifolds or other geometric surfaces (for example, Klein bottle). Therefore, it is necessary to study what characteristics the corresponding connectivity remainder has when the fuzzy graph is embedded into a specific geometric structure.

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