Inequalities Concerning the Rate of Growth of Polynomials Involving the Polar Derivative

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Abstract—**In this paper, we establish inequalities concerning the growth of polynomials involving the polar derivative of the polynomials.**

Index Terms—**polynomial, polar derivative, zero, maximum modulus.**

I. Introduction

Let $p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ be a polynomial of degree n in the complex plane and $p'(z)$ its derivative.

A famous result due to Bernstein ([3], [30], [19]) states that if $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{1}$$

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then inequality (1) can be improved by

$$\max_{|z|=1}|p'(z)|\le \frac{n}{2}\max_{|z|=1}|p(z)|. \tag{2}$$

Inequality (2) was conjectured by Erdös and later verified by Lax [15]. As a generalization of (2), Malik [16] proved that if $p(z)$ does not vanish in $|z|<k$, $k\ge 1$, then

$$\max_{|z|=1}|p'(z)|\le \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{3}$$

Further, Bidkham and Dewan [4] generalized inequality (3) by proving that if $p(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\ge 1$, then

$$\max_{|z|=r}|p'(z)|\le \frac{n(r+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|p(z)|,\quad \text{for}\quad 1\le r\le k. \tag{4}$$

By considering a more general class of polynomials $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, not vanishing in $|z|<k$, $k\ge 1$, Aziz and Zargar [2] generalized (4) and obtained for $0<r\le R\le k$,

$$\max_{|z|=R}|p'(z)|\le \frac{nR^{\mu-1}(k^\mu+R^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+R^\mu)^{\frac{n}{\mu}}}\max_{|z|=r}|p(z)|. \tag{5}$$

Further, Aziz and Shah [1] improved inequality (5) under the same hypothesis that for $0<r\le R\le k$,

$$\max_{|z|=R}|p'(z)|\le \frac{nR^{\mu-1}(k^\mu+R^\mu)^{\frac{n}{\mu}-1}}{(k^\mu+R^\mu)^{\frac{n}{\mu}}}\times\left\{\max_{|z|=r}|p(z)|-\min_{|z|=k}|p(z)|\right\}. \tag{6}$$

For a polynomial $p(z)$ of degree n, we now define the polar derivative of $p(z)$ with respect to a real or complex number α as

$$D_\alpha p(z)=np(z)+(\alpha-z)p^{'}(z).$$

This polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p^{'}(z)$ in the sense that

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p^{'}(z),$$

uniformly with respect to z for $|z|\le R$, $R>0$.

Over the last four decades, a large number of results concerning the polar derivative of polynomials was obtained by many different authors. More information on classical results and polar derivatives can be found in the books of Milovanović et al. [19], Rahman and Schmeisser [27] and Marden [17]. For a better insight, one can refer the literature (for example, [5], [7], [8], [9], [10], [12], [14], [18], [20], [21], [23], [24], [25], [28], [29], [32]) regarding the latest research and development in this direction. Recently, Somsuwan and Nakprasit [31] proved the following polar derivative extension of inequality (6) for the case $k\ge 1$.

Theorem 1. *If $p(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\le\mu\le n$, is a polynomial of degree n having no zeros in $|z|<k$, $k\ge 1$, then for every $\alpha\in\mathbb{C}$ with $|\alpha|\ge R$ and $0<r\le R\le k$,*

$$\max_{|z|=R}|D_\alpha p(z)|\le\frac{n}{1+s_0}\times\left[\max_{|z|=r}|p(z)|E_0-\{E_0-(s_0+1)\}m\right], \tag{7}$$

where

$$E_0=\left(\frac{|\alpha|}{R}+s_0\right)\left(\frac{k^\mu+R^\mu}{k^\mu+r^\mu}\right)^{\frac{n}{\mu}},$$

$$s_0=\left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-m}Rk^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{R(|a_0|-m)}k^{\mu+1}+1}\right\}$$

and $m=\min_{|z|=k}|p(z)|$.

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II. Lemmas

We shall need the following auxiliary results to prove the theorem and verify the claims.

Lemma 2. *If* $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *such that* $p(z)\neq 0$ *in* $|z|<k$, $k>0$, *then for* $|z|\leq k$,

$$|p(z)|\geq m, \tag{8}$$

where $m=\min_{|z|=k}|p(z)|$.

This lemma is due to Gardner et al. [11, see Lemma 2.6].

Lemma 3. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n$, *is a polynomial of degree* n *having no zeros in* $|z|<k$, $k\geq 1$, *then for* $|z|=1$,

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+B_0}\max_{|z|=1}|p(z)|, \tag{9}$$

where

$$B_0=k^{\mu+1}\left\{\frac{(\frac{\mu}{n})\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{(\frac{\mu}{n})\frac{|a_\mu|}{|a_0|}k^{\mu+1}+1}\right\}, \tag{10}$$

$$\left(\frac{\mu}{n}\right)\frac{|a_\mu|k^\mu}{|a_0|}\leq 1 \tag{11}$$

The above lemma is due to Qazi [26].

Lemma 4. *If* $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n$, *is a polynomial of degree* n *having no zeros in* $|z|<k$, $k\geq 1$, *then for every complex number* λ *with* $|\lambda|<1$ *and* $|z|=1$,

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+B_\lambda}\left\{\max_{|z|=1}|p(z)|-\lambda m\right\}, \tag{12}$$

where

$$B_\lambda=k^{\mu+1}\left\{\frac{(\frac{\mu}{n})\frac{|a_\mu|}{||a_0|-\lambda m|}k^{\mu-1}+1}{(\frac{\mu}{n})\frac{|a_\mu|}{||a_0|-\lambda m|}k^{\mu+1}+1}\right\}, \tag{13}$$

$$\left(\frac{\mu}{n}\right)\frac{|a_\mu|k^\mu}{||a_0|-\lambda m|}\leq 1 \tag{14}$$

and $m=\min_{|z|=k}|p(z)|$.

Proof: We consider the polynomial $p(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n$, and $m=\min_{|z|=k}|p(z)|$.

Multiplying both sides of $p(z)$ by e^{-iarga_0}, we have

$$R(z)=e^{-iarga_0}p(z)=|a_o|+e^{-iarga_0}\left(\sum_{\nu=\mu}^{n} a_\nu z^\nu\right)$$

For $\lambda\in\mathbb{C}$ such that $|\lambda|<1$, on $|z|=k$

$$|\lambda m|=|\lambda|m<m\leq|R(z)|.$$

Therefore, by Rouche's Theorem, $R(z)-\lambda m$ has no zero in $|z|<k$, $k\geq 1$. On applying Lemma 3, we complete the proof.

■

Lemma 5. *If* $p(z)=(z-z_0)^s\phi(z)$ *where* $|z_0|<k$ *and* $\phi(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n-s$, *is a polynomial of degree* $n-s$ *having no zeros in* $|z|<k$, $k>0$, *then for* $0<r\leq R\leq k$ *and for every complex number* λ *with* $|\lambda|<1$,

$$\max_{|z|=R}\left|\frac{p(z)}{(z-z_0)^s}\right|\leq exp\left((n-s)\int_r^R A_l dl\right)\times\max_{|z|=r}\left|\frac{p(z)}{(z-z_0)^s}\right|+\left\{1-exp\left((n-s)\int_r^R A_l dl\right)\right\}\times\frac{\lambda m}{(k+|z_0|)^s}, \tag{15}$$

where

$$A_l=\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}l^{\mu-1}k^{\mu+1}+l^\mu}{l^{\mu+1}+(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu+k^{2\mu}l)+k^{\mu+1}}, \tag{16}$$

$m=\min_{|z|=k}|p(z)|$ *and* $m'=\min_{|z|=k}|\phi(z)|$.

Proof: Since $p(z)=(z-z_0)^s\phi(z)$ where $\phi(z)=a_0+\sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1\leq\mu\leq n-s$, has no zeros in $|z|<k$, $k>0$, then for $0<l\leq k$, $\psi(z)=\phi(lz)$ has no zero in $|z|<\frac{k}{l}$, $\frac{k}{l}\geq 1$. Hence using Lemma 4, we have

$$\max_{|z|=1}|\psi'(z)|\leq\frac{n-s}{1+(\frac{k}{l})^{\mu+1}\left\{\frac{(\frac{\mu}{n-s})\frac{|a_\mu|l^\mu}{||a_0|-\lambda m'|}(\frac{k}{l})^{\mu-1}+1}{(\frac{\mu}{n-s})\frac{|a_\mu|l^\mu}{||a_0|-\lambda m'|}(\frac{k}{l})^{\mu+1}+1}\right\}}\times\left\{\max_{|z|=1}|\psi(z)|-\lambda m'\right\},$$

where

$$m'=\min_{|z|=\frac{k}{l}}|\psi(z)|=\min_{|z|=\frac{k}{l}}|\phi(lz)|=\min_{|z|=k}|\phi(z)|\geq\frac{1}{(k+|z_0|)^s}\min_{|z|=k}|p(z)|.$$

Which gives

$$l\max_{|z|=l}|\phi'(z)|\leq n-s\times\left\{\frac{\frac{\mu}{n-s}\frac{|a_\mu|}{||a_0|-\lambda m'|}\frac{k^{\mu+1}}{l}+1}{1+\frac{\mu}{n-s}\frac{|a_\mu|}{||a_0|-\lambda m'|}\frac{k^{\mu+1}}{l}+\frac{\mu}{n-s}\frac{|a_\mu|}{||a_0|-\lambda m'|}\frac{k^{2\mu}}{l^\mu}+\frac{k^{\mu+1}}{l^{\mu+1}}}\right\}\times\left\{\max_{|z|=l}|\phi(z)|-\lambda m'\right\},$$

which is equivalent to

$$\max_{|z|=l}|\phi'(z)|\leq n-s\times\left\{\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}l^{\mu-1}k^{\mu+1}+l^\mu}{l^{\mu+1}+(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu+k^{2\mu}l)+k^{\mu+1}}\right\}\times\left\{\max_{|z|=l}|\phi(z)|-\lambda m'\right\}. \tag{17}$$

Now, for $0<r\leq R\leq k$, and $0\leq\theta<2\pi$, we have

$$|\phi(Re^{i\theta})-\phi(re^{i\theta})|\leq\int_r^R|\phi'(le^{i\theta})|dl,$$

which implies

$$|\phi(Re^{i\theta})| \leq |\phi(re^{i\theta})| + \int_r^R |\phi'(le^{i\theta})|dl,$$

from which it follows that

$$\max_{|z|=R} |\phi(z)| \leq \max_{|z|=r} |\phi(z)| + \int_r^R \max_{|z|=l} |\phi'(z)|dl. \quad (18)$$

Denote $\max_{|z|=r} |\phi(z)|$ by $M(\phi, r)$.
Using (17) to (18), we get

$$M(\phi,R) \leq M(\phi,r) + n - s \times \left[\int_r^R \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} l^{\mu-1}k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu + k^{2\mu}l) + k^{\mu+1}} dl \times M(\phi,l)dl - \int_r^R \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} l^{\mu-1}k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu + k^{2\mu}l) + k^{\mu+1}} dl \times \lambda m'\right]. \quad (19)$$

If we denote R.H.S. of (19) by $f(R)$, then

$$f'(R) = n - s \times \left[\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} R^{\mu-1}k^{\mu+1} + R^\mu}{R^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}R^\mu + k^{2\mu}R) + k^{\mu+1}} \times M(\phi,R) - \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} R^{\mu-1}k^{\mu+1} + R^\mu}{R^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}R^\mu + k^{2\mu}R) + k^{\mu+1}} \times \lambda m'\right]. \quad (20)$$

Also by (19), we have

$$M(\phi,R) \leq f(R). \quad (21)$$

Using (21) to (20), we conclude that

$$f'(R) - (n-s) \times \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} R^{\mu-1}k^{\mu+1} + R^\mu}{R^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}R^\mu + k^{2\mu}R) + k^{\mu+1}} \times \{f(R) - \lambda m'\} \leq 0. \quad (22)$$

Multiplying both sides of (22) by

$$exp\left\{-(n-s) \times \int \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} R^{\mu-1}k^{\mu+1} + R^\mu}{R^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}R^\mu + k^{2\mu}R) + k^{\mu+1}} dR\right\},$$

we get

$$\frac{d}{dR}\left[\{f(R) - \lambda m'\} \times exp\left\{-(n-s) \times \int \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} R^{\mu-1}k^{\mu+1} + R^\mu}{R^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}R^\mu + k^{2\mu}R) + k^{\mu+1}} dR\right\}\right] \leq 0. \quad (23)$$

It is concluded from (23) that the function

$$g(R) = \{f(R) - \lambda m'\}\, exp\left\{-(n-s) \times \int \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} R^{\mu-1}k^{\mu+1} + R^\mu}{R^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}R^\mu + k^{2\mu}R) + k^{\mu+1}} dR\right\}$$

is a non-increasing function of R in $(0, k]$.
Hence for $0 < r \leq R \leq k$,

$$g(r) \geq g(R). \quad (24)$$

Since $f(R) \geq M(\phi, R)$ and $f(r) = M(\phi, r)$, it follows from (24) that

$$M(\phi,r) \geq M(\phi,R) exp\left\{-(n-s) \times \int_r^R \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} l^{\mu-1}k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu + k^{2\mu}l) + k^{\mu+1}} dl\right\} + \left[1 - exp\left\{-(n-s) \times \int_r^R \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} l^{\mu-1}k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu + k^{2\mu}l) + k^{\mu+1}} dl\right\}\right] \lambda m'.$$

or

$$\max_{|z|=r}\left|\frac{p(z)}{(z-z_0)^s}\right| \geq \max_{|z|=R}\left|\frac{p(z)}{(z-z_0)^s}\right| \times exp\left\{-(n-s) \times \int_r^R \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} l^{\mu-1}k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu + k^{2\mu}l) + k^{\mu+1}} dl\right\} + \left[1 - exp\left\{-(n-s) \times \int_r^R \frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|} l^{\mu-1}k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu + k^{2\mu}l) + k^{\mu+1}} dl\right\}\right] \times \lambda \frac{m}{(k+|z_0|)^s}.$$

This completes the proof. ■

Lemma 6. *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$, we have*

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|, \quad (25)$$

where $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$.

The above lemma was obtained by Govil and Rahman [13].

Lemma 7. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$,* *then for every* $\alpha \in \mathbb{C}$ *and* $\lambda \in \mathbb{C}$ *with* $|\alpha| \geq 1$ *and* $|\lambda| \leq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{1+B_\lambda} \times \left\{ (|\alpha| + B_\lambda) \max_{|z|=1} |p(z)| - (|\alpha|-1)|\lambda| m \right\}, \quad (26)$$

where $m = \min_{|z|=k} |p(z)|$ *and* B_λ *is as defined in* (13).

Proof: If $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then it is easy to verify for $|z| = 1$, that

$$|q'(z)| = |np(z) - zp'(z)|. \quad (27)$$

Also, for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, we have

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

Then, by using Lemma 6 and equality (27), for $|z| = 1$ and $|\alpha| \geq 1$, we obtain

$$\begin{aligned} |D_\alpha p(z)| &\leq |np(z) - zp'(z)| + |\alpha||p'(z)| \\ &= |q'(z)| + |\alpha||p'(z)| \\ &= |q'(z)| + |p'(z)| - |p'(z)| + |\alpha||p'(z)| \\ &\leq n \max_{|z|=1} |p(z)| + (|\alpha| - 1)|p'(z)|. \quad (28) \end{aligned}$$

Now, for $|z| = 1$ and $|\alpha| \geq 1$, inequality (28) in conjunction with (12) of Lemma 4 gives

$$|D_\alpha p(z)| \leq n \max_{|z|=1} |p(z)| + (|\alpha|-1) \times \left\{ \frac{n}{1+B_\lambda} \left(\max_{|z|=1} |p(z)| - |\lambda| m \right) \right\},$$

and this ends the proof of Lemma 7. ∎

III. Main Result

In this paper, we establish a generalization of Theorem 1 which yields a result due to Chanam and Dewan [6, Theorem 2] as a particular case.

Theorem 8. *If* $p(z) = (z - z_0)^s \phi(z)$ *where* $|z_0| < k$ *and* $\phi(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n - s$, *is a polynomial of degree* $n-s$ *having no zeros in* $|z| < k$, $k > 0$, *then for every* α *and* $\lambda \in \mathbb{C}$ *with* $|\alpha| \geq R$, $|\lambda| < 1$ *and* $0 < r \leq R \leq k$,,

$$\max_{|z|=R} \left| \frac{D_\alpha p(z)}{(z-z_0)^s} - \frac{(\alpha - z)sp(z)}{(z-z_0)^{s+1}} - \frac{sp(z)}{(z-z_0)^s} \right| \leq \frac{n-s}{1+s_l} \left[\max_{|z|=r} \left| \frac{p(z)}{(z-z_0)^s} \right| E_l - \{E_l - (s_l+1)\} \times \lambda \frac{m}{(k+|z_0|)^s} \right], \quad (29)$$

where

$$E_l = \left(\frac{|\alpha|}{R} + s_l \right) exp\left((n-s) \int_r^R A_l dl \right),$$

$$s_l = \left(\frac{k}{R} \right)^{\mu+1} \left\{ \frac{(\frac{\mu}{n-s}) \frac{|a_\mu|}{||a_0| - \lambda m'|} R k^{\mu-1} + 1}{(\frac{\mu}{n-s}) \frac{|a_\mu|}{R(||a_0| - \lambda m'|)} k^{\mu+1} + 1} \right\},$$

$$A_l = \frac{(\frac{\mu}{n-s}) \frac{|a_\mu|}{||a_0| - \lambda m'|} l^{\mu-1} k^{\mu+1} + l^\mu}{l^{\mu+1} + (\frac{\mu}{n-s}) \frac{|a_\mu|}{||a_0| - \lambda m'|} (k^{\mu+1} l^\mu + k^{2\mu} l) + k^{\mu+1}},$$

$m = \min_{|z|=k} |p(z)|$ *and* $m' = \min_{|z|=k} |\phi(z)|$.

Proof: Let $p(z) = (z - z_0)^s \phi(z)$ where $\phi(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n - s$, is a polynomial of degree $n - s$ having no zeros in $|z| < k$, $k > 0$.
Now, let

$$\begin{aligned} m' &= \min_{|z|=k} |\phi(z)| \\ &= \min_{|z|=k} \left\{ \frac{1}{|z-z_0|^s} |p(z)| \right\} \\ &\geq \frac{1}{(k+|z_0|)^s} \min_{|z|=k} |p(z)| \\ &= \frac{1}{(k+|z_0|)^s} m, \end{aligned}$$

where $m = \min_{|z|=k} |p(z)|$.

Since $\phi(z) \neq 0$ in $|z| < k$, $k > 0$, the polynomial $\psi(z) = \phi(Rz)$, where $0 < R \leq k$, has no zeros in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$. Now applying Lemma 7 to the polynomial $\psi(z)$ and noting that $|\alpha| \geq R$, we get

$$\max_{|z|=1} |D_{\frac{\alpha}{R}} \psi(z)| \leq \frac{n-s}{1+s_l} \left\{ \left(\frac{|\alpha|}{R} + s_l \right) \max_{|z|=1} |\psi(z)| - \left(\frac{|\alpha|}{R} - 1 \right) \lambda \min_{|z|=\frac{k}{R}} |\psi(z)| \right\}, \quad (30)$$

where $s_l = \left(\frac{k}{R}\right)^{\mu+1} \left\{ \frac{(\frac{\mu}{n-s}) \frac{|a_\mu|}{||a_0| - \lambda m'|} R k^{\mu-1} + 1}{(\frac{\mu}{n-s}) \frac{|a_\mu|}{R(||a_0| - \lambda m'|)} k^{\mu+1} + 1} \right\}$.

We apply the relations

$$\max_{|z|=1} |D_{\frac{\alpha}{R}} \psi(z)| = \max_{|z|=R} |D_\alpha \phi(z)|$$

also

$$\begin{aligned} D_\alpha \phi(z) &= (n-s)\phi(z) + (\alpha - z)\phi'(z) \\ &= (n-s)\frac{p(z)}{(z-z_0)^s} + (\alpha - z) \times \left\{ \frac{p'(z)}{(z-z_0)^s} - \frac{sp(z)}{(z-z_0)^{s+1}} \right\} \\ &= \frac{1}{(z-z_0)^s} \{(n-s)p(z) + (\alpha - z)p'(z)\} - \frac{(\alpha - z)sp(z)}{(z-z_0)^{s+1}} \\ &= \frac{D_\alpha p(z)}{(z-z_0)^s} - \frac{(\alpha - z)sp(z)}{(z-z_0)^{s+1}} - \frac{sp(z)}{(z-z_0)^s}, \end{aligned}$$

$$\max_{|z|=1} |\psi(z)| = \max_{|z|=R} \left| \frac{p(z)}{(z-z_0)^s} \right|$$

and

$$\min_{|z|=\frac{k}{R}} |\psi(z)| = \min_{|z|=k} |\phi(z)| \geq \frac{1}{(k+|z_0|)^s} \min_{|z|=k} |p(z)| = \frac{1}{(k+|z_0|)^s} m$$

into (30) and obtain

$$\max_{|z|=R}\left|\frac{D_\alpha p(z)}{(z-z_0)^s}-\frac{(\alpha-z)sp(z)}{(z-z_0)^{s+1}}-\frac{sp(z)}{(z-z_0)^s}\right|\leq \frac{n-s}{1+s_l}\left\{\left(\frac{|\alpha|}{R}+s_l\right)\max_{|z|=R}\left|\frac{p(z)}{(z-z_0)^s}\right| -\left(\frac{|\alpha|}{R}-1\right)\frac{\lambda}{(k+|z_0|)^s}m\right\}. \quad (31)$$

Using Lemma 5 to (31), we conclude the proof of Theorem 8.

■

Remark 9. *Setting $s=0$, Theorem 8 reduces to a generalization of Theorem 1.*

Remark 10. *If $z_0=0$ in Theorem 8, we have*

Corollary 11. *If $p(z)=z^s\phi(z)$ where $\phi(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n-s$, is a polynomial of degree $n-s$ having no zeros in $|z|<k$, $k>0$, then for every α and $\lambda\in\mathbb{C}$ with $|\alpha|\geq R$, $|\lambda|<1$ and $0<r\leq R\leq k$,,*

$$\max_{|z|=R}\left|\frac{D_\alpha p(z)}{z^s}-\frac{(\alpha-z)sp(z)}{z^{s+1}}-\frac{sp(z)}{z^s}\right|\leq\frac{n-s}{1+s_l}\times\left[\max_{|z|=r}\left|\frac{p(z)}{z^s}\right|E_l-\{E_l-(s_l+1)\}\lambda\frac{m}{k^s}\right], \quad (32)$$

where

$$E_l=\left(\frac{|\alpha|}{R}+s_l\right)exp\left((n-s)\int_r^R A_l dl\right),$$

$$s_l=\left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}Rk^{\mu-1}+1}{(\frac{\mu}{n-s})\frac{|a_\mu|}{R(||a_0|-\lambda m'|)}k^{\mu+1}+1}\right\},$$

$$A_l=\left\{\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}l^{\mu-1}k^{\mu+1}+l^\mu}{l^{\mu+1}+(\frac{\mu}{n-s})\frac{|a_\mu|}{||a_0|-\lambda m'|}(k^{\mu+1}l^\mu+k^{2\mu}l)+k^{\mu+1}}\right\},$$

$m=\min_{|z|=k}|p(z)|$ *and* $m'=\min_{|z|=k}|\phi(z)|$.

Remark 12. *Further, we first set λ a real number in Corollary 11 and applying Lemma 2, we have*

$$m'=\min_{|z|=k}|\phi(z)|\leq|\phi(z)| \quad in \quad |z|\leq k.$$

In particular, we have from above

$$m'\leq|a_0|.$$

Since $0<\lambda<1$,

$$m'\lambda\leq|a_0|.$$

Therefore

$$||a_0|-\lambda m'|=|a_0|-\lambda m'.$$

Remark 13. *Putting $s=0$ and using the conclusion of Remark 12, Corollary 11 reduces to a result recently proved by Mir and Malik [22, Theorem 1].*

Remark 14. *Dividing both sides of inequality (32) by $|\alpha|$ and letting $|\alpha|\to\infty$ and following the conclusion of Remark 12, we have*

Corollary 15. *If $p(z)=z^s\phi(z)$ where $\phi(z)=a_0+\sum_{\nu=\mu}^{n}a_\nu z^\nu$, $1\leq\mu\leq n-s$, is a polynomial of degree $n-s$ having no zeros in $|z|<k$, $k>0$, then for every real number λ with $0<\lambda<1$ and $0<r\leq R\leq k$,*

$$\max_{|z|=R}\left|\frac{p'(z)}{z^s}-\frac{sp(z)}{z^{s+1}}\right|\leq\frac{n-s}{R(1+s_l)}\left[\left\{\max_{|z|=r}\left|\frac{p(z)}{z^s}\right|-\lambda\frac{m}{k^s}\right\}\times exp\left((n-s)\int_r^R A_l dl\right)\right], \quad (33)$$

where

$$s_l=\left(\frac{k}{R}\right)^{\mu+1}\left\{\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{|a_0|-\lambda\frac{m}{k^s}}Rk^{\mu-1}+1}{(\frac{\mu}{n-s})\frac{|a_\mu|}{R(|a_0|-\lambda\frac{m}{k^s})}k^{\mu+1}+1}\right\},$$

$$A_l=\left\{\frac{(\frac{\mu}{n-s})\frac{|a_\mu|}{|a_0|-\lambda\frac{m}{k^s}}l^{\mu-1}k^{\mu+1}+l^\mu}{l^{\mu+1}+(\frac{\mu}{n-s})\frac{|a_\mu|}{|a_0|-\lambda\frac{m}{k^s}}(k^{\mu+1}l^\mu+k^{2\mu}\delta)+k^{\mu+1}}\right\}$$

and $m=\min_{|z|=k}|p(z)|$.

Remark 16. *Putting $s=0$ and taking limit as $\lambda\to1$, Corollary 15 yields a result due to Chanam and Dewan [6, Theorem 2]*

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