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Results on Existence and Uniqueness of solutions of Fractional Differential Equations of Caputo-Fabrizio type in the sense of Riemann-Liouville

Dodi Igobi, Wisdom Udogworen

***Abstract*—The Riemann-Liouville notion of the Caputo-Fabrizio fractional differential equation is examined in this paper. The fundamental structure of the fractional calculus is shown. The Laplace transform of the derivative and its related integral are discovered. The extension of the findings to established the existence and uniqueness of the Caputo-Fabrizio fractional differential equation of order $\alpha \in (4,5)$ is examined using the principle of Banach contraction mapping. Examples are employed in order to demonstrate the idea.**

***Index Terms*—fractional calculus, non-singular kernel, Laplace transform, convolution.**

I. Introduction

GIVEN any differential equation either linear or nonlinear, integer or non-integer order, there is a primitive or trajectory that represents the equation's solution. The analytical procedure used in [1] yields this solution.

Even though finding the solution to differential equations is extremely important, the study of the qualitative properties of differential equation solutions has historically piqued mathematicians' interests the most. The existence and uniqueness of the solution are the traditional questions pertaining to differential equations. Modeling of natural phenomena, particularly those in biology, medicine, and economics, results in differential equations that are usually nonlinear It was observed in [2], [3] that they are increasingly complex and the resulting differential equations are highly nonlinear. Hence finding analytical solutions to such models is not easily obtained.

To obtain the solutions of such differential equations in closed form is extremely difficult, if not impossible. In a situation where modeling certain phenomena may result in a differential equation that may not have any solution and the researcher may spend days, weeks, or even months trying to solve an equation that has no solution. This can indeed be very frustrating, yet this kind of situation might be so widespread that several areas of research in science and engineering would suffer severe setbacks. The qualitative theory of differential equations comes in handy in this direction.

Over the past few years, fractional calculus has developed more quickly and attracted a great deal of attention. Numerous outstanding findings from the fractional models were published in [4] [5] [6] [7] [8] and effectively used in numerous scientific and engineering domains.

According to our assessment, there are currently several major forms of fractional differential equations that are extensively researched, enabling researchers to select the type of fractional derivative that best fits a certain real-world scenario. In various branches of science and engineering, it has become important for certain researchers to develop new fractional derivatives with non-singular kernels in order to model more real-world issues. Study was done on a new fractional differential equation that has the advantage of non-singularity kernel and can explain materials and structures of varied time scales that are different for fractional models with singular kernels to adequately depict as seen in [9] [10] [11] [12] [13] [15] [14]. In this study, we extend on the findings about the existence and uniqueness of solutions examined in [15] [16] [17] [18] [19] [20] [21] to fractional differential equation where $\alpha \in (4,5)$

Definition 1. *[16] Let $n < \alpha \leq n+1$, $n \in \mathbb{N}$ and f be a real function defined on $t \in [a,b]$ the Riemann-Liouville fractional integral of order α is defined by*

$$({}_aI_t^{\alpha})f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds \tag{1}$$

where

$$\Gamma(\alpha) = \int_0^{\infty}(t)^{\alpha-1}e^{-t}dt \qquad \alpha > 0 \tag{2}$$

is the gamma function
(1) is the generalization of the n-iterated integral given as

$$({}_aI_t^{n})f(t) == \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}f(s)ds, t \in [a,b] \tag{3}$$

Definition 2. *[15] Let $f \in H^{'}(a,b)$, $a < b$, $\alpha \in [0,1)$ then the definition of the new (left Caputo) fractional derivative in the sense of Caputo and Fabrizio becomes*

$$({}_a^{CFC}D_t^{\alpha})f(t) = \frac{B(\alpha)}{1-\alpha}\int_a^t f^{'}(x)e^{[-\alpha\frac{(t-x)}{1-\alpha}]}dx \tag{4}$$

where $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$

Definition 3. *In the right case we have*

$$({}^{CFC}D_b^{\alpha}f)(t) = \frac{-B(\alpha)}{1-\alpha}\int_t^b f^{'}(x)e^{[-\alpha\frac{(x-t)}{1-\alpha}]}dx \tag{5}$$

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D. Igobi is a Senior Lecturer in the Department of Mathematics, University of Uyo, Nigeria; e-mail: dodiigobi@uniuyo.edu.ng.
W. Udogworen is a researcher in the Department of Mathematics, University of Uyo, Nigeria; e-mail: udogworenwisdom@gmail.com.

Definition 4. *[15] Let $f \in H^{'}(a,b)$, $a < b$, $\alpha \in [0,1)$, then, Caputo and Fabrizio, the definition of the new (left) Riemann-Liouville fractional derivative becomes*

$$\left({}_{a}^{CFR}D^{\alpha}f\right)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{a}^{t} f(x)e^{[-\alpha\frac{(t-x)}{1-\alpha}]}dx \quad (6)$$

The associated fractional integral is

$$\left({}_{a}^{CF}I^{\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_{a}^{t} f(s)ds \quad (7)$$

Thus in the meaning of Caputo and Fabrizio, right Riemann-Liouville fractional derivative is

$$\left({}^{CFR}D_{b}^{\alpha}f\right)(t) = \frac{-B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{t}^{b} f(x)e^{[-\alpha\frac{(t-x)}{1-\alpha}]}dx \quad (8)$$

The associated fractional integral is

$$\left({}_{t}^{CF}I_{b}^{\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_{a}^{t} f(s)ds \quad (9)$$

Definition 5. *[16] Let $n < \alpha \le n+1$ and f be such that $f^{n} \in H'[a,b]$, $a < b$, set $\beta = \alpha - n$, $\beta \in (0,1)$, then Caputo-Fabrizio type's Riemann-Liouville fractional derivative has the following form*

$$\left({}_{a}^{CFR}D_{t}^{\alpha}f\right)(t) = \left({}_{a}^{CFR}D_{t}^{\beta}f^{(n)}\right)(t). \quad (10)$$

The associated fractional integral is

$$\left({}_{a}^{CF}I_{t}^{\alpha}f\right)(t) = \left({}_{a}I^{n}\left({}_{a}^{CF}I_{t}{}^{\beta}f\right)\right)(t). \quad (11)$$

If we use the convention that $({}_{a}I_{t}^{0}f)(t) = f(t)$ then for the case $0 < \alpha \le 1$ we have $\beta = \alpha$ and hence $({}_{a}I_{t}^{\alpha}f)(t) = ({}_{a}I_{t}^{\alpha}f)(t)$ Also, the convention $f^{(0)}(t) = f(t)$ lead to $({}_{a}^{CFR}D_{t}^{\alpha}f)(t) = ({}_{a}^{CFR}D_{t}^{\alpha}f)(t)$ for $0 < \alpha \le 1$

Definition 6. *[15] Let $n < \alpha \le n+1$ and f be such that $f^{n} \in H'[a,b]$, $a < b$, set $\beta = \alpha - n$, $\beta \in (0,1)$, then the right Riemann-Liouville fractional derivative of Caputo-Fabrizio type has the following form*

$$\left({}_{t}^{CFR}D_{b}^{\alpha}f\right)(t) = \left({}_{t}^{CFR}D_{b}^{\beta}(-1)^{n}f^{(n)}\right)(t). \quad (12)$$

The corresponding fractional integral is

$$\left({}_{t}^{CF}I_{b}^{\alpha}f\right)(t) = \left({}_{t}I_{b}^{n}\left({}_{t}^{CF}I_{b}^{\beta}f\right)\right)(t). \quad (13)$$

If we use the convention that $({}_{t}I_{b}^{0}f)(t) = f(t)$ then for the case $0 < \alpha \le 1$ we have $\beta = \alpha$ and hence $({}_{t}I_{b}^{\alpha}f)(t) = ({}_{t}I_{b}^{\alpha}f)(t)$ Also, the convention $f^{(0)}(t) = f(t)$ lead to $({}_{t}^{CFR}D_{b}^{\alpha}f)(t) = ({}_{t}^{CFR}D_{b}^{\alpha}f)(t)$ for $0 < \alpha \le 1$

II. Preliminaries

proposition 1. *[16] For $u(t) \in [a,b]$ and $n \le \alpha \le n+1$ we have*

$$\left({}_{a}^{CFR}D^{\alpha}\cdot{}_{a}^{CF}I^{\alpha}u\right)(t) = u(t) \quad (14)$$

and

$$\left({}_{a}^{CF}I^{\alpha}{}_{a}^{CFR\alpha}u\right)(t) = u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^{k} \quad (15)$$

proposition 2. *[15] For $u(t) \in [a,b]$ and $n \le \alpha \le n+1$ we have*

$$\left({}^{CFR}D_{b}^{\alpha}\cdot{}^{CF}I_{b}^{\alpha}u\right)(t) = u(t) \quad (16)$$

and

$$\left({}^{CF}I_{b}^{\alpha}{}^{CFR}D_{b}^{\alpha}u\right)(t) = u(t) - \sum_{k=0}^{n-1}\frac{(-1)^{k}u^{(k)}(b)}{k!}(b-t)^{k} \quad (17)$$

Lemma 1. *Let $f \in H^{'}(a,b)$, $a < b$, $\alpha \in [0,1)$ such that*

$$\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{0}^{x} f(t)e^{\frac{-\alpha(x-t)}{1-\alpha}}dt$$

Hence, in the Riemann-Liouville sense, the Laplace transform of the Caputo-Fabrizio derivative is given as

$$\mathcal{L}\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t) = \frac{sF(s)}{s(1-\alpha)+\alpha} \quad (18)$$

a) Proof:: From Caputo-Fabrizio's definition we have

$$\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{0}^{x} f(t)e^{\frac{-\alpha(x-t)}{1-\alpha}}dt$$

$$\mathcal{L}\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t) = \mathcal{L}\left(\frac{1}{1-\alpha}\frac{d}{dt}\int_{0}^{x} f(t)e^{\frac{-\alpha(x-t)}{1-\alpha}}dt\right)$$

$$= \frac{1}{1-\alpha}\mathcal{L}\left\{\frac{d}{dt}\int_{0}^{x} f(t)e^{\frac{-\alpha(x-t)}{1-\alpha}}dt\right\}$$

$$= \frac{1}{1-\alpha}\left\{\mathcal{L}\{\frac{d}{dt}\}\cdot\mathcal{L}\int_{0}^{x} f(t)e^{\frac{-\alpha(x-t)}{1-\alpha}}dt\right\}.$$

Using the convolution theorem [1] given as

$$f*g = \int_{0}^{x} f(t)g(x-t)dt$$

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\} = F(s)G(s)$$

we have

$$\mathcal{L}\left({}_{0}^{CFR\alpha}{}_{x}f\right)(t) = \frac{1}{1-\alpha}\left[\mathcal{L}\{\frac{d}{dt}\}\cdot\mathcal{L}\{e^{\frac{-\alpha}{1-\alpha}}t\}\cdot\mathcal{L}\{f(t)\}\right]$$

$$= \frac{1}{1-\alpha}\left[s\cdot\frac{1}{s+\frac{\alpha}{1-\alpha}}\cdot F(s)\right] = \frac{1}{1-\alpha}\left[\frac{s(1-\alpha)F(s)}{S(1-\alpha)+\alpha}\right]$$

$$= \frac{sF(s)}{s(1-\alpha)+\alpha}.$$

Therefore

$$\mathcal{L}\left\{\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t)\right\} = \frac{sF(s)}{s(1-\alpha)+\alpha}$$

Example 1. *Consider the equation $\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t) = sint$. By* **Lemma 1.** *we apply Laplace transform starting at a to both sides and making use of convolution theorem we have*

$$\mathcal{L}\left({}_{0}^{CFR}D_{x}^{\alpha}f\right)(t) = \mathcal{L}\{\sin kt\}$$

$$\frac{B(\alpha)sF(s)}{s(1-\alpha)+\alpha} = \frac{k}{s^{2}+k^{2}}$$

$$B(\alpha)sF(s) = [s(1-\alpha)+\alpha]\frac{k}{s^{2}+k^{2}}$$

$$B(\alpha)sF(s) = s(1-\alpha)\frac{k}{s^{2}+k^{2}} + \alpha\frac{k}{s^{2}+k^{2}}$$

$$F(s) = \frac{s(1-\alpha)k}{sB(\alpha)(s^{2}+k^{2})} + \frac{\alpha k}{sB(\alpha)(s^{2}+k^{2})}$$

$$F(s) = \frac{(1-\alpha)k}{B(\alpha)(s^{2}+k^{2})} + \frac{\alpha}{B(\alpha)}\frac{k}{s(s^{2}+k^{2})}$$

Then applying the inverse Laplace transform we obtain the associated integral form

$$\mathcal{L}^{-1}\{F(s)\} = \frac{(1-\alpha)}{B(\alpha)}\mathcal{L}^{-1}\{\frac{k}{(s^2+k^2)}\}$$

$$+\frac{\alpha}{B(\alpha)}\mathcal{L}^{-1}\{\frac{k}{s(s^2+k^2)}\}$$

$$f(t) = \frac{1-\alpha}{B(\alpha)}\sin kt + \frac{\alpha}{B(\alpha)}\int_a^t \sin ksds$$

$$\left({}_a^{CF}D^{\alpha}f\right)(t) = \frac{1-\alpha}{B(\alpha)}\sin kt + \frac{\alpha}{B(\alpha)}\int_a^t \sin ksds$$

Example 2. *Consider the Caputo-Fabrizio derivative given as*

$$\left({}_a^{CFR}D^{\alpha}y\right)(t) = \cos kt \tag{19}$$

Then by Lemma1, the Laplace transform is given as

$$F(s) = \frac{(1-\alpha)s}{B(\alpha)(s^2+k^2)} + \frac{\alpha}{B(\alpha)}\frac{s}{s(s^2+k^2)}. \tag{20}$$

From equation (19), we have

$$\mathcal{L}\left({}_0^{CFR}D_x^{\alpha}f\right)(t) = \mathcal{L}\{\cos kt\}$$

$$\frac{B(\alpha)sF(s)}{s(1-\alpha)+\alpha} = \frac{s}{s^2+k^2}$$

$$B(\alpha)sF(s) = [s(1-\alpha)+\alpha]\frac{s}{s^2+k^2}$$

$$B(\alpha)sF(s) = s(1-\alpha)\frac{s}{s^2+k^2} + \alpha\frac{s}{s^2+k^2}$$

$$F(s) = \frac{s(1-\alpha)s}{sB(\alpha)(s^2+k^2)} + \frac{\alpha s}{sB(\alpha)(s^2+k^2)}$$

$$F(s) = \frac{(1-\alpha)s}{B(\alpha)(s^2+k^2)} + \frac{\alpha}{B(\alpha)}\frac{s}{s(s^2+k^2)}$$

The related integral form is then obtained by performing the inverse Laplace transform as

$$\mathcal{L}^{-1}\{F(s)\} = \frac{(1-\alpha)}{B(\alpha)}\mathcal{L}^{-1}\{\frac{s}{(s^2+k^2)}\}$$

$$+\frac{\alpha}{B(\alpha)}\mathcal{L}^{-1}\{\frac{s}{s(s^2+k^2)}\}$$

$$f(t) = \frac{1-\alpha}{B(\alpha)}\cos kt + \frac{\alpha}{B(\alpha)}\int_a^t \cos ksds$$

Therefore we can defined the corresponding higher fractional integral of $\left({}_0^{CFR}D_x^{\alpha}\right)$ *by* $\left({}_0^{CFR}D_x^{\beta}\right)$ *where* $\beta = \alpha - n$

$$\left({}_a^{CF}I^{\beta}f^{(n)}\right)(t) = \frac{1-\beta}{B(\beta)}coskt + \frac{\beta}{B(\beta)}\int_a^t cosksds \tag{21}$$

$$\left({}_aI^n\left({}_a^{CF}I_t^{\beta}f^{(n)}\right)\right)(t) = \left({}_aI^n\right)\Big[\frac{1-\beta}{B(\beta)}coskt + \frac{\beta}{B(\beta)}\int_a^t cosksds\Big] \tag{22}$$

Theorem 1. *Let* $n < \alpha \leq n+1$ *and* f *be such that* $f^n \in H'[a,b]$, $a < b$, *set* $\beta = \alpha - n$, $\beta \in (0,1)$, *such that*

$$\left({}_a^{CFR}D_t^{\alpha}f\right)(t) = \left({}_a^{CFR}D_t^{\beta}f^{(n)}\right)(t) \tag{23}$$

then Laplace transform of the higher order fractional derivatives is given as

$$\mathcal{L}\Big\{\left({}_0^{CFR}D_x^{\beta}f^{(n)}\right)(t)\Big\} = \frac{B(\beta)\big[ss^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0)\big]}{s(1-\beta)+\beta}$$

b) Proof::

$$\left({}_0^{CFR}D_x^{\beta}f^{(n)}\right)(t) = \frac{B(\beta)}{1-\beta}\frac{d}{dt}\int_0^t f^{(n)}(t)e^{\frac{-\beta(x-t)}{1-\beta}}dt$$

$$\mathcal{L}\left({}_0^{CFR}D_x^{\beta}f\right)(t) = \mathcal{L}\left(\frac{B(\beta)}{1-\beta}\frac{d}{dt}\int_0^t f^{(n)}(t)e^{\frac{-\beta(x-t)}{1-\beta}}dt\right)$$

$$= \frac{B(\beta)}{1-\beta}\mathcal{L}\left\{\frac{d}{dt}\int_0^t f^{(n)}(t)e^{\frac{-\alpha(x-t)}{1-\alpha}}dt\right\}$$

$$= \frac{B(\beta)}{1-\beta}\left\{\mathcal{L}\{\frac{d}{dt}\}\cdot\mathcal{L}\int_0^t f^{(n)}(t)e^{\frac{-\beta(x-t)}{1-\beta}}dt\right\}.$$

Using the convolution theorem we have

$$f * g = \int_0^t f(t)g(x-t)dt$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\} = F(s)G(s)$$

$$\mathcal{L}\left({}_0^{CFR}D_x^{\beta}f^{(n)}\right)(t) = \frac{B(\beta)}{1-\beta}\Big[\mathcal{L}\{\frac{d}{dt}\}\cdot\mathcal{L}\{e^{\frac{-\beta}{1-\beta}}t\}\cdot\mathcal{L}\{f^{(n)}(t)\}\Big]$$

$$= \frac{B(\beta)}{1-\beta}\big[s\cdot\frac{1}{s+\frac{\beta}{1-\beta}}\cdot s^nF(s) - \sum_{k=1}^n F^{(k-1)}(0)\big]$$

$$= \frac{B(\beta)}{1-\beta}\Big[\frac{s(1-\beta)s^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0)}{s(1-\beta)+\beta}\Big]$$

$$= \frac{B(\beta)\big[ss^nF(s) - \sum_{k=1}^n s^{n-k}F(k-1)(0)\big]}{s(1-\beta)+\beta}$$

Therefore

$$\mathcal{L}\Big\{\left({}_0^{CFR}D_x^{\beta}f^{(n)}\right)(t)\Big\} = \frac{B(\beta)\big[ss^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0)\big]}{s(1-\beta)+\beta}$$

Consider the equation $\left({}_0^{CFR}D_x^{\alpha}f\right)(t) = u(t)$. *If we apply Laplace transform starting at* a *to both sides and use the convolution theorem we have*

$$\left({}_a^{CFR}D^{\beta}f\right)(t) = u(t) \tag{24}$$

$$\mathcal{L}\left({}_0^{CFR}{}_x^{\beta}f\right)(t) = \mathcal{L}\{u(t)\}$$

$$\frac{B(\beta)\big[ss^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0)\big]}{s(1-\beta)+\beta} = U(s)$$

$$B(\beta)\big[ss^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0)\big] = [s(1-\beta)+\beta]U(s)$$

$$B(\beta)\big[ss^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0)\big] = s(1-\beta)U(s) + \beta U(s$$

$$s^nF(s) - \sum_{k=1}^n s^{n-k}F^{(k-1)}(0) = \frac{s(1-\beta)U(s)}{sB(\beta)} + \frac{\beta U(s)}{sB(\beta)}$$

$$s^nF(s)-\sum_{k=1}^{n}s^{n-k}F^{(k-1)}(0)=\frac{(1-\beta)U(s)}{B(\beta)}+\frac{\beta}{B(\beta)}\frac{U(s)}{s}$$

Then applying the inverse Laplace transform we have

$$\mathcal{L}^{-1}\{s^nF(s)-\sum_{k=1}^{n}s^{n-k}F^{(k-1)}(0)\}=$$

$$\frac{(1-\beta)}{B(\beta)}\mathcal{L}^{-1}\{U(s)\}+\frac{\beta}{B(\beta)}\mathcal{L}^{-1}\{\frac{U(s)}{s}\}$$

$$f^{(n)}(t)=\frac{1-\beta}{B(\beta)}u(t)+\frac{\beta}{B(\beta)}\int_a^t u(s)ds$$

Therefore we can define the corresponding fractional integral of $\left({}_0^{CFR\beta}{}_x\right)$ *by*

$$\left({}_a^{CF}I^{\beta}f^{(n)}\right)(t)=\frac{1-\beta}{B(\beta)}u(t)+\frac{\beta}{B(\beta)}\int_a^t u(s)ds \tag{25}$$

$$\left({}_aI^n\left({}_a^{CF}I_t^{\beta}f^{(n)}\right)\right)(t)=\left({}_aI^n\right)\left[\frac{1-\beta}{B(\beta)}u(t)+\frac{\beta}{B(\beta)}\int_a^t u(s)ds\right] \tag{26}$$

We defined the right fractional integral as

$$\left({}_tI_b^n\left({}^{CF}I_b^{\beta}f^{(n)}\right)\right)(t)=\left({}_tI_b^n\right)\left[\frac{1-\beta}{B(\beta)}f(t)+\frac{\beta}{B(\beta)}\int_t^b u(s)ds\right] \tag{27}$$

Theorem 2. *[15] Consider the system*

$$\left({}_a^{CFR}D^{\alpha}y\right)(t)=f(t,y(t)),\quad t\in[a,b],\ 0<\alpha\leq 1, \tag{28}$$

with the initial condition
$y(a)=c$
such that $A\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha(b-a)}{B(\alpha)}\right)<1$ *and* $|f(t,y_1)-f(t,y_2)|\leq A|y_1-y_2|$, $A>0$. *Here* $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ *and* $y:[a,b]\to\mathbb{R}$, *then the system (28) has a unique solution of the form*

$$y(t)=c+{}_a^{CF}I^{\alpha}f(t,y(t)) \tag{29}$$

Example 3. *Consider the system*

$$\left({}_a^{CFR\alpha}y\right)(t)=\sin t,\quad t\in[a,b],\ 0<\alpha\leq 1, \tag{30}$$

with the initial condition
$y(a)=c$
Such that
$A\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}(b-a)\right)<1$ *and* $|siny_1-siny_2|\leq A|y_1-y_2|$, $A>0$. *Here* $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ *and* $y:[a,b]\to\mathbb{R}$ *then the system (30) has a unique solution of the form*

$$y(t)=c+{}_a^{CF}I^{\alpha}sint \tag{31}$$

First, we apply ${}_a^{CF}I^{\alpha}$ *to the system (30) to have*

$$\left({}_a^{CF}I^{\alpha}\right)\left({}_a^{CFR}D^{\alpha}y\right)(t)=\left({}_a^{CF}I^{\alpha}\right)\sin t, \tag{32}$$

we recall that

$$\left({}_a^{CF}I^{\alpha}{}_a^{CFR}D^{\alpha}y\right)(t)=y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k \tag{33}$$

for $k=0$ *we have*

$$\left({}_a^{CF}I^{\alpha}{}_a^{CFR}D^{\alpha}y\right)(t)=y(t)-y(a)$$

$$y(t)-y(a)={}_a^{CF}I^{\alpha}\sin t,$$

$$y(t)=y(a)+{}_a^{CF}I^{\alpha}\sin t,$$

but $y(a)=c$ *therefore*

$$y(t)=c+{}_a^{CF\alpha}\sin t, \tag{34}$$

Conversely if we apply ${}_a^{CFR\alpha}$ *on (31) we have*

$${}_a^{CFR}D^{\alpha}y(t)={}_a^{CFR}D^{\alpha}c+{}_a^{CFR}D^{\alpha}{}_a^{CF}I^{\alpha}\sin t,$$

we also recall that

$$\left({}_a^{CFR}D^{\alpha}\cdot{}_a^{CF}I^{\alpha}y\right)(t)=y(t)$$

so

$${}_a^{CFR}D^{\alpha}y(t)=sint,$$

Hence $y(t)$ *satisfies (30) if and only if it satisfies (31).*
Construction of the Banach space $C[a,b]$*.*
Let

$$B(c,r)=\{y\in\Omega:|y-c|<r\}$$

$$\bar{B}(c,r)=\{y\in\Omega:|y-c|\leq r\}.$$

Since Ω *is an open subset of* $\mathbb{R}$ *and* $c\in\Omega$, *there exists* $k>0$ *such that* $B(c,k)\subset\Omega$. *This implies* $\bar{B}(c,\frac{K}{2})\subset\Omega$. *So without loss of generality, we can choose* $R>0$ *for* $R\leq k$ *such that* $\bar{B}(c,R)\subset[a,b]$. $\bar{B}(c,R)$ *is a compact subset of* $\mathbb{R}$ *and* f *is continuous implies* f *has a maximum on* $\bar{B}(c,R)$. *Let* $\max_{t\in[a,b]}\{|f(z)|:z\in\bar{B}(c,R)\}=M<\infty$, $\epsilon<\frac{1}{A}$ *and* $0<\epsilon\leq\frac{R}{M}$, *where* A *is the local Lipschitz constant.*
Let $C[a,b]=\{y|y:[a,b]\to\mathbb{R}\text{is continuous}\}$.
Endow $C[a,b]$ *with the maximum norm* $\|x\|=\max_{t\in[a,b]}|y(t)|$, $\forall y\in C[a,b]$.
This norm induces a metric ρ *on* $C[a,b]$ *given by* $\rho(y_1,y_2)=|y_1-y_2|_{\infty}$, $\forall y_1,y_2\in C[a,b]$. *Therefore,* $C[a,b]$ *is a complete metric space.*
Construction of the subset K*.*
Define

$$K=\{y\in C[a,b]:y(t)\in\bar{B}(c,R),\forall t\in[a,b]\}$$

$$=\{y\in C[a,b]:|y(t)-c|\leq R,\ \forall t\in[a,b]\}.$$

This shows that K *is a closed ball, and therefore it is closed, bounded, convex, and nonempty. Therefore* K *is a closed subset of the complete metric space* $C[a,b]$ *and so* K *is a complete metric space.*
Definition of a map φ *from* K *into* K*.*

$$y(t)=c+{}_a^{CF\alpha}\sin t$$

and defined a map φ *on* K *by*

$$\varphi(y(t))=c+{}_a^{CF\alpha}\sin t\ \forall y(t)\in\bar{B}(c,R).$$

We observe that y^* *is a fixed point of* φ *if and only if* y^* *is a solution of equation (30).*
We now show that φ *maps* K *to* K. *Since* f *is locally Lipschitzian then* f *is continuous.*
Let $y\in K$, *it is enough to show that* $\varphi(y)\in K$. *But* $y\in K\implies y(t)\in\bar{B}(c,R),\forall t\in[a,b]$. *Using the definition of* φ, *we have*

$$|\varphi(y(t))-c|=|{}_a^{CF}I^{\alpha}\sin sds|$$

$$|\varphi(y(t))-c|=|\frac{1-\alpha}{B(\alpha)}\sin t+\frac{\alpha}{B(\alpha)}\int_a^t \sin sds|$$
$$\leq \frac{1-\alpha}{B(\alpha)}|\sin t|+\frac{\alpha}{B(\alpha)}\int_a^t |\sin sds|$$
$$\leq \frac{1-\alpha}{B(\alpha)}\max_{t\in[a,b]}|\sin t|+\frac{\alpha}{B(\alpha)}\int_a^t \max_{t\in[a,b]}|\sin s|ds$$
$$=\frac{1-\alpha}{B(\alpha)}M+\frac{M\alpha}{B(\alpha)}\int_a^t ds$$
$$\leq M\left(\frac{\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}(t-a)\right)\leq M\epsilon\leq R$$

Since $t\in[a,b]$ and we put $\epsilon=\frac{R}{M}$. But $|\varphi(y(t))-c|\leq R \implies \varphi(y(t))\in \overline{B}(c,R)$. Thus φ maps K into K.
We next show that φ is a contraction map on K. Let $t\in[a,b]$ then for arbitrary $y_1,y_2\in K$ we have

$$|(\varphi y_1)(t)-(\varphi y_2)(t)|=|{}_a^{CF}I^{\alpha}[\sin y_1-\sin y_2]|$$
$$\leq A_a^{CF\alpha}|\sin y_1-\sin y_2|$$
$$=A({}_a^{CF}I^{\beta}|\sin y_1-\sin y_2|)$$
$$=A(\frac{1-\beta}{B(\beta)}\max_{t\in[a,b]}|\sin y_1-\sin y_2|)+$$
$$A(\frac{\beta}{B(\beta)}\int_a^t\max_{t\in[a,b]}|\sin y_1-\sin y_2|ds)$$
$$=A(\frac{1-\alpha}{B(\alpha)}|y_1-y_2|_\infty+\frac{\beta}{B(\beta)}(t-a)|y_1-y_2|_\infty)$$
$$\leq A(\frac{1-\beta}{B(\beta)}+\frac{\beta}{B(\beta)}(b-a))|y_1-y_2|_\infty$$
$$|(\varphi y_1)(t)-(\varphi y_2)(t)|\leq$$
$$A(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}(b-a))\|y_1-y_2\|$$

where $A\big(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}(b-a)\big)<1$. Hence φ is a contraction map. By the contraction mapping principle, φ has a unique fixed point. This implies that the fractional differential equation (30) has a unique solution y, $(y:[a,b]\to\mathbb{R})$ defined on $[a,b]$ which satisfies the initial condition $y(a)=c$. The proof is complete.

Theorem 3. *[15] Consider the system*

$$({}_a^{CFR}D^{\alpha}y)(t)=f(t,y(t)),\quad t\in[a,b],\,1<\alpha\leq 2 \tag{35}$$

with initial condition

$$y(a)=c \tag{36}$$

such that $\frac{A}{B(\alpha-1)}\big[(2-\alpha)(b-a)+\frac{(\alpha-1)(b-a)^2}{2}\big]<1$ and $|f(t,y_1)-f(t,y_2)|\leq A|y_1-y_2|$, $A>0$. Also $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ and $y:[a,b]\to\mathbb{R}$, then the system has a unique solution of the form

$$\begin{aligned}y(t)&=c+{}_a^{CF}I^{\alpha}f(t,y(t))\\&=c+\frac{2-\alpha}{B(\alpha-1)}\int_a^t f(s,y(s))ds\\&+\frac{\alpha-1}{B(\alpha-1)}\big({}_aI^2f(s,y(s))ds\big)\end{aligned} \tag{37}$$

Example 4. *Consider the system*

$$\big({}_a^{CFR}D^{\alpha}y\big)(t)=\cos t,\quad t\in[a,b],\,1<\alpha\leq 2,\;y(a)=c \tag{38}$$

Such that $A\big(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}(b-a)\big)<1$ and $|cosy_1-cosy_2|\leq A|y_1-y_2|$, $A>0$. Here $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ and $y:[a,b]\to\mathbb{R}$ then the system (38) has a unique solution of the form

$$\begin{aligned}y(t)&=c+{}_a^{CF}I^{\alpha}\cos t\\&=c+\frac{2-\alpha}{B(\alpha-1)}\int_a^t\cos sds+\frac{\alpha-1}{B(\alpha-1)}\big({}_aI^2\cos sds\big)\end{aligned} \tag{39}$$

we begin the proof by obtaining the Laplace transform of the derivative First, we apply ${}_a^{CF}I^{\alpha}$ to system (38) with $\beta=\alpha-1$ to have

$$\big({}_a^{CF}I^{\beta}\big)\big({}_a^{CFR}D^{\beta}y\big)(t)=\big({}_a^{CF}I^{\beta}\big)\cos t,$$

we recall that

$$\big({}_a^{CF}I^{\beta}{}_a^{CFR}I^{\beta}y\big)(t)=y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k$$

for $k=0$ we have

$$\big({}_a^{CF}I^{\beta}{}_a^{CFR}D^{\beta}y\big)(t)=y(t)-y(a)$$
$$y(t)-y(a)={}_a^{CF}I^{\beta}\cos t,$$
$$y(t)=y(a)+{}_a^{CF}I^{\beta}\cos t,$$

but $y(a)=c$ therefore

$$y(t)=c+\big({}_aI^n\big)\big[\frac{1-\beta}{B(\beta)}coskt+\frac{\beta}{B(\beta)}\int_a^t cosksds\big] \tag{40}$$

Conversely, if we apply ${}_a^{CFR}D^{\beta}$ on (39) we have

$${}_a^{CFR}D^{\beta}y(t)={}_a^{CFR}D^{\beta}c+{}_a^{CFR}D^{\beta}{}_a^{CF}I^{\beta}\cos t,$$

Hence $y(t)$ satisfies (38) if and only if it satisfies (39).
Construction of the Banach space $C[a,b]$.
Let

$$B(c,r)=\{y\in\Omega:|y-c|<r\}$$
$$\bar{B}(c,r)=\{y\in\Omega:|y-c|\leq r\}.$$

Since Ω is an open subset of $\mathbb{R}$ and $c\in\Omega$, there exists $k>0$ such that $B(c,k)\subset\Omega$. This implies $\bar{B}(c,\frac{K}{2})\subset\Omega$. So without loss of generality, we can choose $R>0$ for $R\leq k$ such that $\bar{B}(c,R)\subset[a,b]$. $\bar{B}(c,R)$ is a compact subset of $\mathbb{R}$ and f is continuous implies f has a maximum on $\bar{B}(c,R)$. Let $\max_{t\in[a,b]}\{|f(z)|:z\in\bar{B}(c,R)\}=M<\infty$, $\epsilon<\frac{1}{A}$ and $0<\epsilon\leq\frac{R}{M}$, where A is the local Lipschitz constant.
Let $C[a,b]=\{y|y:[a,b]\to\mathbb{R}is\ continuous\}$.
Endow $C[a,b]$ with the maximum norm $\|x\|=\max_{t\in[a,b]}|y(t)|$, $\forall y\in C[a,b]$.
This norm induces a metric ρ on $C[a,b]$ given by $\rho(y_1,y_2)=|y_1-y_2|_\infty$, $\forall y_1,y_2\in C[a,b]$. Therefore, $C[a,b]$ is a complete metric space.
Construction of the subset K.
Define

$$\begin{aligned}K&=\{y\in C[a,b]:y(t)\in\bar{B}(c,R),\forall t\in[a,b]\}\\&=\{y\in C[a,b]:|y(t)-c|\leq R,\,\forall t\in[a,b]\}.\end{aligned}$$

This shows that K is a closed ball, and therefore it is closed, bounded, convex, and nonempty. Therefore K is a closed subset of the complete metric space $C[a,b]$ and so K is a complete metric space.
Definition of a map φ from K into K.

$$y(t) = c + {}_{a}^{CF}I^{\beta}\cos t$$

and defined a map φ on K by

$$\varphi(y(t)) = c + {}_{a}^{CF}I^{\beta}\cos t \ \forall y(t) \in \bar{B}(c,R).$$

We observe that y^* is a fixed point of φ if and only if y^* is a solution of equation (38).
We now show that φ maps K to K. Since f is locally Lipschitzian then f is continuous.
Let $y \in K$, it is enough to show that $\varphi(y) \in K$. But $y \in K \implies y(t) \in \bar{B}(c,R), \forall t \in [a,b]$. Using the definition of φ, we have

$$|\varphi(y(t)) - c| = |{}_{a}^{CF}I^{\beta}\cos s ds|$$

$$\begin{aligned}
|\varphi(y(t)) - c| &= |\frac{1-\beta}{B(\beta)}\cos t + \frac{\beta}{B(\beta)}\int_a^t \sin s ds| \\
&\leq \frac{1-\beta}{B(\beta)}|\cos t| + \frac{\beta}{B(\beta)}\int_a^t |\cos s ds| \\
&\leq \frac{1-\beta}{B(\beta)}\max_{t\in[a,b]}|\cos t| + \frac{\beta}{B(\beta)}\int_a^t \max_{t\in[a,b]}|\cos s| ds \\
&= \frac{1-\beta}{B(\beta)}M + \frac{M\beta}{B(\beta)}\int_a^t ds \\
&\leq M\left(\frac{\beta}{B(\beta)} + \frac{\beta}{B(\beta)}(t-a)\right) \leq M\epsilon \leq R
\end{aligned}$$

Since $t \in [a,b]$ and we put $\epsilon = \frac{R}{M}$. But $|\varphi(y(t)) - c| \leq R \implies \varphi(y(t)) \in \overline{B}(c,R)$. Thus φ maps K into K.
We next show that φ is a contraction map on K. Let $t \in [a,b]$ then for arbitrary $y_1, y_2 \in K$ we have

$$\begin{aligned}
|(\varphi y_1)(t) - (\varphi y_2)(t)| &= |{}_{a}^{CF}I^{\beta}[\cos y_1 - \cos y_2]| \\
&\leq A{}_{a}^{CF}I^{\beta}|\cos y_1 - \cos y_2| = A({}_{a}^{CF}I^{\beta}|\cos y_1 - \cos y_2|) \\
&= A(\frac{1-\beta}{B(\beta)}\max_{t\in[a,b]}|\cos y_1 - \cos y_2| \\
&\quad + \frac{\beta}{B(\beta)}\int_a^t \max_{t\in[a,b]}|\cos y_1 - \cos y_2| ds) \\
&= A(\frac{1-\beta}{B(\beta)}|y_1 - y_2|_\infty + \frac{\beta}{B(\beta)}(t-a)|y_1 - y_2|_\infty) \\
&\leq A(\frac{1-\beta}{B(\beta)} + \frac{\beta}{B(\beta)}(b-a))|y_1 - y_2|_\infty
\end{aligned}$$

$$|(\varphi y_1)(t) - (\varphi y_2)(t)| \leq A\big(\frac{1-\beta}{B(\beta)} + \frac{\beta}{B(\beta)}(b-a)\big)\|y_1 - y_2\|$$

where $A\big(\frac{1-\beta}{B(\beta)} + \frac{\beta}{B(\beta)}(b-a)\big) < 1$. Hence φ is a contraction map. By the contraction mapping principle, φ has a unique fixed point. This implies that the fractional differential equation (38) has a unique solution y, ($y : [a,b] \to \mathbb{R}$) defined on $[a,b]$ which satisfies the initial condition $y(a) = c$. The proof is complete.

Theorem 4. *[16] Consider the initial value problem*

$$({}_{a}^{CFR}D^{\alpha}y)(t) = f(t,y(t)),\ a < t < b,\ 2 < \alpha < 3 \quad (41)$$

$$y(a) = c_1$$

$$y'(a) = c_2$$

such that $\frac{A}{B(\alpha-2)}[\frac{(3-\alpha)(b-a)^2}{2} + \frac{(\alpha-2)(b-a)^3}{6}] < 1$ and $|f(t,y_1) - f(t,y_2)| \leq A|y_1 - y_2|$, $A > 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, $y : [a,b] \to \mathbb{R}$. Then the system (41) has a unique solution of the form

$$y(t) = c_1 + c_2(t-a) + ({}_{a}^{CFR}I^{\alpha}f(t,y(t)) \quad (42)$$

Theorem 5. *[16] Consider the initial value problem*

$$({}_{a}^{CFR}D^{\alpha}y)(t) = f(t,y(t)),\ a < t < b, \quad 3 < \alpha < 4 \quad (43)$$

$$y(a) = c_1$$

$$y'(a) = c_2$$

$$y''(a) = c_3$$

such that $\frac{A}{B(\alpha-3)}[\frac{(4-\alpha)(b-a)^3}{6} + \frac{(\alpha-3)(b-a)^4}{24}] < 1$. Then the system (42) has a unique solution of the form

$$y(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + {}_{a}^{CF}I^{\alpha}f(t,y(t)). \quad (44)$$

III. Main Results

We extend on the finding from section (2) regarding the existence and uniqueness of Caputo-Fabrizio fractional differential equations to $\alpha \in (4,5)$

Theorem 6. *Consider the initial value problem*

$$({}_{a}^{CFR}D^{\alpha}y)(t) = f(t,y(t)),\ a < t < b \quad 4 < \alpha < 5 \quad (45)$$

$$y(a) = c_1$$

$$y'(a) = c_2$$

$$y''(a) = c_3$$

$$y'''(a) = c_4$$

such that $\frac{A}{B(\alpha-4)}[\frac{(5-\alpha)(b-a)^4}{24} + \frac{(\alpha-4)(b-a)^5}{120}] < 1$. Then the system (45) has a unique solution of the form

$$\begin{aligned} y(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3 \\ + {}_{a}^{CF}I^{\alpha}f(t,y(t)). \end{aligned} \quad (46)$$

First, we apply ${}_{a}^{CF}I^{\alpha}$ to the system (45) to have

$$\big({}_{a}^{CF}I^{\alpha}\big)\big({}_{a}^{CFR}D^{\alpha}y\big)(t) = \big({}_{a}^{CF}I^{\alpha}\big)f(t,y(t)), \quad (47)$$

we recall that

$$\big({}_{a}^{CF}I^{\alpha}{}_{a}^{CFR}D^{\alpha}y\big)(t) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k \quad (48)$$

for $k = 0,1,2,3$ we have

$$\begin{aligned}
\big({}_{a}^{CF}I^{\alpha}{}_{a}^{CFR}D^{\alpha}y\big)(t) = y(t) - y(a) + y^{'}(a)(t-a) \\
+ y^{''}(a)(t-a)^2 + y^{'''}(a)(t-a)^3
\end{aligned}$$

$$y(t) - y(a) = {}_{a}^{CF}I^{\alpha}\sin t,$$

$$\begin{aligned}
y(t) - y(a) + y^{'}(a)(t-a) + y^{''}(a)(t-a)^2 \\
+ y^{'''}(a)(t-a)^3 + {}_{a}^{CF}I^{\alpha}f(t,y(t),
\end{aligned}$$

but $y(a) = c$ therefore

$$y(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(a)(t-a)^3 + {}_a^{CF}I^\alpha f(t, y(t), \quad (49)$$

Conversely, if we apply ${}_a^{CFR}D^\alpha$ on (46)
we have

$$\begin{aligned} {}_a^{CFR}D^\alpha y(t) &= {}_a^{CFR}D^\alpha c_1 + {}_a^{CFR}D^\alpha c_2(t-a) \\ &+ {}_a^{CFR}D^\alpha c_3(t-a)^2 + {}_a^{CFR}D^\alpha c_4(t-a)^3 \\ &+ {}_a^{CFR}D^\alpha {}_a^{CF}I^\alpha f(t, y(t)). \end{aligned}$$

we also recall that

$$\big({}_a^{CFR}D^\alpha \cdot {}_a^{CF}I^\alpha y\big)(t) = y(t)$$

so

$${}_a^{CFR}D^\alpha y(t) = f(t, y(t),$$

Hence $y(t)$ satisfies (45) if and only if it satisfies (46).
Construction of the Banach space $C[a,b]$.
Let

$$B(c,r) = \{y \in \Omega : |y-c| < r\}$$

$$\bar{B}(c,r) = \{y \in \Omega : |y-c| \leq r\}.$$

Since Ω is an open subset of $\mathbb{R}$ and $c \in \Omega$, there exists $k > 0$ such that $B(c,k) \subset \Omega$. This implies $\bar{B}(c, \frac{K}{2}) \subset \Omega$. So without loss of generality, we can choose $R > 0$ for $R \leq k$ such that $\bar{B}(c,R) \subset [a,b]$. $\bar{B}(c,R)$ is a compact subset of $\mathbb{R}$ and f is continuous implies f has a maximum on $\bar{B}(c,R)$. Let $\max\limits_{t\in[a,b]}\{|f(z)| : z \in \bar{B}(c,R)\} = M < \infty$, $\epsilon < \frac{1}{A}$ and $0 < \epsilon \leq \frac{R}{M}$, where A is the local Lipschitz constant.
Let $C[a,b] = \{y | y : [a,b] \to \mathbb{R} \text{is continuous}\}$.
Endow $C[a,b]$ with the maximum norm $\|x\| = \max\limits_{t\in[a,b]} |y(t)|$, $\forall y \in C[a,b]$.
This norm induces a metric ρ on $C[a,b]$ given by $\rho(y_1, y_2) = |y_1 - y_2|_\infty, \forall y_1, y_2 \in C[a,b]$. Therefore, $C[a,b]$ is a complete metric space.
Construction of the subset K.
Define

$$\begin{aligned} K &= \{y \in C[a,b] : y(t) \in \bar{B}(c,R), \forall t \in [a,b]\} \\ &= \{y \in C[a,b] : |y(t) - c| \leq R, \forall t \in [a,b]\}. \end{aligned}$$

This shows that K is a closed ball, and therefore it is closed, bounded, convex, and nonempty. Therefore K is a closed subset of a complete metric space $C[a,b]$ and so K is a complete metric space.
Definition of a map φ from K into K.

$$y(t) = c + {}_a^{CF}I^\alpha f(t, y(t)$$

and defined a map φ on K by

$$\varphi(y(t)) = c + {}_a^{CF\alpha} f(t, y(t) \ \forall y(t) \in \bar{B}(c,R).$$

We observe that y^* is a fixed point of φ if and only if y^* is a solution of equation (45).
We now show that φ maps K to K. Since f is locally Lipschitzian then f is continuous.
Let $y \in K$, it is enough to show that $\varphi(y) \in K$. But $y \in K$ $\implies y(t) \in \bar{B}(c,R), \forall t \in [a,b]$. we can rewrite (46) with $\beta = \alpha - 4$ as

$$\begin{aligned} y(t) &= c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3 \\ &+ {}_a^{CF}I^4\Big(\frac{5-\alpha}{B(\alpha-4)} f(t) + \frac{\alpha-4}{B(\alpha-4)} \int_a^t f(s, y(s)) ds\Big) \end{aligned}$$

and using the definition of φ, we have

$$\begin{aligned} |\varphi(y(t)) - c| &= |{}_a^{CF}I^\alpha f(s, y(s)) ds| \\ |\varphi(y(t)) - c_1 + c_2(t-a) &+ c_3(t-a)^2 \\ + c_4(t-a)^3| &= {}_a^{CF}I^4 \Big|\frac{5-\alpha}{B(\alpha-4)} f(t) \\ &+ \frac{\alpha-4}{B(\alpha-4)} \int_a^t f(s, y(s)) ds\Big| \\ &\leq {}_a^{CF}I^4\Big(\frac{5-\alpha}{B(\alpha-4)} |f(t)| + \frac{\alpha-4}{B(\alpha-4)} \int_a^t |f(s, y(s)) ds|\Big) \\ &\leq {}_a^{CF}I^4\Big(\frac{5-\alpha}{B(\alpha-4)} \max_{t\in[a,b]} |f(t)| + \\ &\frac{\alpha-4}{B(\alpha-4)} \int_a^t \max_{t\in[a,b]} |f(s, y(s))| ds\Big) \\ &= {}_a^{CF}I^4\Big(\frac{5-\alpha}{B(\alpha-4)} M + \frac{M\alpha-4}{B(\alpha-4)} \int_a^t ds\Big) \\ &\leq \frac{M}{B(\alpha-4)}\Big(\frac{(5-\alpha)(b-a)^4}{24} + \frac{(\alpha-4)(b-a)^5}{120}\Big) \\ &\leq M\epsilon \leq R \end{aligned}$$

we endow the set $C[a,b]$ with the norm $\|y\| = \max\limits_{t\in[a,b]} |y(t)|$. This norm induces a metric ρ on $C[a,b]$ given by $\rho(y_1, y_2) = |y_1 - y_2|_\infty \ \forall y_1, y_2 \in C[a,b]$. Therefore $C[a,b]$ is a complete metric space. We define an operator φ

$$(\varphi y)(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3 + {}_a^{CF}I^\alpha f(t, y(t)). \quad (50)$$

Then for arbitrary $y_1, y_2 \in [a,b]$,$\beta = \alpha - 4$, we have

$$\begin{aligned} |(\varphi(y_1))(t) - (\varphi y_2)(t)| &= |{}_a^{CF}I^\alpha [f(t, y(t)) - f(t, y_2(t))]| \\ &\leq A {}_a^{CF}I^\alpha |y_1 - y_2| = A_a I^4({}_a^{CF}I^\beta |y_1 - y_2|) \\ &= A_a I^4\Big(\frac{1-\beta}{B(\beta)} |y_1 - y_2| + \frac{\beta}{B(\beta)} \int_a^t |y_1 - y_2| ds\Big) \\ &= A\Big(\frac{1-\beta}{6B(\beta)} \int_a^t (t-s)^3 |y_1 - y_2| ds \\ &+ \frac{\beta}{24B(\beta)} \int_a^t (t-s)^4 |y_1 - y_2| ds\Big) \\ &= A\Big(\frac{5-\alpha}{6B(\alpha-4)} \int_a^t (t-s)^3 |y_1 - y_2| ds \\ &+ \frac{\alpha-4}{24B(\alpha-4)} \int_a^t (t-s)^4 |y_1 - y_2| ds\Big) \\ &\leq A\Big[\frac{5-\alpha}{6B(\alpha-4)} \int_a^t (t-s)^3 ds \\ &+ \frac{\alpha-4}{24B(\alpha-4)} \int_a^t (t-s)^4\Big] \|y_1 - y_2\| \end{aligned}$$

$$\leq \frac{A}{B(\alpha-4)}[\frac{(5-\alpha)(b-a)^4}{24}+\frac{(\alpha-4)(b-a)^5}{120}]\|y_1-y_2\|.$$

Hence φ is a contraction. According to the Banach contraction principle, there exists a unique fixed point. This implies that equation (45) has a unique solution y such that $\varphi y = y$. The proof is complete.

Example 5. *Consider the system*

$$\left({}_{a}^{CFR}D^{\alpha}y\right)(t) = \sin t, \quad t \in [a,b],\, 4 < \alpha \leq 5 \tag{51}$$

$$y(a) = c_1$$

$$y'(a) = c_2$$

$$y''(a) = c_3$$

$$y'''(a) = c_4$$

such that $\frac{A}{B(\alpha-4)}[\frac{(5-\alpha)(b-a)^4}{24}+\frac{(\alpha-4)(b-a)^5}{120}] < 1$ *and* $|siny_1 - siny_2| \leq A|y_1-y_2|$, $A>0$. *Here* $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ *and* $y:[a,b]\to\mathbb{R}$, *then the system (51) has a unique solution of the form*

$$y(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3 + {}_{a}^{CF}I^{\alpha}\sin t. \tag{52}$$

we can rewrite (52) with $\beta = \alpha - 4$ *as*

$$y(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3 + {}_{a}^{CF}I^{4}\Big(\frac{5-\alpha}{B(\alpha-4)}\sin t + \frac{\alpha-4}{B(\alpha-4)}\int_a^t \sin s ds\Big)$$

and using the definition of φ, *we have*

$$|\varphi(y(t)) - c| = |{}_{a}^{CF}I^{\alpha}\sin s ds|$$

$$\varphi(y(t)) - c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3| = {}_{a}^{CF}I^{4}|\frac{5-\alpha}{B(\alpha-4)}\sin t + \frac{\alpha-4}{B(\alpha-4)}\int_a^t \sin s ds| \leq {}_{a}^{CF}I^{4}\Big(\frac{5-\alpha}{B(\alpha-4)}|\sin t| + \frac{\alpha-4}{B(\alpha-4)}\int_a^t |\sin s ds|\Big)$$

$$\leq {}_{a}^{CF}I^{4}\Big(\frac{5-\alpha}{B(\alpha-4)}\max_{t\in[a,b]}|\sin t| + \frac{\alpha-4}{B(\alpha-4)}\int_a^t \max_{t\in[a,b]}|\sin s| ds\Big) = {}_{a}^{CF}I^{4}\Big(\frac{5-\alpha}{B(\alpha-4)}M + \frac{M\alpha-4}{B(\alpha-4)}\int_a^t ds\Big)$$

$$\leq \frac{M}{B(\alpha-4)}\Big(\frac{(5-\alpha)(b-a)^4}{24} + \frac{(\alpha-4)(b-a)^5}{120}\Big) \leq M\epsilon \leq R$$

we endow the set $C[a,b]$ *with the norm* $\|y\| = \max_{t\in[a,b]}|y(t)|$. *This norm induces a metric* ρ *on* $C[a,b]$ *given by* $\rho(y_1,y_2) = |y_1-y_2|_\infty \ \forall y_1,y_2 \in C[a,b]$. *Therefore* $C[a,b]$ *is a complete metric space. We define an operator* φ

$$(\varphi y)(t) = c_1 + c_2(t-a) + c_3(t-a)^2 + c_4(t-a)^3 + {}_{a}^{CF}I^{\alpha}\sin t. \tag{53}$$

Then for arbitrary $y_1, y_2 \in [a,b]$,$\beta = \alpha - 4$, *we have*

$$|(\varphi(y_1))(t) - (\varphi y_2)(t)| = |{}_{a}^{CF}I^{\alpha}[\sin y_1 - \sin y_2]|$$

$$\leq A{}_{a}^{CF}I^{\alpha}|y_1-y_2| = A_aI^4({}_{a}^{CF}I^{\beta}|y_1-y_2|)$$

$$= A_aI^4\Big(\frac{1-\beta}{B(\beta)}|y_1-y_2| + \frac{\beta}{B(\beta)}\int_a^t |y_1-y_2|ds\Big)$$

$$= A\Big(\frac{1-\beta}{3B(\beta)}\int_a^t (t-s)^3|y_1-y_2|ds + \frac{\beta}{24B(\beta)}\int_a^t (t-s)^4|y_1-y_2|ds\Big)$$

$$= A\Big(\frac{5-\alpha}{6B(\alpha-4)}\int_a^t (t-s)^3|y_1-y_2|ds + \frac{\alpha-4}{24B(\alpha-4)}\int_a^t (t-s)^4|y_1-y_2|ds\Big)$$

$$\leq A[\frac{5-\alpha}{6B(\alpha-4)}\int_a^t (t-s)^3 ds + \frac{\alpha-4}{24B(\alpha-4)}\int_a^t (t-s)^4]\|y_1-y_2\|$$

$$\leq \frac{A}{B(\alpha-4)}[\frac{(5-\alpha)(b-a)^4}{24}+\frac{(\alpha-4)(b-a)^5}{120}]\|y_1-y_2\|.$$

for $\alpha = \frac{9}{2}$

$$= \frac{A}{B(\frac{9}{2}-4)}[\frac{(5-\frac{9}{2})(b-a)^4}{24}+\frac{(\frac{9}{2}-4)(b-a)^5}{120}]\|y_1-y_2\|.$$

$$= \frac{A}{B(\frac{1}{2})}[\frac{(\frac{1}{2})(b-a)^4}{24}+\frac{(\frac{1}{2})(b-a)^5}{120}]\|y_1-y_2\|.$$

$$= \frac{A}{24B(\frac{1}{2})}[(\frac{1}{2})(b-a)^4+\frac{(\frac{1}{2})(b-a)^5}{5}]\|y_1-y_2\|.$$

if $a = 4, b = 5$

$$= \frac{A}{24B(\frac{1}{2})}[(\frac{1}{2})(5-4)^4+\frac{(\frac{1}{2})(5-4)^5}{5}]\|y_1-y_2\|.$$

$$= \frac{A}{24B(\frac{1}{2})}[(\frac{1}{2})+\frac{(\frac{1}{2})}{5}]\|y_1-y_2\|.$$

$$= \frac{A}{24B(\frac{1}{2})}[(\frac{1}{2})+(\frac{1}{10})]\|y_1-y_2\|.$$

$$= \frac{A}{24B(\frac{1}{2})}[(\frac{6}{10})]\|y_1-y_2\|.$$

$$= \frac{A}{40B(\frac{1}{2})}\|y_1-y_2\|.$$

therefore

$$|(\varphi(y_1))(t) - (\varphi y_2)(t)| = |{}_{a}^{CF}I^{\alpha}[\sin y_1 - \sin y_2]|$$

$$\leq \frac{A}{40B(\frac{1}{2})}\|y_1-y_2\|.$$

Hence φ *is a contraction. According to the Banach contraction principle, there exists a unique fixed point. This implies that equation (51) has a unique solution* y *such that* $\varphi y = y$. *The proof is complete.*

IV. Conclusion

Within the framework of the Caputo-Fabrizio fractional differential equation in the Riemann-Liouville sense, the essential structure of the of the fractional differential equations was examined. Results on existence and uniqueness were shown, and they were extended to include order $\alpha \in (4, 5)$. Examples were used to confirm the suitability of the result.

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