Enhancing Two-Warehouse Inventory Models for Perishable Goods: Time-Price Dependent Demand, Inflation, and Partial Backlogging

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***Abstract*—Depending on demand, the selling price is the price at which the goods are offered for sale in response to their demand level. It is a pricing strategy that considers the relationship between supply and demand. The two-warehouse inventory model in the proposed model deals with perishable goods. The demands for the goods are time- and price-dependent, taking inflationary impacts into account. To improve operational effectiveness and customer happiness, the model allows for partial backlogs of shortages. The proposed model uses a positive definite Hessian matrix condition to get the best quantity and cycle length while minimising the nonlinear average total cost. This research paper conducts a numerical evaluation of the highly nonlinear average total cost using MATLAB software. Additionally, the study includes an analysis of the robustness of the results through sensitivity analysis. The conclusion drawn from the study helps summarise the essential findings and highlights the efficiency of the proposed model.**

***Index Terms*—Two-warehouse, Inflation, Partial backlogging, Time and price dependent demand, Perishable goods**

I. Introduction

IN the current dynamic business environment, proficient inventory management is pivotal in optimising operations and mitigating overall costs, particularly in industries dealing with perishable goods. These goods, prone to rapid deterioration, demand meticulous attention to inventory strategies to curtail losses and efficiently meet customer demands. This research paper delves into optimising a two-warehouse inventory model tailored to address the unique challenges of managing deteriorating goods. The model comprehensively considers crucial factors such as time- and price-dependent demand, inflationary impacts, shortages, and partial backlogs, providing decision-makers with practical strategies to enhance inventory control.

Perishable commodities, due to their short shelf life, require careful management of stock levels and demand to minimise waste and obsolescence. The proposed two-warehouse inventory model provides a methodical methodology for strategically allocating perishable commodities across two warehouses. This allocation technique maximises cost efficiency by optimising storage, transportation, and distribution expenses, hence enhancing operational effectiveness.

Inflation has a substantial impact on decisions about inventory management, as changes in currency value affect the expenses associated with holding, purchasing, and selling inventory. By incorporating the effects of inflation into the model, firms may accurately evaluate inventory expenses and create flexible pricing plans to ensure long-term profitability.

Shortages provide a frequent obstacle for organisations that handle perishable commodities since demand may exceed supply, resulting in stockouts and significant revenue declines. The proposed inventory model allows for partial backlogs caused by shortages, which enables businesses to meet client orders when there is not enough stock available. This feature helps to reduce revenue losses and improve client happiness and loyalty.

Temporal and monetary variables intrinsically connect to the demand for perishable items. The suggested model incorporates both temporal and price-sensitive demand, enabling organisations to customise pricing strategies based on demand changes, hence improving revenue and inventory levels.

This research study provides a comprehensive analysis of the two-warehouse inventory model, specifically designed for perishable items that experience fluctuations in demand based on time and price. Numerical simulations and sensitivity assessments using MATLAB software have validated the suggested model for robustness and efficacy. The findings offer useful insights and practical consequences for firms aiming to enhance inventory management techniques, decrease overall costs, and ensure a consistent supply of perishable commodities to fulfil client demands. The text also explores potential areas for future research, emphasising the need for efficient inventory management in the perishable commodities sector..

Positive Definite Hessian Matrix

An essential component of the suggested investigation involves the application of the positive-definite Hessian matrix condition. By implementing this mathematical optimisation technique, the cost function in the framework of

Manuscript received September 30, 2023; revised April 10, 2024.

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inventory is guaranteed to be convex, leading to efficient convergence towards the global minimum. By satisfying this condition, the model is able to infer the most efficient quantity and duration of the cycle, which ultimately results in decreased average total expenses and increased operational efficiency.

In demonstrating the positive definiteness of a matrix, it becomes imperative to establish that all its principal minors possess positive determinants. These principal minors are derived by selecting specific subsets of rows and columns, forming smaller square matrices from the chosen elements. i.e. Let $M_{n\times n}$ is a hessian matrix, then $M_{n\times n}$ are positive definite matrix iff $M_{n\times n}, M_{(n-1)\times(n-1)}, \ldots, M_{(n-(n-1))\times(n-(n-1))}$ is positive.

II. Literature Review

The literature review covers a range of research papers on inventory models for deteriorating items, showcasing different deterioration patterns, shortage policies, demand functions, inflation considerations, and warehouse systems. The studies contribute valuable insights into optimising inventory management and cost reduction in dynamic market scenarios, as shown in Table I.

Chandra K. Jaggi [1] proposed a model for instantaneous deterioration with two warehouses. The research highlights the significance of screening defective items to fulfil the demand for high-quality products. Yadav *et al.* [2] extended the model to incorporate exponentially time-varying demand and partial backlogging while considering inflation. They aimed to derive optimal replenishment policies to minimise total inventory costs.

Yadav and Swami [3] investigated non-instantaneous deterioration in the presence of wholly backlogged shortages, two warehouses, and linearly time-dependent demand. Mashud et al. [4] devised a model for non-instantaneous items with partial backlogging and demand that is dependent on stock and price. During shortages, they analysed the effect of price and supply on demand.

Ali Akbar Shaikh et al. [5] examined two warehouse systems that incorporated partial backlog, stock-dependent demand, and non-instantaneous commodities, all while accounting for inflation. In a single warehouse, Sumit Saha and Nabendu Sen [6] investigated instantaneous deterioration with a partial backlog, taking into account time-dependent and price-dependent demand.

Adrian Macı'as-Lo'pez et al. [7] examined a non-instantaneous inventory model within a single warehouse that incorporated time-dependent demand, price, and stock. In their study, Biman Kanti Nath and Nabendu Sen [8] introduced a two-warehouse model to account for selling price-dependent demand and entirely backlogged shortages of non-instantaneous goods.

Udayakumar et al. [9] investigated the conditions under which advertisement-dependent demand, partial backlogging, and non-instantaneous items operate in a single warehouse. In a single warehouse, Chayanika Rout et al. [10] examined instantaneous commodities with a partial backlog and demand that was both constant and backlog-dependent.

Md. Al-Amin Khan et al. [11] investigated non-instantaneous items with non-linear stock-dependent demand and a partial backlog in a single warehouse. In a single warehouse, Boina Anil Kumar and Susanta Kumar Paikray [12] examined instantaneous commodities with completely backlogged shortages and trapezoidal demand.

In their study, Neeraj Kumar et al. [13] introduced a two-warehouse model to account for stock-dependent demand and non-instantaneous items with a partial backlog. In a two-warehouse system, Avijit Duary et al. [14] investigated instantaneous items with partial backlogging, selling price, duration, and frequency of advertisement-dependent demand.

In a single warehouse, Chaman Singh and Gurudatt Rao Ambedkar [15] investigated instantaneous items with a complete backlog, inventory, selling price, and lifetime-dependent demand. In a two-warehouse system, Jagadeesan Viswanath et al. [16] analysed instantaneous items with fully backlogged shortages, ramp-type demand, and a constant deterioration rate.

Nurnadiah Binti Nurhasril et al. [17] introduced a two-warehouse model to account for the demand for instantaneous items that exhibit a linear increase and no shortages. Nita H. Shah et al. [18] examined the demand for instantaneous items that exhibited no shortages in a single warehouse, determined by factors such as selling price, goodwill effect, and market potential.

Sadaf Fatma et al. [19] investigated instantaneous items with a partial backlog and linear time-sensitive demand. Karan Pathak et al. [20] investigates optimal replenishment strategies for managing shelf-life stock. This study adopts a two-warehouse framework and thoughtfully structures a model that accounts for nuanced dynamics during shortages. Additionally, it explicitly addresses inflationary forces.

Investigating the optimal value of the inventory model in a complex environment characterised by instantaneous deterioration, partial backlog, time-and-price-varying demand, and a two-warehouse system is the impetus for this research. The objective of the researchers is to develop optimal strategies that efficiently decrease total inventory expenses, taking into account practical market circumstances and diverse business limitations.

By examining this particular confluence of factors utilising a variety of methodologies, the research endeavours to provide insightful perspectives on effective inventory management for collapsible products. The suggested framework assumes that people must have a better understanding of what happens when demand changes, be able to handle shortages and backlogs effectively, understand how inflation affects the cost of inventory, and find the best warehouse module to boost productivity.

In general, the research aims to provide practical strategies that can be implemented by industry decision-makers to enhance their inventory management procedures, increase profitability, ensure a steady supply of goods to satisfy customer needs, and minimise expenses and potential losses associated with deteriorating products.

TABLE I
LITERATURE REVIEW FOR THE PROPOSED MODEL

Authors/ Year	Deterioration	Shortages	Demand	Inflation	Warehouse System
Chandra K. Jaggi (2015)	Instantaneous	Not allowed	Constant	Not allowed	Two-warehouse
Yadav AS *et al.* (2017)	Instantaneous	Partial backlogging	Exponentially time varying demand	Allowed	Two-warehouse
Yadav AS & Swami A. (2019)	Non-instantaneous	Completely backlogged	Linearly time dependent	Not allowed	Two-warehouse
Mashud A. *et al.* (2018)	Non-instantaneous	Partial backlogging	Stock and price dependent	Not allowed	Single
Ali Akbar Shaikh *et al.* (2019)	Non-instantaneous	Partial backlogging	Stock-dependent	Allowed	Two-warehouse
Sumit Saha & Nabendu Sen (2019)	Instantaneous	Partial backlogging	Time and price dependent	Allowed	Single
Adria´n Macı´as-Lo´pez *et al.* (2021)	Non-instantaneous	Not allowed	Price, Stock, and time-dependent	Not allowed	Single
Biman Kanti Nath & Nabendu Sen (2021)	Non-instantaneous	Completely backlogged	Selling Price Dependent	Not allowed	Two-warehouse
R. Udayakumar *et al.* (2021)	Non-instantaneous	Partial backlogging	Price and advertisement dependent	Allowed	Single
Chayanika Rout *et al.* (2021)	Instantaneous	Partial backlogging	Constant and backlog-dependent	Not allowed	Single
Md Al-Amin Khan *et al.* (2022)	Non- instantaneous	Partial backlogging	Non-linear stock-dependent	Not allowed	Single
Boina Anil Kumar & Susanta Kumar Paikray (2022)	Instantaneous	Completely backlogged	Trapezoidal type	Not allowed	Single
Neeraj Kumar *et al.* (2022)	Non- instantaneous	Partial backlogging	Stock dependent	Not allowed	Two-warehouse
Avijit Duary *et al.* (2022)	Instantaneous	Partial backlogging	Selling price, time and frequency of advertisement dependent	Not allowed	Two-warehouse
Chaman Singh & Gurudatt Rao Ambedkar (2023)	Instantaneous	Not allowed	Stock selling cost and lifetime dependent	Allowed	Single
Jagadeesan Viswanath et al. (2023)	Instantaneous	Completely backlogged	Ramp type	Not allowed	Two-warehouse
Nurnadiah Binti Nurhasril *et al.* (2023)	Instantaneous	Not allowed	Linearly increasing demand rate	Not allowed	Two-warehouse
Nita H. Shah *et al.* (2023)	Instantaneous	Not allowed	Function of the selling price, goodwill effect, and market potential	Not allowed	Single
Sadaf Fatma *et al.* (2023)	Instantaneous	Partial backlogging	Linear time-sensitive	Not allowed	Single
Karan Pathak *et al.* (2024)	Non- instantaneous	Partial backlogging	Biquadratic time-dependent	Allowed	Two-warehouse
In this paper	Instantaneous	Partial backlogging	Time- and price- dependent	Allowed	Two-warehouse

III. NOTATIONS AND ASSUMPTIONS

The formulation of the model in this study is based on the subsequent notation and assumptions.

A. Notations

Notations and their descriptions for the proposed model are shown in Table II.

B. Assumptions

The assumptions for the proposed model are being followed as:

1. The demand for the goods fluctuates based on price and changes over time. i.e.

$$Demand = \begin{cases} (\beta t - \alpha) p \; ; & t \geq 0 \\ 0 \; ; & t < 0 \end{cases}, \text{ where } \alpha, \beta, p > 0 .$$

2. No replacement or repair of deteriorated items takes place within a given cycle.
3. Deterioration occurs in both warehouses (rented and owned), with constant rates of λ_{wr} and λ_{wo} at time $0 \leq t \leq t_r$ and $0 \leq t \leq t_o$, respectively.
4. Owned warehouses experience a greater deterioration rate than rented warehouses.
5. Replenishment occurs at an infinite rate.
6. The owned warehouse has limited capacity *Y*, and the rented warehouse has unlimited capacity.

IV. MATHEMATICAL MODEL FORMULATION

The inventory system is established as follows: At the initiation of each cycle, *X* units of items are introduced into the inventory system. Within this process, *Y* units are designated for the owned warehouse, while the remaining units are allocated to the rented warehouse. A crucial distinction emerges in the consumption pattern, where items from the owned warehouse are engaged only after the goods from the rented warehouse have been thoroughly utilised.

During the interval $[0, t_r]$ in the rented warehouse, the inventory level experiences depletion and ultimately reaches zero at time t_r. This outcome result from the combined influences of deterioration and the demand rate. Simultaneously, within the owned warehouse, during the same interval $[0, t_r]$, inventory depletion is attributed solely to the impact of deterioration. Furthermore, within the interval $[t_r, t_o]$, the inventory within the owned warehouse undergoes a reduction and eventually reaches zero at time t_2. This depletion results from the compounded effects of both demand and deterioration. By the time t_2 arrives, both warehouses will be devoid of inventory. This intricate interplay leads to a shortage within the interval $[t_o, T]$, resulting from demand and partial backlogging. The dynamic behaviour of this inventory model is vividly illustrated in Fig. 1.

TABLE II
NOTATIONS AND THEIR DESCRIPTION FOR THE PROPOSED MODEL

Notations	Description
A	Ordering cost
α	Demand's parameter
β	Demand's parameter
$I_{wr}(t)$	Inventory level in a rented warehouse at the time $0 \le t \le t_r$
$I_{wo}(t)$	Inventory level in an owned warehouse at the time $0 \le t \le t_o$
$I_{ws}(t)$	Inventory level of backlogging at the time $t_o \le t \le T$
X	Inventory level in a rented warehouse at the initial time
Y	Inventory level in an owned warehouse at the initial time
Z	Inventory level of backlogging at time T
λ_{wr}	Deterioration rate in a rented warehouse at the time $0 \le t \le t_r$, where $0 < \lambda_{wr} << 1$
λ_{wo}	Deterioration rate in an owned warehouse at the time $0 \le t \le t_o$, where $0 < \lambda_{wo} << 1$
$\hbar_{rc}$	Holding per unit cost in a rented warehouse
$\hbar_{oc}$	Holding per unit cost in owned warehouse
s_{ws}	Shortage per unit cost
ls_{ws}	Lost sale per unit cost
d_{wr}	Deterioration per unit cost in rented warehouse
d_{wo}	Deterioration per unit cost in owned warehouse
ℓ	Inflation rate at the time $0 \le t \le T$
p_u	Purchasing per unit cost
p	Selling price
λ	Shortage rate at the time $t_o \le t \le T$
Decision Variables	
t_r	Time at which rented warehouse becomes zero
t_o	Time at which owned warehouse becomes zero
T	Total cycle length

The following differential equations for the proposed model can be written as:

$$\frac{dI_{wr}(t)}{dt}+\lambda_{wr}I_{wr}(t)=-(-\alpha+\beta t)p \qquad ,0\le t\le t_r \quad (1)$$

$$\frac{dI_{wo}(t)}{dt}+\lambda_{wo}I_{wo}(t)=0 \qquad ,0\le t\le t_r \quad (2)$$

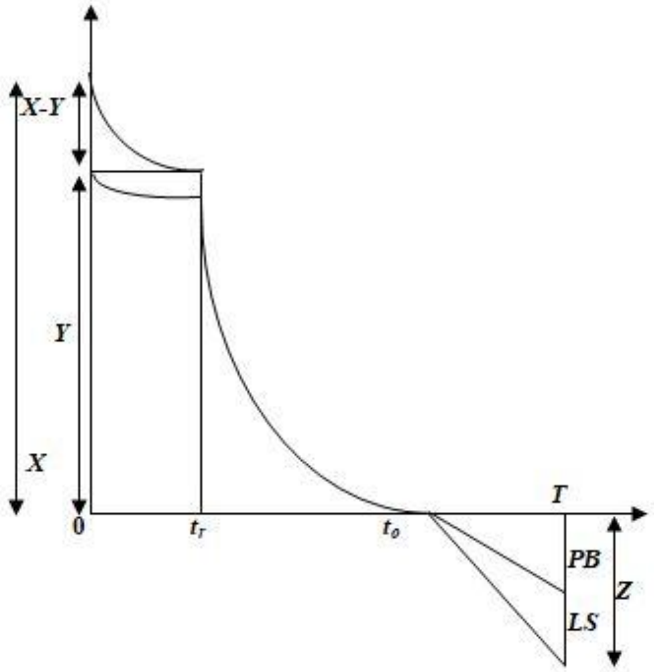

Fig. 1. Two-warehouse inventory model for the proposed model

$$\frac{dI_{wo}(t)}{dt}+\lambda_{wo}I_{wo}(t)=-(-\alpha+\beta t)p \qquad ,t_r\le t\le t_o \quad (3)$$

$$\frac{dI_{ws}(t)}{dt}=-(-\alpha+\beta t)pe^{-\lambda(T-t)} \qquad ,t_o\le t\le T \quad (4)$$

On solving the equations (1)-(4) using the boundary condition, $I_{wr}(t_r)=0,\ I_{wo}(t_r)=Y,\ I_{wo}(t_o)=0,\ I_{ws}(t_o)=Z$ respectively, we get

$$I_{wr}(t)=\left\{\begin{matrix}\dfrac{(\alpha-\beta t)p}{\lambda_{wr}}+\dfrac{\beta p}{\lambda_{wr}^2}\\ -\dfrac{e^{-\lambda_{wr}t}\left(\begin{matrix}\beta pe^{\lambda_{wr}t_r}+\alpha\lambda_{wr}pe^{\lambda_{wr}t_r}\\ -\lambda_{wr}\beta pt_re^{\lambda_{wr}t_r}\end{matrix}\right)}{\lambda_{wr}^2}\end{matrix}\right\} \qquad ,0\le t\le t_r$$

(5)

$$I_{wo}(t)=Ye^{-t} \qquad ,0\le t\le t_r \quad (6)$$

$$I_{wo}(t)=\left\{\begin{matrix}\dfrac{(\alpha-\beta t)p}{\lambda_{wo}}+\dfrac{\beta p}{\lambda_{wo}^2}-\dfrac{e^{-\lambda_{wo}t}p}{\lambda_{wo}^2}\\ \left(\beta e^{\lambda_{wo}t_o}+\alpha\lambda_{wo}e^{\lambda_{wo}t_o}-\beta\lambda_{wo}t_oe^{\lambda_{wo}t_o}\right)\end{matrix}\right\} \qquad ,t_r\le t\le t_o$$

(7)

$$I_{ws}(t)=\left\{\begin{matrix}\dfrac{pe^{-\lambda(T-t)}(\beta+\alpha\lambda-b\lambda t)}{\lambda^2}\\ -\dfrac{pe^{-\lambda(T-t_o)}(\beta+a\lambda-\beta\lambda t_o)}{\lambda^2}\end{matrix}\right\} \qquad ,t_o\le t\le T$$

(8)

Now, considering the continuity at $t = t_r$, it's followed by (6) and (7), leading to:

$$Y=\frac{e^{t_r}}{\lambda_{wo}^2}\left\{\begin{matrix}\dfrac{(\alpha p-\beta pt_r)}{\lambda_{wo}}+\beta p-e^{-\lambda_{wo}t_r}p\\ \left(\beta e^{\lambda_{wo}t_o}+\alpha\lambda_{wo}e^{\lambda_{wo}t_o}-\beta\lambda_{wo}t_oe^{\lambda_{wo}t_o}\right)\end{matrix}\right\}$$

The total quantity placed in both warehouses (owned and rented) at the initial time can be expressed in standard form as follows:

$$X = I_{wr}(0) + I_{wo}(0) = I_{wr}(0) + Y$$

$$X = Y + \frac{\left(\beta - \beta e^{\lambdabar_{wr} t_r} - \alpha \lambdabar_{wr} e^{\lambdabar_{wr} t_r} + \lambdabar_{wr} \beta t_r e^{\lambdabar_{wr} t_r}\right) p}{\lambdabar_{wr}^2} + \frac{\alpha p}{\lambdabar_{wr}}$$

When the value of "t" is substituted with "T" in (8), the result gives the overall shortages.

$$Z = \left\{ \frac{p\left(\beta + \alpha\lambda - T\beta\lambda\right)}{\lambda^2} - \frac{p e^{-\lambda(T - t_r)}\left(\beta + \alpha\lambda - \beta\lambda t_r\right)}{\lambda^2} \right\}$$

Now, the total quantity to be replenished for the next cycle can be expressed as:

$$TQ = X + Z$$

$$TQ = \left\{ \begin{array}{l} Y + \dfrac{\left(\beta - \beta e^{\lambdabar_{wr} t_r} - \alpha \lambdabar_{wr} e^{\lambdabar_{wr} t_r} + \lambdabar_{wr} \beta t_r e^{\lambdabar_{wr} t_r}\right) p}{\lambdabar_{wr}^2} \\ + \dfrac{\alpha p}{\lambdabar_{wr}} + \left\{ \begin{array}{l} \dfrac{p\left(\beta + \alpha\lambda - T\beta\lambda\right)}{\lambda^2} \\ - \dfrac{p e^{-\lambda(T - t_r)}\left(\beta + \alpha\lambda - \beta\lambda t_r\right)}{\lambda^2} \end{array} \right\} \end{array} \right\} \quad (9)$$

The total cost comprises various elements that contribute to the overall expenses and can be expressed as:

A. Ordering cost

$$OC = \mathrm{A}$$

B. Worth holding cost in a rented warehouse

$$HC_{wr} = \hbar_{rc} \int_0^{t_r} I_{wr}(t) e^{-\ell t}\, dt$$

$$HC_{wr} = \frac{\hbar_{rc} p e^{-\ell t_r}}{\lambdabar_{wr}^2 \ell^2 \left(\lambdabar_{wr} + \ell\right)} \left\{ \begin{array}{l} \lambdabar_{wr}^2 \beta - \lambdabar_{wr}^2 \beta e^{\ell t_r} + \beta \ell^2 e^{\ell t_r} + \lambdabar_{wr}^2 \beta \ell t_r \\ -\alpha \lambdabar_{wr}^2 \ell - \beta \ell^2 e^{t_r(\lambdabar_{wr} + \ell)} - \alpha \lambdabar_{wr} \ell^2 e^{t_r(\lambdabar_{wr} + \ell)} \\ + \alpha \lambdabar_{wr} \ell^2 e^{\ell t_r} + \alpha \lambdabar_{wr}^2 \ell e^{\ell t_r} + \lambdabar_{wr} \beta \ell^2 t_r e^{t_r(\lambdabar_{wr} + \ell)} \end{array} \right\}$$

C. Worth holding cost in the owned warehouse

$$HC_{wo} = \hbar_{oc} \int_0^{t_o} I_{wo}(t) e^{-\ell t} dt$$

$$HC_{wo} = \hbar_{oc} \left[\int_0^{t_r} I_{wo}(t) e^{-\ell t} dt + \int_{t_r}^{t_o} I_{wo}(t) e^{-\ell t} dt \right]$$

$$HC_{wo} = \hbar_{oc} \left\{ \begin{array}{l} \dfrac{Y\left(e^{-t_r(\ell+1)} - 1\right)}{\ell + 1} + \dfrac{\beta p}{\lambdabar_{wo}} \left(\dfrac{e^{-\ell t_r}\left(\ell t_r + 1\right) - e^{-\ell t_o}\left(\ell t_o + 1\right)}{\ell^2} \right) \\ - \dfrac{\alpha p\left(e^{-\ell t_r} - e^{-\ell t_o}\right)}{\lambdabar_{wo} \ell} + \dfrac{\beta p e^{\lambdabar_{wo} t_o}\left(e^{-t_r(\lambdabar_{wo} + \ell)} - e^{-t_o(\lambdabar_{wo} + \ell)}\right)}{\lambdabar_{wo}^2 \left(\lambdabar_{wo} + \ell\right)} \\ + \dfrac{\alpha p e^{\lambdabar_{wo} t_o}\left(e^{-t_r(\lambdabar_{wo} + \ell)} - e^{-t_o(\lambdabar_{wo} + \ell)}\right)}{\lambdabar_{wo}\left(\lambdabar_{wo} + \ell\right)} - \dfrac{\beta p\left(e^{-\ell t_r} - e^{-\ell t_o}\right)}{\lambdabar_{wo}^2 \ell} \\ - \dfrac{\beta p t_o e^{\lambdabar_{wo} t_o}\left(e^{-t_r(\lambdabar_{wo} + \ell)} - e^{-t_o(\lambdabar_{wo} + \ell)}\right)}{\lambdabar_{wo}\left(\lambdabar_{wo} + \ell\right)} \end{array} \right\}$$

D. Deterioration cost in a rented warehouse

$$DC_{wr} = d_{wr} \lambdabar_{wr} \int_0^{t_r} I_{wr}(t) e^{-\ell t} dt$$

$$DC_{wr} = \frac{d_{wr} p e^{-\ell t_r}}{\lambdabar_{wr} \ell^2 \left(\lambdabar_{wr} + \ell\right)} \left\{ \begin{array}{l} \lambdabar_{wr}^2 \beta - \lambdabar_{wr}^2 \beta e^{\ell t_r} - \alpha \lambdabar_{wr}^2 \ell \\ -\alpha \lambdabar_{wr} \ell^2 e^{t_r(\lambdabar_{wr} + \ell)} + \lambdabar_{wr}^2 \beta \ell t_r \\ + \alpha \lambdabar_{wr}^2 \ell e^{\ell t_r} + \lambdabar_{wr} \beta \ell^2 t_r e^{t_r(\lambdabar_{wr} + \ell)} \\ - \beta \ell^2 e^{t_r(\lambdabar_{wr} + \ell)} + \alpha \lambdabar_{wr} \ell^2 e^{\ell t_r} + \beta \ell^2 e^{\ell t_r} \end{array} \right\}$$

E. Deterioration cost in the owned warehouse

$$DC_{wo} = d_{wo} \lambdabar_{wo} \int_0^{t_o} I_{wo}(t) e^{-\ell t} dt$$

$$DC_{wo} = d_{wo} \lambdabar_{wo} \left\{ \int_0^{t_r} I_{wo}(t) e^{-\ell t} dt + \int_{t_r}^{t_o} I_{wo}(t) e^{-\ell t} dt \right\}$$

$$DC_{wo} = -\lambdabar_{wo} d_{wo} \left\{ \begin{array}{l} \lambdabar_{wo} \left[\begin{array}{l} \dfrac{\beta p}{\lambdabar_{wo}} \left(\dfrac{e^{-\ell t_r}\left(\ell t_r + 1\right) - e^{-\ell t_o}\left(\ell t_o + 1\right)}{\ell^2} \right) \\ - \dfrac{\alpha p\left(e^{-\ell t_r} - e^{-\ell t_o}\right)}{\lambdabar_{wo} \ell} - \dfrac{\beta p\left(e^{-\ell t_r} - e^{-\ell t_o}\right)}{\lambdabar_{wo}^2 \ell} \\ + \dfrac{\alpha p e^{\lambdabar_{wo} t_o}\left(e^{-(\lambdabar_{wo} + \ell) t_r} - e^{-(\lambdabar_{wo} + \ell) t_o}\right)}{\lambdabar_{wo}\left(\lambdabar_{wo} + \ell\right)} \\ + \dfrac{\beta p e^{\lambdabar_{wo} t_o}\left(e^{-(\lambdabar_{wo} + \ell) t_r} - e^{-(\lambdabar_{wo} + \ell) t_o}\right)}{\lambdabar_{wo}^2\left(\lambdabar_{wo} + \ell\right)} \\ - \dfrac{\beta p t_o e^{\lambdabar_{wo} t_o}\left(e^{-(\lambdabar_{wo} + \ell) t_r} - e^{-(\lambdabar_{wo} + \ell) t_o}\right)}{\lambdabar_{wo}\left(\lambdabar_{wo} + \ell\right)} \end{array} \right] \\ + \dfrac{Y\left(e^{-(1+\ell) t_r} - 1\right)}{\ell + 1} \end{array} \right\}$$

F. The present value of the backlogging cost.

$$SC_{ws} = s_{ws} \int_{t_o}^{T} I_{ws}(t) e^{-\ell t} dt$$

$$SC_{ws} = s_{ws} \left\{ \begin{array}{l} \dfrac{\beta p}{\lambda} \left[\dfrac{e^{-T\ell}\left(\left(\ell - \lambda\right)T + 1\right) - e^{(\lambda - \ell)t_o - T\lambda}\left(\left(\ell - \lambda\right)t_o + 1\right)}{\left(\lambda - \ell\right)^2} \right] \\ - \dfrac{\alpha p\left(e^{(\lambda - \ell)t_o - T\lambda} - e^{-T\ell}\right)}{\lambda\left(\lambda - \ell\right)} - \dfrac{\beta p\left(e^{(\lambda - \ell)t_o - T\lambda} - e^{-T\ell}\right)}{\lambda^2\left(\lambda - \ell\right)} + \\ \dfrac{\alpha p e^{-\lambda(T - t_o)}\left(e^{-T\ell} - e^{-\ell t_o}\right)}{\lambda \ell} + \dfrac{\beta p e^{-\lambda(T - t_o)}\left(e^{-T\ell} - e^{-\ell t_o}\right)}{\lambda^2 \ell} \\ - \dfrac{\beta p t_o e^{-\lambda(T - t_o)}\left(e^{-T\ell} - e^{-\ell t_o}\right)}{\lambda \ell} \end{array} \right\}$$

G. The cost incurred from lost sales is attributable to the presence of opportunity cost.

$$LS_{ws} = ls_{ws} e^{-rT} \int_{t_o}^{T} \alpha\left(1 - \Upsilon\right) dt$$

$$LS_{ws} = \frac{ls_{ws} e^{-T\ell}}{\lambda}\left(T - t_o + e^{-\lambda(T - t_o)} - 1\right)$$

Thus, the total average cost per unit time for the proposed model can be expressed as:

$$ATCU(t_r,t_o,T)=\frac{OC+HC_{wr}+HC_{wo}+DC_{wr}+DC_{wo}+SC_{ws}+LS_{ws}}{T}$$

$$ATCU(t_r,t_o,T)=\left\{\begin{array}{l}
\mathrm{A}+\dfrac{\hbar_{rc}\,p e^{-\ell t_r}}{\bar{\lambda}_{wr}^2\ell^2\left(\bar{\lambda}_{wr}+\ell\right)}\left\{\begin{array}{l}\bar{\lambda}_{wr}^2\beta-\bar{\lambda}_{wr}^2\beta e^{\ell t_r}+\beta\ell^2 e^{\ell t_r}+\bar{\lambda}_{wr}^2\beta\ell t_r\\ -\alpha\bar{\lambda}_{wr}^2\ell-\beta\ell^2 e^{t_r\left(\bar{\lambda}_{wr}+\ell\right)}-\alpha\bar{\lambda}_{wr}\ell^2 e^{t_r\left(\bar{\lambda}_{wr}+\ell\right)}\\ +\alpha\bar{\lambda}_{wr}\ell^2 e^{\ell t_r}+\alpha\bar{\lambda}_{wr}^2\ell e^{\ell t_r}+\bar{\lambda}_{wr}\beta\ell^2 t_r e^{t_r\left(\bar{\lambda}_{wr}+\ell\right)}\end{array}\right\}\\
+s_{ws}\left\{\begin{array}{l}\dfrac{\beta p}{\lambda}\left[\dfrac{e^{-T\ell}\left(\left(\ell-\lambda\right)T+1\right)-e^{(\lambda-\ell)t_o-T\lambda}\left(\left(\ell-\lambda\right)t_o+1\right)}{\left(\lambda-\ell\right)^2}\right]\\ -\dfrac{\alpha p\left(e^{(\lambda-\ell)t_o-T\lambda}-e^{-T\ell}\right)}{\lambda\left(\lambda-\ell\right)}-\dfrac{\beta p\left(e^{(\lambda-\ell)t_o-T\lambda}-e^{-T\ell}\right)}{\lambda^2\left(\lambda-\ell\right)}+\\ \dfrac{\alpha p e^{-\lambda(T-t_o)}\left(e^{-T\ell}-e^{-\ell t_o}\right)}{\lambda\ell}+\dfrac{\beta p e^{-\lambda(T-t_o)}\left(e^{-T\ell}-e^{-\ell t_o}\right)}{\lambda^2\ell}\\ -\dfrac{\beta p t_o e^{-\lambda(T-t_o)}\left(e^{-T\ell}-e^{-\ell t_o}\right)}{\lambda\ell}\end{array}\right.\\
\hbar_{oc}\left\{\begin{array}{l}\dfrac{Y\left(e^{-t_r(\ell+1)}-1\right)}{\ell+1}+\dfrac{\beta p}{\bar{\lambda}_{wo}}\left(\dfrac{e^{-\ell t_r}\left(\ell t_r+1\right)-e^{-\ell t_o}\left(\ell t_o+1\right)}{\ell^2}\right)-\dfrac{\alpha p\left(e^{-\ell t_r}-e^{-\ell t_o}\right)}{\bar{\lambda}_{wo}\ell}\\ -\dfrac{\beta p\left(e^{-\ell t_r}-e^{-\ell t_o}\right)}{\bar{\lambda}_{wo}^2\ell}\\ +\dfrac{\alpha p e^{\bar{\lambda}_{wo}t_o}\left(e^{-t_r\left(\bar{\lambda}_{wo}+\ell\right)}-e^{-t_o\left(\bar{\lambda}_{wo}+\ell\right)}\right)}{\bar{\lambda}_{wo}\left(\bar{\lambda}_{wo}+\ell\right)}+\dfrac{\beta p e^{\bar{\lambda}_{wo}t_o}\left(e^{-t_r\left(\bar{\lambda}_{wo}+\ell\right)}-e^{-t_o\left(\bar{\lambda}_{wo}+\ell\right)}\right)}{\bar{\lambda}_{wo}^2\left(\bar{\lambda}_{wo}+\ell\right)}\\ -\dfrac{\beta p t_o e^{\bar{\lambda}_{wo}t_o}\left(e^{-t_r\left(\bar{\lambda}_{wo}+\ell\right)}-e^{-t_o\left(\bar{\lambda}_{wo}+\ell\right)}\right)}{\bar{\lambda}_{wo}\left(\bar{\lambda}_{wo}+\ell\right)}\end{array}\right\}\\
+\dfrac{d_{wr}\,p e^{-\ell t_r}}{\bar{\lambda}_{wr}\ell^2\left(\bar{\lambda}_{wr}+\ell\right)}\left\{\begin{array}{l}\bar{\lambda}_{wr}^2\beta-\bar{\lambda}_{wr}^2\beta e^{\ell t_r}-\alpha\bar{\lambda}_{wr}^2\ell-\beta\ell^2 e^{t_r\left(\bar{\lambda}_{wr}+\ell\right)}\\ -\alpha\bar{\lambda}_{wr}\ell^2 e^{t_r\left(\bar{\lambda}_{wr}+\ell\right)}+\bar{\lambda}_{wr}^2\beta\ell t_r+\alpha\bar{\lambda}_{wr}\ell^2 e^{\ell t_r}\\ +\alpha\bar{\lambda}_{wr}^2\ell e^{\ell t_r}+\bar{\lambda}_{wr}\beta\ell^2 t_r e^{t_r\left(\bar{\lambda}_{wr}+\ell\right)}+\beta\ell^2 e^{\ell t_r}\end{array}\right\}\\
+\dfrac{ls_{ws}e^{-T\ell}}{\lambda}\left(T-t_o+e^{-\lambda(T-t_o)}-1\right)\\
-\bar{\lambda}_{wo}d_{wo}\left\{\begin{array}{l}\bar{\lambda}_{wo}\left[\begin{array}{l}\dfrac{\beta p}{\bar{\lambda}_{wo}}\left(\dfrac{e^{-\ell t_r}\left(\ell t_r+1\right)-e^{-\ell t_o}\left(\ell t_o+1\right)}{\ell^2}\right)-\dfrac{\alpha p\left(e^{-\ell t_r}-e^{-\ell t_o}\right)}{\bar{\lambda}_{wo}\ell}\\ +\dfrac{\alpha p e^{\bar{\lambda}_{wo}t_o}\left(e^{-\left(\bar{\lambda}_{wo}+\ell\right)t_r}-e^{-\left(\bar{\lambda}_{wo}+\ell\right)t_o}\right)}{\bar{\lambda}_{wo}\left(\bar{\lambda}_{wo}+\ell\right)}+\dfrac{\beta p e^{\bar{\lambda}_{wo}t_o}\left(e^{-\left(\bar{\lambda}_{wo}+\ell\right)t_r}-e^{-\left(\bar{\lambda}_{wo}+\ell\right)t_o}\right)}{\bar{\lambda}_{wo}^2\left(\bar{\lambda}_{wo}+\ell\right)}\\ -\dfrac{\beta p t_o e^{\bar{\lambda}_{wo}t_o}\left(e^{-\left(\bar{\lambda}_{wo}+\ell\right)t_r}-e^{-\left(\bar{\lambda}_{wo}+\ell\right)t_o}\right)}{\bar{\lambda}_{wo}\left(\bar{\lambda}_{wo}+\ell\right)}-\dfrac{\beta p\left(e^{-\ell t_r}-e^{-\ell t_o}\right)}{\bar{\lambda}_{wo}^2\ell}\end{array}\right]\\ +\dfrac{Y\left(e^{-(1+\ell)t_r}-1)\right)}{\ell+1}\end{array}\right.
\end{array}\right\} \tag{10}$$

Format To minimise the total cost of inventory per unit of time in present value, it is necessary to consider the following factors for optimisation: $ATCU\left(t_r,t_o,T\right)$

$$\frac{\partial ATCU\left(t_r,t_o,T\right)}{\partial t_r}=0\ ,\ \frac{\partial ATCU\left(t_r,t_o,T\right)}{\partial t_o}=0\ \&\ \frac{\partial ATCU\left(t_r,t_o,T\right)}{\partial T}=0 \tag{11}$$

which also satisfies the condition,

$$\frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r^2}>0\ ,\ \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_o^2}>0\ \&\ \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial T^2}>0 \tag{12}$$

A hessian matrix M_{33} is formed by (10) and can be written as

$$M_{33}=\begin{pmatrix} \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r^2} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r\partial t_o} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r\partial T} \\ \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_o\partial t_r} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_o^2} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_o\partial T} \\ \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial T\partial t_r} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial T\partial t_o} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial T^2} \end{pmatrix}$$

$$M_{11}=\left(\frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r^2}\right)$$

$$M_{22}=\begin{pmatrix} \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r^2} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_r\partial t_o} \\ \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_o\partial t_r} & \frac{\partial^2 ATCU\left(t_r,t_o,T\right)}{\partial t_o^2} \end{pmatrix}$$

The total average cost equation can be minimised if and only if the Hessian matrix 'M_{33}' is positive-definite. i.e., the determinant of all the minors of a hessian matrix M_{33} is positive.

$$\left|M_{11}\right|>0,\ \ \left|M_{22}\right|>0\ \ \&\ \ \left|M_{33}\right|>0 \tag{13}$$

V. Solution Procedure

The solution of equation (10) can be systematically obtained by following a sequence of seven steps.

Step 1:

Initiate the process by inputting all the pertinent parameters into the average total cost function.

Step 2:

Proceed to derive the first partial derivative of the average total cost function concerning the decision variable, as illustrated in equation (11).

Step 3:

Solve the resultant system of equations to determine the decision variables.

Step 4:

Affirm the correctness of equations (12) and (13) by including the decision variable obtained in Step 3.

Step 5:

If a solution does not meet the requirements of either equation (12) or (13), it suggests that the proposed model is erroneous or that it is not possible to minimise the total average cost. When faced with such situations, modify the existing parameters and then repeat the three procedures that were done before.

Step 6:

If the solution corresponds with the equations, these decision variables signify the optimal choices.

Step 7:

Determine the mean of the total cost per unit of time per cycle by substituting the values of the decision variables in equation (10).

VI. Numerical Analysis

Let us examine the parameter values as outlined in the table:

TABLE III
PARAMETER VALUES AND UNITS IN THE PROPOSED MODEL

Parameters	Values	Units
A	120	\$/order
α	10	-----
β	20	-----
Y	100	unit
λ_{wr}	0.03	%
λ_{wo}	0.09	%
$\hbar_{rc}$	2	\$/unit
$\hbar_{oc}$	3	\$/unit
s_{ws}	8	\$/unit
ls_{ws}	10	\$/unit
d_{wr}	0.8	\$/unit
d_{wo}	2	\$/unit
ℓ	0.06	%
p_u	1	\$/unit
p	2.6 p_u	\$/unit
λ	0.07	%

Upon solving equation (11) with the provided data, we determine the optimal values for t_1*, t_2*, and T* to be t_1*= 1.003, t_2* = 2.056, and T* = 2.329, respectively. Using these values in (10), we ascertain that the optimal order quantity is X* = 124.0173 units, resulting in a minimised average total cost of $ATCU$*=\$245.214.

$$M_{33}=\begin{pmatrix} 82.9118 & -108.4740 & -3.6534 \\ -108.4740 & 340.5169 & -233.0856 \\ -3.6534 & -233.0856 & 285.0893 \end{pmatrix}$$

$$\left|M_{11}\right|=82.9118>0,\ \left|M_{22}\right|=2.8548\text{e}+04>0\ \&\ \left|M_{33}\right|=5.6057\text{e}+05>0$$

The above example underscores a key observation: the total cost function demonstrates strict convexity. As a result, utilising the total cost function of the model allows us to ascertain the optimal value of decision variables. This optimisation revolves around minimising the total cost per unit of time within the inventory system.

VII. Sensitivity Analysis

A thorough sensitivity analysis was conducted across various scenarios to gain a comprehensive understanding of how parameter variations affect key inventory metrics and cost components. Within each scenario, specific parameters within the model were systematically adjusted, meticulously assessing their impact on critical factors such as the total average cost $\left(ATCU^*\left(t_r,t_o,T\right)\right)$, quantity $\left(Q^*\right)$, and time parameters $\left(t_r^*,t_o^*,T^*\right)$.

This approach allowed for a comprehensive exploration of the intricate relationships between these parameters and the resulting implications for inventory management. In the sections that follow, the findings will be presented, offering valuable insights into the dynamic nature of the proposed model under varying conditions.

For A: In Fig. 2, one can observe that an increase in ordering cost leads to a higher average total cost and quantity. Additionally, t_o^* and T^* exhibit slight increases, while t_r^* experiences a minor decrease.

For α: Both the total average cost and quantity demonstrated a consistent downward trend with increasing values of the α. Time parameters t_r^*, t_o^*, and T^* showed a minimal increase. These observation are illustrated in Fig. 3.

For β: The total average cost and quantity showed a consistent upward trend with increasing β. All time parameters exhibited slight decreases, as shown in Fig. 4.

For Y: When the quantity in the owned warehouse increases, the average cost and quantity strictly increase. Furthermore, all the time parameters also demonstrate an increase, as illustrated in Fig. 5.

For λ_{wr}: When the deterioration rate in the rented warehouse increases, both the average cost and the quantity exhibit a decrease, along with the time parameters t_r^* and T^*. Notably, t_o^* shows an increase in this scenario. These trends are illustrated in Fig. 6.

For λ_{wo}: An increase in the deterioration rate within the owned warehouse leads to higher average costs and quantities. Specifically, t_r^* and T^* experience an increase, while t_o^* remaining unaltered. These findings are visualized in Fig. 7, where minor adjustments in the time parameters can be observed.

For λ: An increase in the shortage rate resulted in a notable decrease in the average cost and a significant increase in the quantity held. Furthermore, all-time parameters exhibited slight increments, as shown in Fig. 8.

For $\hbar_{rc}$: An increase in the holding cost per unit in the rented warehouse led to a significant reduction in the average total cost and the quantity held in inventory. Moreover, this adjustment resulted in decreased values for all relevant parameters. These observations are graphically presented in Fig. 9.

For $\hbar_{oc}$: When the holding cost per unit in the owned warehouse increased, it led to a significant and consistent rise in the average total cost and inventory quantity. Moreover, all relevant time parameters exhibited noticeable increases in response to this adjustment. These observations are visually represented in Fig. 10.

For s_{ws}: When the shortage per unit cost increased, it significantly impacted the inventory system. Specifically, it led to a noteworthy increase in the average cost and a simultaneous decrease in inventory quantity. Furthermore, when examining the time parameters, it was observed that t_r^* and T^* experienced decreases, whereas t_o^* showed an increase in response to this adjustment. These findings are visually presented in Fig. 11.

For d_{wr}: When the deterioration per unit cost in the rented warehouse increased, it had a discernible impact on the inventory system. Specifically, it resulted in a notable increase in the average cost and a simultaneous increase in inventory quantity. Furthermore, upon a comprehensive examination of the time parameters, it became evident that t_r^* and t_o^* experienced increases, while T^* exhibited a decrease in response to this alteration. These significant observations are visually presented in Fig. 12.

For d_{wo}: When the deterioration per unit cost in the owned warehouse increased, it significantly impacted the inventory system. This change led to a notable increase in average cost and inventory quantity. Furthermore, all-time parameters exhibited slight increments in response to this adjustment. These observations and trends are visually depicted in Fig. 13.

For ℓ: When the inflation rate increases, it has a noteworthy impact on the inventory system. This change results in a substantial decrease in the total cost and a significant increase in inventory quantity. Additionally, all time parameters experience increases in response to this alteration. These observations and trends are visually represented in Fig. 14.

For p_u: When the purchasing per unit cost experiences an increase, the consequences are significant for the inventory system. Specifically, this change results in a substantial and strict increase in the average total cost while simultaneously leading to a decrease in inventory quantity. Additionally, all time parameters exhibit decreases in response to this modification. These observations and trends are visually represented in Fig. 15.

For p: When the selling price per item increases, it has a noteworthy impact on the inventory system. Specifically, this change significantly reduces the average total cost, indicating improved cost efficiency. Additionally, it results in a slight decrease in the quantity of inventory. Furthermore, all-time parameters exhibit decreases in response to this modification. These effects are visually depicted in Fig. 16, illustrating the system's sensitivity to changes in the selling price per item.

For p_u: When the purchasing per unit cost experiences an increase, the consequences are significant for the inventory system. Specifically, this change results in a substantial and strict increase in the average total cost while simultaneously leading to a decrease in inventory quantity. Additionally, all time parameters exhibit decreases in response to this modification. These observations and trends are visually represented in Fig. 15.

For p: When the selling price per item increases, it has a noteworthy impact on the inventory system. Specifically, this change leads to a significant reduction in the average total cost, indicating improved cost efficiency. Additionally, it results in a slight decrease in the quantity of inventory.

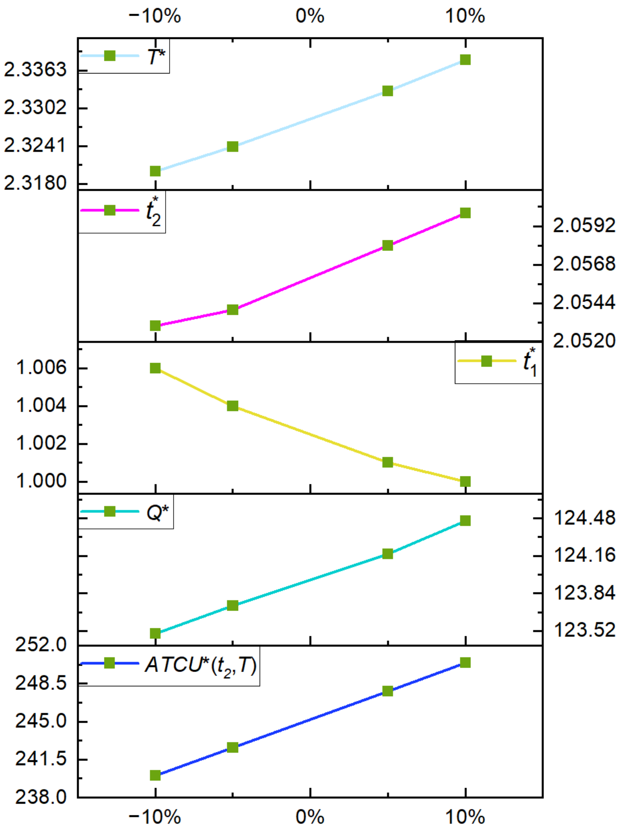

Fig. 2. Variation of Parameter A

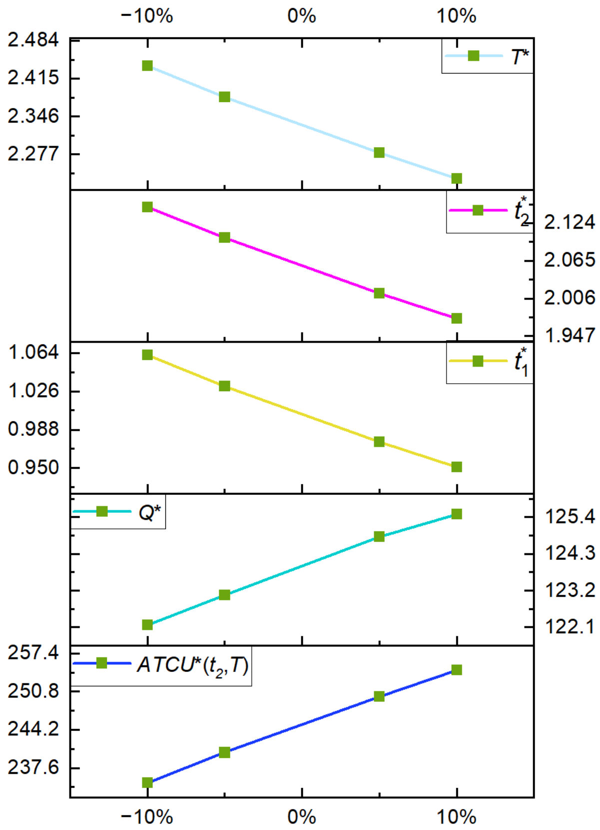

Fig. 4. Variation of Parameter β

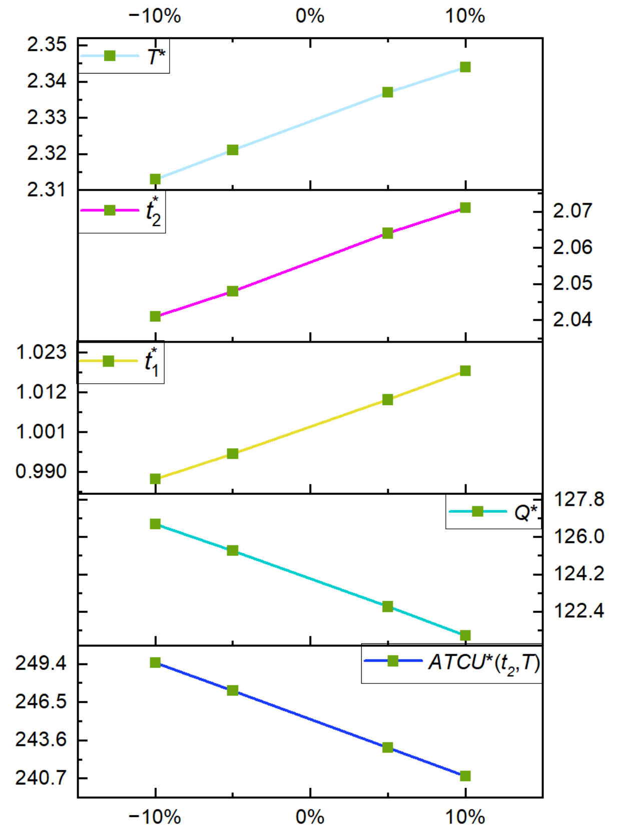

Fig. 3. Variation of Parameter α

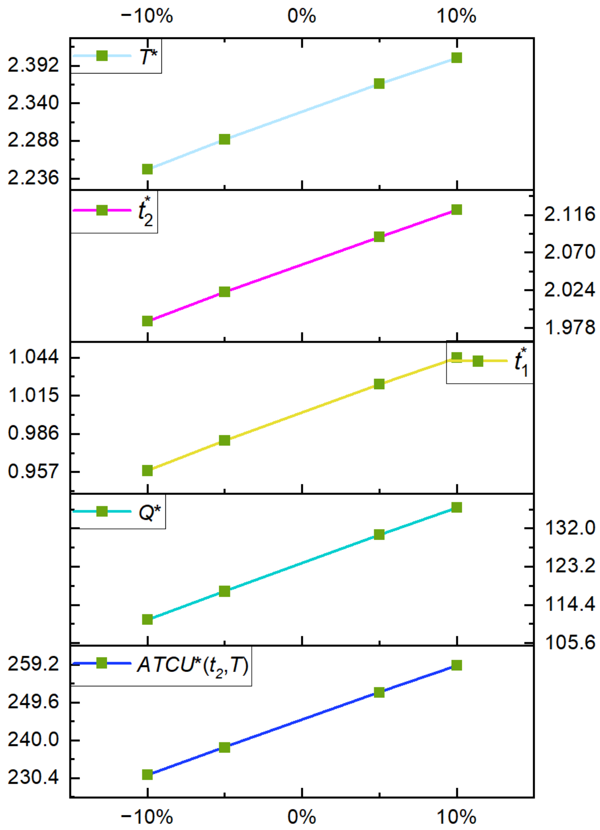

Fig. 5. Variation of Parameter Y

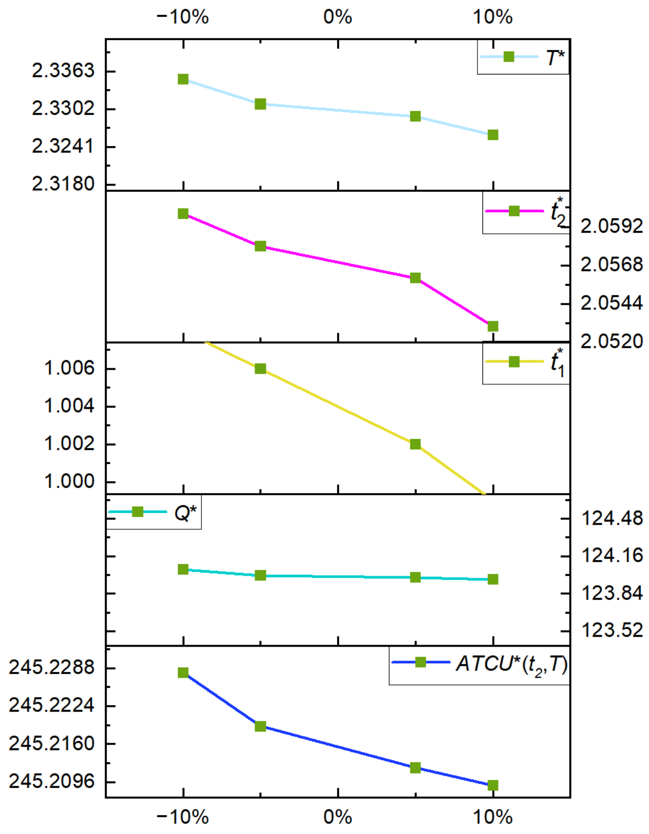

Fig. 6. Variation of Parameter $\hbar_{oc}$

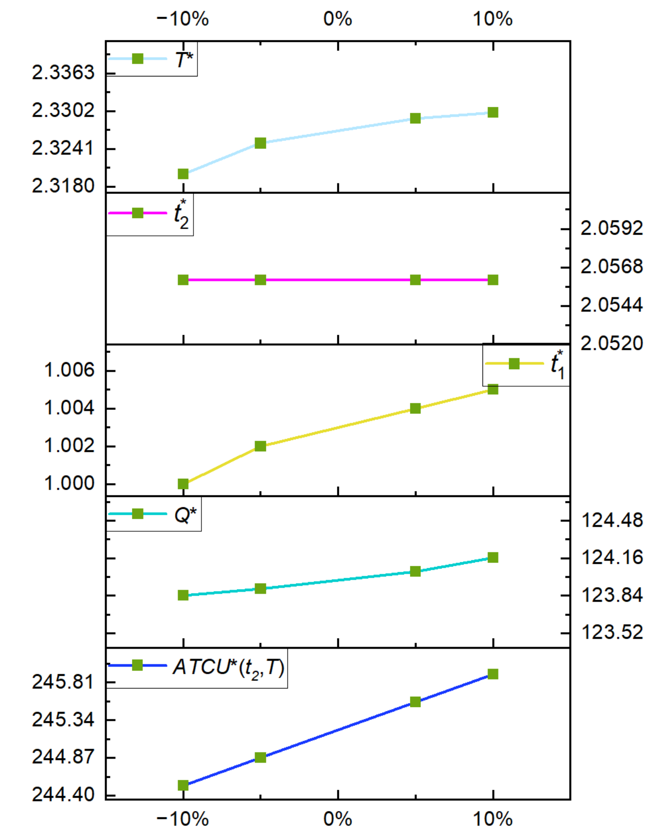

Fig. 7. Variation of Parameter $\lambdabar_{wo}$

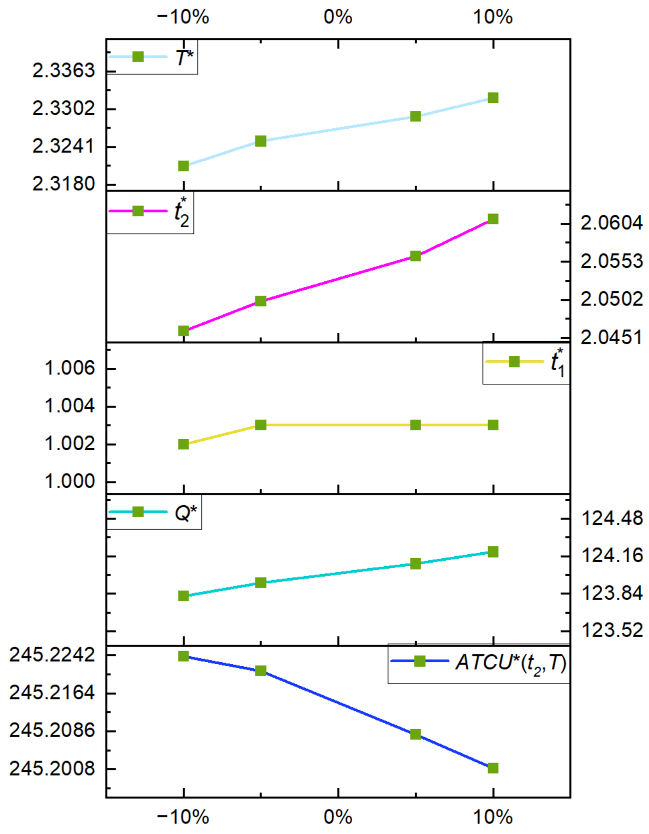

Fig. 8. Variation of Parameter λ

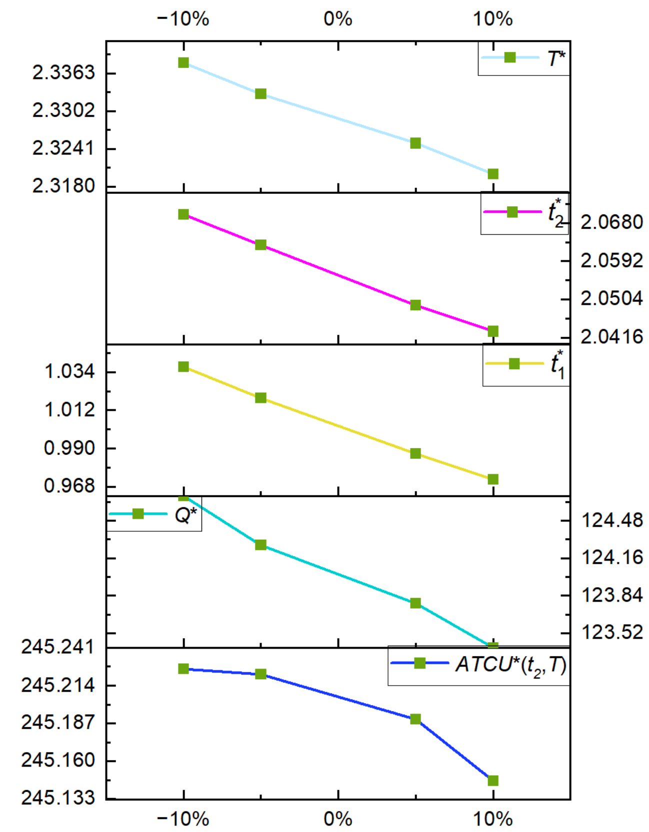

Fig. 9. Variation of Parameter $\hbar_{rc}$

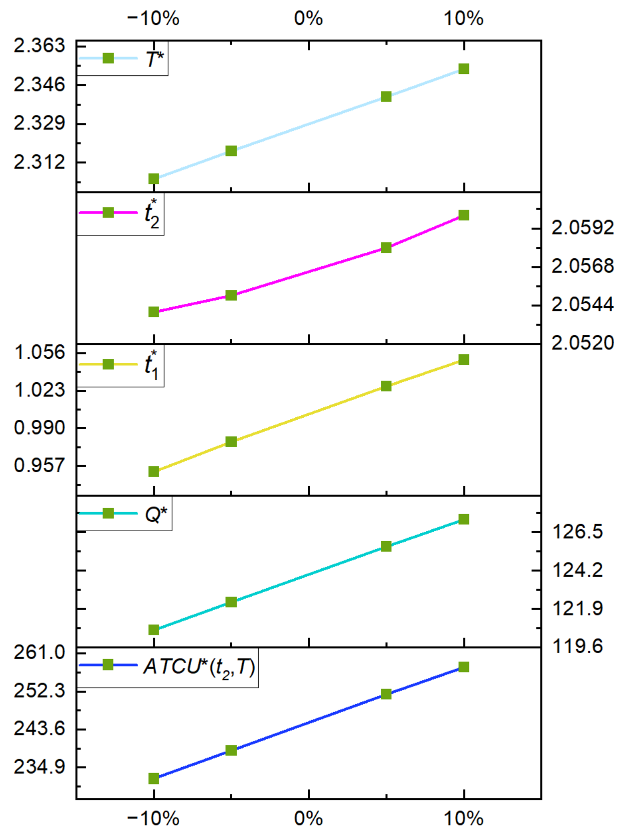

Fig. 10. Variation of Parameter $\hbar_{oc}$

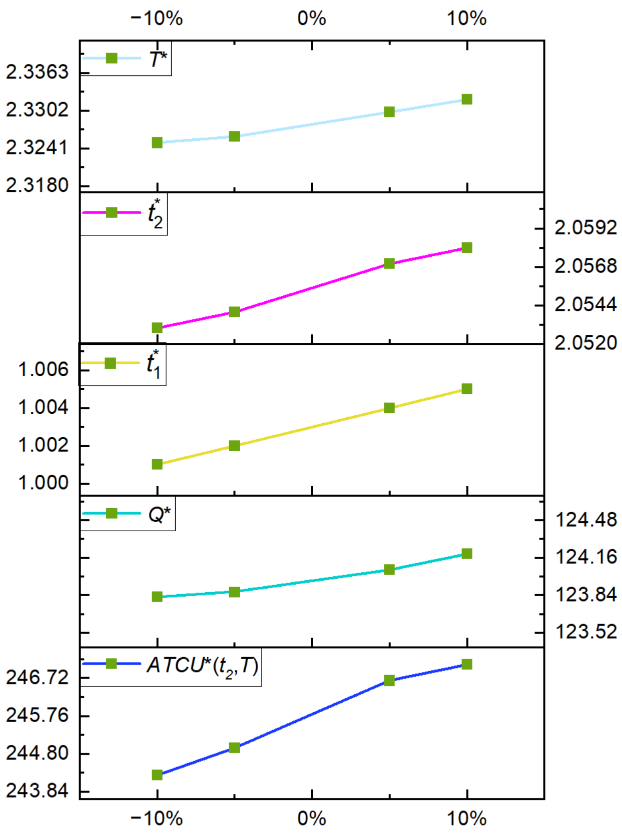

Fig. 12. Variation of Parameter d_{wr}

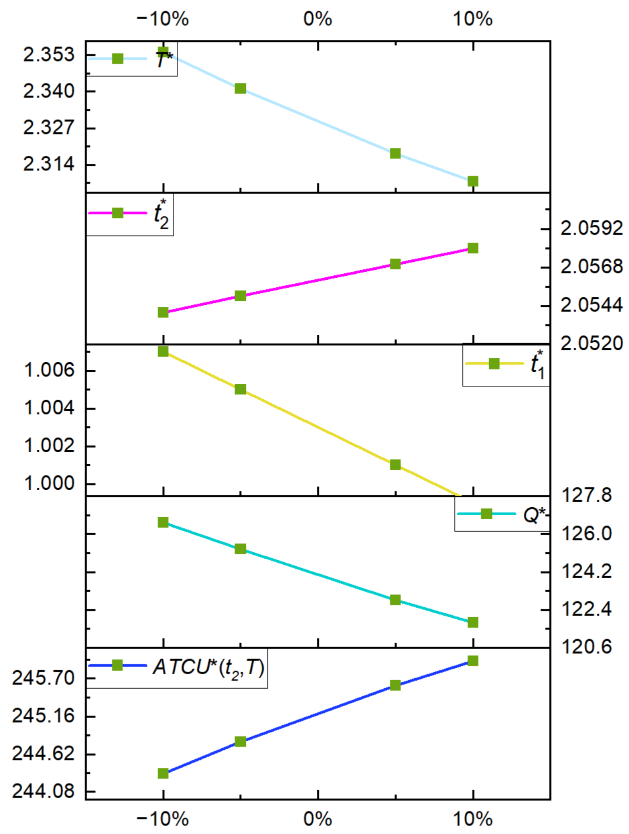

Fig. 11 Variation of Parameter s_{ws}

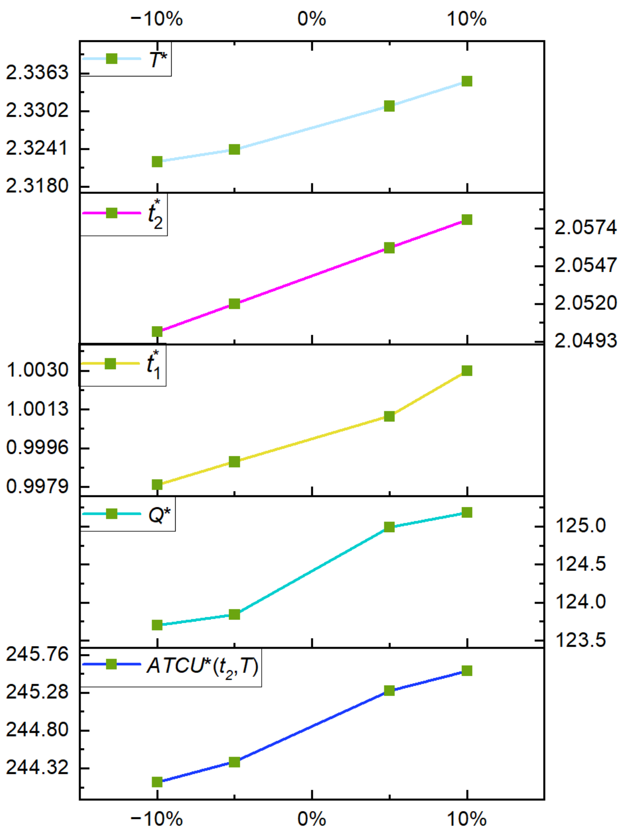

Fig. 13. Variation of Parameter d_{wo}

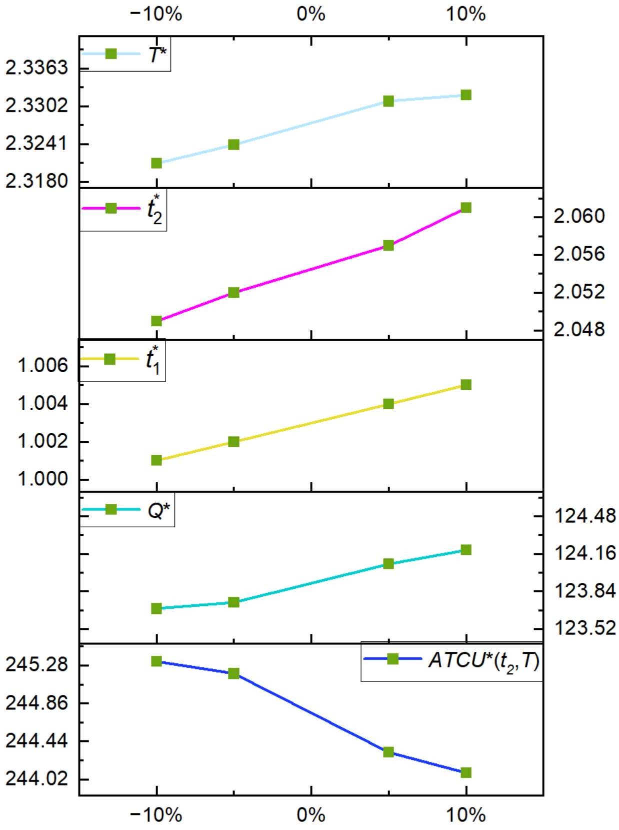

Fig. 14 Variation of Parameter ℓ

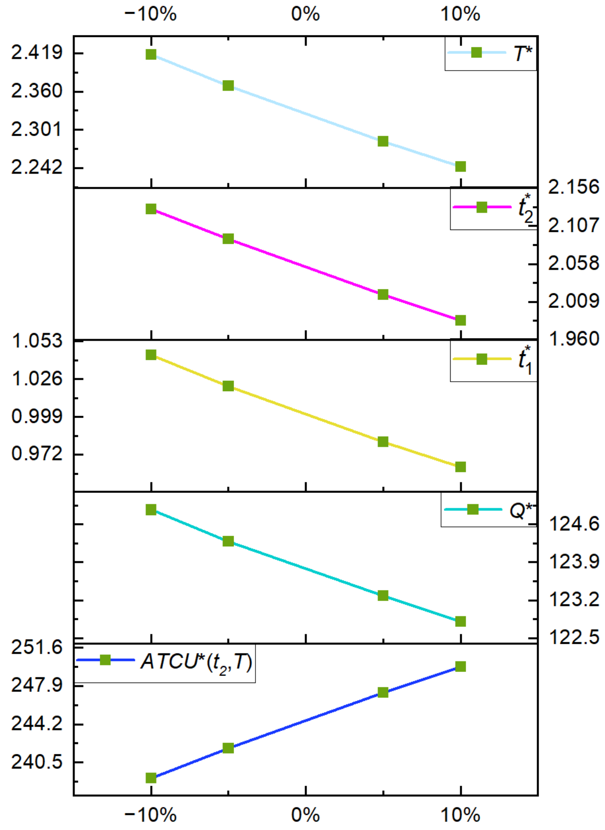

Fig. 16. Variation of Parameter p

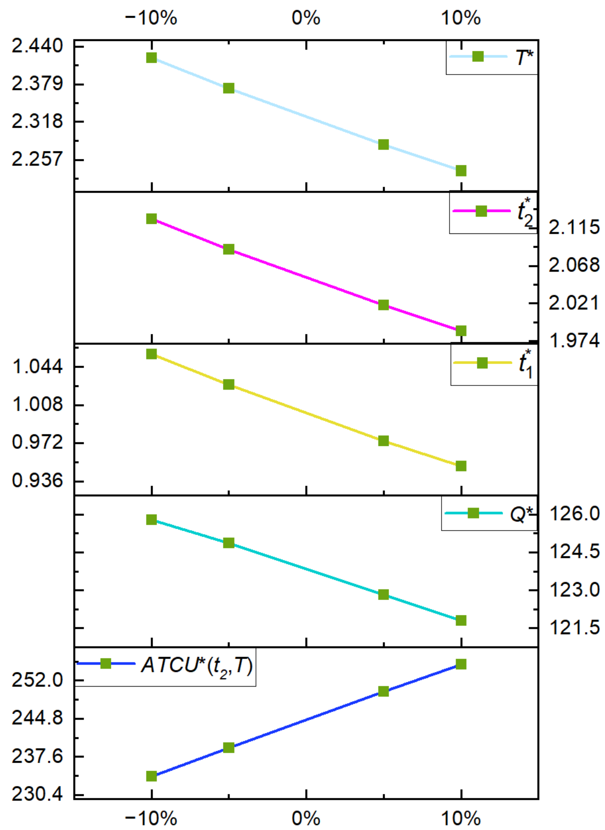

Fig. 15. Variation of Parameter p_u

Furthermore, all-time parameters exhibit decreases in response to this modification. These effects are visually depicted in Fig. 16, illustrating the system's sensitivity to changes in the selling price per item.

VIII. Conclusion

This research paper is a distinctive contribution to the landscape of two-warehouse inventory management. Its uniqueness lies in meticulously exploring deteriorating items, considering the interplay of time and price-dependent demand amidst inflationary trends and supply shortages. By delving into these complexities, this study offers insights that set it apart from existing works in the field.

The benefits of this research are multifaceted. Firstly, it sheds light on a niche aspect of inventory management that can potentially impact industries dealing with perishable goods extensively. Secondly, the comprehensive analysis of various factors influencing the inventory system provides practitioners with a nuanced understanding to enhance decision-making processes. Moreover, the considerations of inflation and shortage introduce a realistic dimension that mirrors real-world scenarios, thereby rendering the findings more applicable.

Numerous exciting avenues for future research and development emerge from the findings of this study. The current research has uncovered valuable insights and illuminated potential directions that can expand the scope and impact of two-warehouse inventory management research.

Integrating advanced machine learning and predictive analytics techniques is an area with significant promise for further exploration. By harnessing the power of data-driven models, researchers could create more accurate demand forecasts, optimise inventory replenishment strategies, and even develop adaptive systems that respond in real-time to changing market dynamics.

Furthermore, extending the analysis to encompass a more comprehensive range of industries and contexts could provide a broader understanding of the applicability of the proposed inventory model. Each sector may have unique variables and constraints that influence inventory decisions, and by studying these differences, we can refine the model to be even more versatile and robust.

Lastly, the study could benefit from collaborative efforts between academia and industry. Real-world implementation and validation of the proposed inventory model in various business settings could provide valuable feedback and insights, ultimately bridging the gap between theoretical research and practical applications.

In essence, the future scope of research in dual-warehouse inventory management is rich and diverse. By capitalising on the groundwork laid by this study, researchers can contribute to enhanced decision-making processes, innovative technological solutions, and more sustainable inventory management practices across industries.

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