Inventory Models with Compounding Interest Revisited

Yung-Ning Cheng, Kou-Huang Chen

Abstract—**We provide a detailed examination of inventory models with compounding interest that derived an approximated closed-form optimal solution. The goal of our paper is threefold. First, we find a criterion to guarantee the existence of the original optimal solution. Under the criterion, we prove the uniqueness of the original optimal solution. Second, because the source paper did not consider the zeros of the first derivative, we point out that the complicated discussion of the second derivative proposed by the source paper is useless in his derivations. Third, we provide our approximated closed-form optimal solution that is more accurate than the approximated closed-form optimal solution in the source paper. Numerical examples cited from the source paper support our claim. Our paper will provide significant improvements to the source paper.**

Index Terms—**Inventory systems, Compounding interest, Closed-form solution, Approximated solution**

I. Introduction

RECENTLY Çalışkan published several papers with inventory models to reveal that his works provided significant contributions to this kind of research topic. Çalışkan [1] studied inventory systems related to Ghare and Schrader [2] and Widyadana et al. [3] by an alternative method to obtain the optimal replenishment cycle length with a nested radical expression and a closed-form solution for cubic polynomial in an arccosine formation. Çalışkan [4] studied the approximated inventory system proposed by Chung and Ting [5] and then derived a new approximated optimal solution such that his new formulated optimal solution has a simple expression than that of Chung and Ting [5]. Çalışkan [6] examined the approximated inventory system considered by Widyadana et al. [3] with a simplified derivation to avoid applying the cost-difference comparison method developed by Widyadana et al. [3] which was originally constructed by Wee et al. [7]. Therefore, we can claim that those papers published by Çalışkan stand for an important research trend lately.

In this paper, we will focus on the paper of Çalışkan [8] that developed an inventory system containing (a) setup cost, (b) estimated holding cost, and (c) the monetary value of the inventory with compounding interest. Çalışkan [8] first considered the interest for the first replenishment cycle, and then the interests will be generated continuous interests for the rest time horizon of the year, and then examined the partial cycles and then developed a novel inventory system. The goal of Çalışkan [8] is to derive an approximated closed-form optimal solution for his inventory system. Çalışkan [9] is another paper to study this new inventory system with compounding interest developed by Çalışkan [8]. Çalışkan [9] mentioned $\frac{d}{dQ}\mathrm{TC}(Q)$ as Equation (11) in this paper that was derived in Çalışkan [8]. However, Çalışkan [9] did not study $\frac{d}{dQ}\mathrm{TC}(Q)$, instead, he directly applied $e^x \approx \frac{2+x}{2-x}$ to simply $\mathrm{TC}(Q)$ (as Equation (10) in this paper) to derive an approximated inventory model (as Equation (16) in this paper). Consequently, the closed-form minimum solution obtained in Çalışkan [9] is identical to the closed-form minimum solution derived in Çalışkan [8] as Equation (17) in this paper. Hence, we can claim that Çalışkan [9] did not provide any improvement for Çalışkan [8].

Remark. Çalışkan [8] was published in 2021 that did not cite Çalışkan [9] that was published in 2020. However, Çalışkan [9] had cited Çalışkan [8] in his References. Hence, we treat Çalışkan [8] as the first paper and Çalışkan [9] as the second paper for inventory models with compounding interest proposed by Çalışkan.

In this paper, we will study Çalışkan [8] to show the following three issues:

(a) We will point out that Çalışkan [8] did not consider the zeros for his first derivative. Hence, he did not prove the existence of his original optimal solution.

(b) Çalışkan [8] developed a discussion for the second derivative for his objective function to verify its convexity. Convexity can help researchers to know that if the zero of the first derivative indeed exists then it is unique. However, convexity cannot guarantee existence. Owing to Çalışkan [8] only deriving an approximated optimal solution, his results with the convexity are useless.

(c) We provide a more accurate approximated optimal solution that was supported by the numerical examples as provided in Çalışkan [8].

Hence, our paper will help researchers realize this important paper developed by Çalışkan [8] and related issues to construct approximated models to generate closed-form solutions for approximated inventory models.

II. Assumptions and Notation

To be compatible with Çalışkan [8], we adopted the same notation and assumption as that of Çalışkan [8] and several

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auxiliary functions developed by us to simplify the expressions during our derivations.

Notation

D: the constant demand per year.
S: the setup cost per replenishment.
T: the duration for one replenishment.
I(t): the inventory level, with $I(0) = Q$ and $I(T) = 0$.
Q: the ordering quantity per replenishment with $Q = DT$.
c: the purchasing cost per item.
i: the holding cost that is expressed as a fraction of c per year.
r: the annual interest rate. x: a new variable assumed by Çalışkan [8], with $x = e^{-rQ/D}$.
$f(x) = e^{-x}$: an auxiliary function assumed by us, for $x > 0$.
$g(x) = (x-2)^2$, an auxiliary function assumed by us, for $x > 0$.
$h(x) = \frac{1-x+x\ln x}{(1-x)^2}$: an auxiliary function assumed by Çalışkan [8].
$m(x) = \begin{cases} (x-2)^2, & \text{for } 0 < x \le 4, \\ 16\sqrt{x} - 28, & \text{for } 4 \le x. \end{cases}$ an auxiliary function assumed by us.
$U(x) = \frac{ic}{2} + c(e^r - 1)\left[\frac{1-x+x\ln x}{(1-x)^2}\right]$, $W(x) = \frac{r^2 S}{D\ln^2 x}$, and $P(x) = x\ln x + \ln x + 2 - 2x$: three auxiliary functions assumed by us, for $e^{-r} \le x < 1$.
$F(x) = U(x) - W(x)$: an auxiliary function assumed by us, for $e^{-r} \le x < 1$, such that to solve $\frac{d}{dQ}TC(Q) = 0$ is equivalent to solving $F\left(\frac{-rQ}{D}\right) = 0$.
$V(r) = \frac{e^r(e^r - 1 - r)}{e^r - 1}$: an auxiliary function, for $r > 0$, assumed by us.
$V_1(r) = e^{2r} - 3e^r + 2 + r$, and $V_2(r) = 2e^{2r} - 3e^r + 1$: are two auxiliary functions, for $r \ge 0$, assumed by us.
Q_C^*: the approximated closed-form solution proposed by Çalışkan [8] for his approximated inventory system.
Q_α^*: the closed-form solution proposed by us, with rational expression, for our approximated inventory model.
Q_β^* : the closed-form solution proposed by us, with trigonometric expression, for our approximated inventory model.
Q_γ^*: the limit of our sequenced solution, for our approximated inventory model.
Q^*: the exact optimal solution derived by us for original inventory system developed by Çalışkan [8].

Assumptions

The lead time is neglected.
The setup cost is S and the annual average setup cost is $\frac{S}{T} = \frac{DS}{Q}$.
The annual purchasing cost with compounding is shown as $c(e^r - 1)\left[\frac{Q}{1-e^{-rQ/D}} - \frac{D}{r}\right]$ that was derived by Çalışkan [8], under the restriction $0 < Q \le D$.
The average holding cost is estimated as the average of the maximum inventory level, $I(0) = Q$, and the ending inventory level, $I(T) = 0$, to imply the total holding cost per replenishment is $ic\frac{Q}{2}T$ and the annual average holding cost is $ic\frac{Q}{2}$.

III. Background Explanation

In this section, we will use examples to explain two different solution procedures between Çalışkan [8] and ours as developed in this study.

For the first example, we assume an objective function,

$$g(x) = (x-2)^2, \tag{3.1}$$

for $x > 0$. The solution procedure of Çalışkan [8] is to obtain $\frac{d^2}{dx^2}g(x)$ and then analyzed its property to show that $\frac{d^2}{dx^2}g(x) = 2 > 0$ such that g(x) is a convex function.
Çalışkan [8] did not directly solve $\frac{d}{dx}g(x) = 0$ to search for the exact optimal solution. Instead, he constructed an approximated closed-form optimal solution.
We adopt another solution process to solve $\frac{d}{dx}g(x) = 0$ to locate a critical point, $x^\Delta = 2$, and then we prove that $\frac{d}{dx}g(x) < 0$, for $0 < x < x^\Delta$ and $\frac{d}{dx}g(x) > 0$, for $x^\Delta < x < \infty$ such that g(x) is a decreasing function for $0 < x < x^\Delta$ and g(x) is an increasing function for $x^\Delta < x < \infty$. Hence, the critical point, $x^\Delta = 2$, becomes the minimum solution that is the optimal solution.
We must point out that the derivations of Çalışkan [8] for his second derivative are redundant in his solution procedure because he did not try to find the zeros for the first derivative, that is, Çalışkan [8] did not solve the minimum solution for his original model.

For the second example, we assume that

$$f(x) = e^{-x}, \tag{3.2}$$

for $x > 0$. From $\frac{d^2}{dx^2}f(x) = e^{-x} > 0$, f(x) is convex, for $x > 0$. However, f(x) does not have a minimum solution for $x > 0$.
After we demonstrate that the convexity property cannot guarantee the existence of the minimum solution. We illustrate our solution process as follows.
We examine $\frac{d}{dx}f(x) = -e^{-x} < 0$, for $x > 0$, to imply that f(x) is a decreasing function, and then the inferior value is attained when $\lim_{x\to\infty} f(x) = 0$. Our solution process is scrutinized the first derivative, without referring to the second derivative.

We provide a third example, denoted as m(x), where

$$m(x) = \begin{cases} (x-2)^2, & \text{for } 0 < x \le 4, \\ 16\sqrt{x} - 28, & \text{for } 4 \le x. \end{cases} \tag{3.3}$$

We know that

$$\lim_{x\to 4^-}(x-2)^2 = 4 = \lim_{x\to 4^+} 16\sqrt{x} - 28. \tag{3.4}$$

We derive that

$$\frac{d}{dx}m(x) = \begin{cases} 2(x-2), & \text{for } 0 < x < 4, \\ 8/\sqrt{x}, & \text{for } 4 < x. \end{cases} \tag{3.5}$$

and

$$\frac{d^2}{dx^2}m(x) = \begin{cases} 2, & \text{for } 0 < x < 4, \\ -4/x^{3/2}, & \text{for } 4 < x. \end{cases} \tag{3.6}$$

From Equation (3.6), it shows that m(x) is not a convex function for $0 < x$ and then applying the convexity property is not worked with m(x).
By our approach, we study the first derivative to examine

$$2(x-2) = 0, \tag{3.7}$$

for $0 < x < 4$, and

$$8/\sqrt{x} = 0, \quad (3.8)$$

for $4 < x$. We will derive that $\frac{d}{dx}m(x) = 0$ has only one solution, denoted as x^{Δ}, then $x^{\Delta} = 2$.

Moreover, we show that $\frac{d}{dx}m(x) < 0$, for $0 < x < 2$, and $\frac{d}{dx}m(x) > 0$, for $2 < x < 4$ and $4 < x$, with

$$\lim_{x\to 4^-} 2(x-2) = 4 = \lim_{x\to 4^+} 8/\sqrt{x}. \quad (3.9)$$

It follows that $m(x)$ decreases for $0 < x < 2$ and $m(x)$ increases for $2 < x$ to yield that $x^{\Delta} = 2$ is the minimum solution.

Hence, by our approach, working on the first derivative, without referring to the second derivative, can handle more objective functions in inventory systems.

IV. Review of Çalışkan

In this section, we provide a brief review of Çalışkan [8] to illustrate how did he develop his approximated inventory model.

Çalışkan [8] developed an inventory model with compounding interest such that the annual total cost is expressed as follows,

$$TC(Q) = \frac{DS}{Q} + ic\frac{Q}{2} + c(e^r - 1)\left[\frac{Q}{1-e^{-rQ/D}} - \frac{D}{r}\right], \quad (4.1)$$

under the condition, $0 < Q \le D$.

He took the first derivative of Equation (4.1) to derive that

$$\frac{d}{dQ}TC(Q) = \frac{-DS}{Q^2} + \frac{ic}{2},$$

$$+c(e^r - 1)\left[\frac{1-e^{-rQ/D} - (rQ/D)e^{-rQ/D}}{\left(1-e^{-rQ/D}\right)^2}\right]. \quad (4.2)$$

He assumed a new variable, x, with

$$x = e^{-r\frac{Q}{D}}, \quad (4.3)$$

and then Çalışkan [8] rewrote $\frac{d}{dQ}TC(Q) = 0$ as

$$\frac{ic}{2} + c(e^r - 1)\left[\frac{1-x+x\ln x}{(1-x)^2}\right] = \frac{r^2S}{D\ln^2 x}. \quad (4.4)$$

For further discussion, he assumed an auxiliary function, $h(x)$, with

$$h(x) = \frac{1-x+x\ln x}{(1-x)^2}. \quad (4.5)$$

He showed that $\lim_{x\to 1} h(x) = 1/2$, $\lim_{x\to 0} h(x) = 1$, and $\lim_{x\to\infty} h(x) = 0$ and then mentioned that $h(x)$ is a monotone decreasing function, for $x \ge 0$.

He replaced

$$h(x) = \frac{1}{2}, \quad (4.6)$$

in Equation (4.4) to simplify Equation (4.3) as

$$\frac{ic}{2} + \frac{c(e^r-1)}{2} = \frac{DS}{Q^2}. \quad (4.7)$$

Based on Equation (4.7), he claimed that he obtained an approximated closed-form optimal solution, denoted as Q_C^*, then

$$Q_C^* = \sqrt{\frac{2DS}{c(i+e^r-1)}}. \quad (4.8)$$

V. Our Proof of the Existence and Uniqueness of the Optimal Solution

In this section, we will provide two theorems to prove the existence and uniqueness of the optimal solution for the approximated inventory model proposed by Çalışkan [8].

We recall that Çalışkan [8] worked very hard to prove that his objective function of Equation (4.1) is convex. However, we already demonstrate that convexity cannot guarantee the existence of a minimum solution.

We check Çalışkan [8] to discover that he did not explain why the equation of $\frac{d}{dQ}TC(Q) = 0$ has a unique solution. That is an answer to the question of why there is a unique solution of x that satisfies Equation (4.4) is not provided in Çalışkan [8].

We recall the restriction of $0 < Q \le D$, and the assumption of the new variable, $x = e^{-r\frac{Q}{D}}$ proposed by Çalışkan [8] in Equation (4.3), and then we find the domain for the variable x as

$$e^{-r} \le x < 1. \quad (5.1)$$

Hence, the discussion of two limits of $\lim_{x\to 0} h(x) = 1$ and $\lim_{x\to\infty} h(x) = 0$ derived by Çalışkan [8] are redundant.

Based on Equation (4.4), we will assume two auxiliary functions, $U(x)$ and $W(x)$, where $U(x)$ is the left-hand side of Equation (4.4), and $W(x)$ is the right-hand side of Equation (4.4) such that

$$U(x) = \frac{ic}{2} + c(e^r - 1)\left[\frac{1-x+x\ln x}{(1-x)^2}\right], \quad (5.2)$$

and

$$W(x) = \frac{r^2S}{D\ln^2 x}, \quad (5.3)$$

for $e^{-r} \le x < 1$.

We begin to prove the following four issues:

(i) $U(x)$ is a decreasing function,

(ii) $W(x)$ is an increasing function,

(iii) $U(e^{-r}) > W(e^{-r})$, and

(iv) $\lim_{x\to 1} U(x) < \lim_{x\to 1} W(x)$.

We derive that

$$\frac{d}{dx}U(x) = c(e^r - 1)\left[\frac{x\ln x + \ln x + 2 - 2x}{(1-x)^3}\right], \quad (5.4)$$

for $e^{-r} < x < 1$.

Based on the numerator term in the bracket of Equation (5.4), we assume another auxiliary function, denoted as $P(x)$, where

$$P(x) = x\ln x + \ln x + 2 - 2x, \quad (5.5)$$

for $e^{-r} \le x \le 1$. We extend the domain of $P(x)$ to its boundary to simplify the expressions.

We find that

$$\frac{d}{dx}P(x) = \ln x + \frac{1}{x} - 1, \quad (5.6)$$

and

$$\frac{d^2}{dx^2}P(x) = \frac{x-1}{x^2} < 0, \quad (5.7)$$

for $e^{-r} < x < 1$.

Based on Equation (5.7), we know that $\frac{d}{dx}P(x)$ is a decreasing function.

We compute that

$$\lim_{x\to e^{-r}} \frac{d}{dx}P(x) = e^r - 1 - r > 0, \quad (5.8)$$

owing to Taylor's series expansion of the exponential function, $e^r = \sum_{k=0}^{\infty}\frac{r^k}{k!}$. Moreover, we obtain that $\lim_{x\to 1}\frac{d}{dx}P(x) = 0$. Hence, $\frac{d}{dx}P(x)$ decreases from $\lim_{x\to e^{-r}}\frac{d}{dx}P(x) > 0$ to $\lim_{x\to 1}\frac{d}{dx}P(x) = 0$ to imply that $\frac{d}{dx}P(x) > 0$, for $e^{-r} < x < 1$.

Therefore, $P(x)$ is an increasing function.

We find that $P(1) = 0$ and

$$P(e^{-r}) = (2 + r)\left[\frac{2-r}{2+r} - e^{-r}\right]. \quad (5.9)$$

We recall that the proof of Wan and Chu [10], Lemma 1, claimed that Rachamadugu [11] and Chung and Lin [12] already have examined

$$e^{-x} > \frac{2-x}{2+x}, \quad (5.10)$$

for $x > 0$.

Based on Equation (5.10), we imply that $P(e^{-r}) < 0$.

Hence, $P(x)$ is an increasing function from $P(e^{-r}) < 0$ to $P(1) = 0$ such that we prove that $P(x) < 0$, for $e^{-r} < x < 1$.

We recall Equation (5.4) to show that $\frac{d}{dx}U(x) < 0$, for $e^{-r} < x < 1$. Hence, we finish the verification for (i) $U(x)$ being a decreasing function.

The logarithm function $\ln(x)$ is an increasing function. When $e^{-r} < x < 1$, then $\ln(x) < 0$ to imply that $\ln^2(x)$ is a decreasing function and then $1/\ln^2(x)$ and $W(x)$ are both increasing functions that finish the proof for (ii).

We derive that $U(e^{-r}) = \frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1}$, and $W(e^{-r}) = \frac{S}{D}$. Based on Tables 1 and 2 of Çalışkan [8], we will execute comparisons between $U(e^{-r})$ and $W(e^{-r})$, with $i = 0$, $c = 10$, $D \in \{10000, 500\}$, $S \in \{100, 50, 20, 10\}$, and $r \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.5, 0.95\}$.

The maximum value occurs when $D = 500$and $S = 100$, we imply that

$$\max\left\{\frac{S}{D}\right\} = 0.2. \quad (5.11)$$

We assume another auxiliary function, denoted as $V(r)$, with

$$V(r) = \frac{e^r(e^r-1-r)}{e^r-1}, \quad (5.12)$$

for $r > 0$.

We will try to prove that $V(r)$ is an increasing function.

We show that

$$\frac{d}{dr}V(r) = \frac{e^r(e^{2r}-3e^r+2+r)}{(e^r-1)^2}. \quad (5.13)$$

Refer to Equation (5.13), we assume that $V_1(r) = e^{2r} - 3e^r + 2 + r$ to derive that

$$\frac{d}{dr}V_1(r) = 2e^{2r} - 3e^r + 1. \quad (5.14)$$

Based on Equation (5.14), we further assume that

$$V_2(r) = 2e^{2r} - 3e^r + 1, \quad (5.15)$$

then

$$\frac{d}{dr}V_2(r) = e^{2r} + 3e^r(e^r - 1) > 0, \quad (5.16)$$

for $r > 0$.

From Equation (5.16), we know that $V_2(r)$ is an increasing function with $V_2(0) = 0$ to yield that $V_2(r) > 0$, for $r > 0$. Using Equation (5.14), we obtain that $V_1(r)$ is an increasing function with $V_1(0) = 0$ to yield that $\frac{d}{dr}V(r) > 0$, for $r > 0$. Hence, $V(r)$ is an increasing function.

After we show that $V(r)$ is an increasing function, we know that the minimum value occurs when $r = 0.05$, we imply that

$$\min\left\{\frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1}\right\} = 0.261. \quad (5.17)$$

Now, we observe Equations (5.11) and (5.17), we prove that for every set of parameters in the numerical examples of Çalışkan [8], we numerically check that

$$U(e^{-r}) > W(e^{-r}). \quad (5.18)$$

Remark. We recall the data from Tables 1 and 2 from Çalışkan [8], $D \in \{10000, 500\}$, $S \in \{100, 50, 20, 10\}$, and $r \in \{0.05, 0.1, 0.15, 0.2, 0.25, 0.5, 0.95\}$, researchers usually will provide a lengthy comparison between $U(e^{-r}) = \frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1}$ and $W(e^{-r}) = \frac{S}{D}$.

Through our analytic results, we can directly compare (a) The minimum of $U(e^{-r})$, and (b) The maximum of $W(e^{-r})$, to derive the relationship of Equation (5.18).

At last, not least, we recall that Çalışkan [8] derived $\lim_{x\to1} h(x) = 1/2$, and then

$$\lim_{x\to1} U(x) = \frac{c(i+e^r-1)}{2}. \quad (5.19)$$

On the other hand, $\lim_{x\to1} W(x) = \infty$, to show that

$$\lim_{x\to1} U(x) < \lim_{x\to1} W(x). \quad (5.20)$$

We assume that a new auxiliary function denoted as $F(x)$, with

$$F(x) = U(x) - W(x). \quad (5.21)$$

We have already verified that $U(x)$ is a decreasing function and $W(x)$ is an increasing function such that $F(x)$ is a decreasing function.

We recall that

$$F(e^{-r}) = U(e^{-r}) - W(e^{-r}) > 0, \quad (5.22)$$

and

$$\lim_{x\to1} F(x) = \lim_{x\to1} U(x) - \lim_{x\to1} W(x) < 0. \quad (5.23)$$

Hence, there is a unique point, denoted as x^* that satisfies, $F(x^*) = 0$.

We show that Equation (4.4) has a unique solution, Q^*, with

$$Q^* = \frac{-D}{r}\ln(x^*). \quad (5.24)$$

By Equation (4.2), we know that

$$\frac{d}{dQ}TC(Q) = F\left(\frac{-rQ}{D}\right) = F(x). \quad (5.25)$$

We show that $F(x) > 0$, for $e^{-r} < x < x^*$ and $F(x) < 0$, for $x^* < x < 1$. We rewrite the above results in the variable Q, then $F\left(\frac{-rQ}{D}\right) > 0$, for $e^{-r} < e^{-rQ/D} < e^{-rQ^*/D}$ and $F\left(\frac{-rQ}{D}\right) < 0$, for $e^{-rQ^*/D} < e^{-rQ/D} < 1$.

The logarithm function is an increasing function for $e^{-r} < e^{-rQ/D} < e^{-rQ^*/D}$, using $\ln(x)$, we derive that $-r < \frac{-rQ}{D} < \frac{-r}{D}Q^*$ and then rewrite it as $Q^* < Q < D$.

Similarly, for $e^{-rQ^*/D} < e^{-rQ/D} < 1$, using $\ln(x)$, we obtain that $\frac{-r}{D}Q^* < \frac{-rQ}{D} < 0$ and then rewrite it as $0 < Q < Q^*$.

Based on Equation (5.25), we prove that $\frac{d}{dQ}TC(Q) > 0$, for $Q^* < Q < D$. On the other hand, we verify that $\frac{d}{dQ}TC(Q) < 0$, for $0 < Q < Q^*$ to imply that Q^* is the minimum solution. We summarize our findings in the next theorem.

Theorem 1.

If the parameters satisfy $\frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1} > \frac{S}{D}$, then we prove that there is a point, denoted as Q^*, with $0 < Q^* < D$, that satisfies $F\left(\frac{-r}{D}Q^*\right) = 0$ which is the minimum solution for $TC(Q)$ of Equation (4.1).

On the other hand, if the parameters satisfy

$$\frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1} \le \frac{S}{D}, \tag{5.26}$$

we imply that $F(x) = 0$ has a unique solution at $x = e^{-r}$ and $F(x) < 0$, for $e^{-r} < x < 1$ such that $F\left(\frac{-rQ}{D}\right) < 0$, for $e^{-r} < e^{-rQ/D} < 1$ to imply that $\frac{d}{dQ}TC(Q) < 0$, for $0 < Q < D$ and result in $Q^* = D$. Hence, we derive the second main result.

Theorem 2.

If the parameters satisfy $\frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1} \le \frac{S}{D}$, then we prove that there is a point, denoted as Q^*, with $Q^* = D$, that is the minimum solution for $TC(Q)$ of Equation (4.1).

Based on our above discussion, we only use $\frac{d}{dQ}TC(Q)$ to prove that $\frac{d}{dQ}TC(Q) = 0$ has a unique solution which is the minimum solution, without referring to $\frac{d^2}{dQ^2}TC(Q)$.

Moreover, we must point out that even researchers proved his objective function is convex, for example, $f(x) = e^{-x}$ that is convex. The convexity property cannot guarantee $\frac{d}{dx}f(x) = -e^{-x}$ having a root for $x > 0$.

In fact, the inferior value of $f(x) = e^{-x}$ is zero that is attained as

$$\lim_{x\to\infty} f(x) = 0. \tag{5.27}$$

However, this inferior value cannot be attained by any point, x, with $x > 0$.

We recall the goal of Çalışkan [8] is to derive an approximated closed-form optimal solution for his new inventory model, under the restriction $0 < Q \le D$. Consequently, his derivations with $\frac{d^2}{dQ^2}TC(Q)$ and convexity of $TC(Q)$ are meaningless owing to the following two reasons:

(a) Çalışkan [8] did not try to solve $\frac{d}{dQ}TC(Q) = 0$, which means that Çalışkan [8] overlooked the existence problem of $\frac{d}{dQ}TC(Q) = 0$.

(b) His lengthy proof of convexity was not used to verify the uniqueness of $\frac{d}{dQ}TC(Q) = 0$.

Based on our above discussion, we not only solve the existence and uniqueness of $\frac{d}{dQ}TC(Q) = 0$, but also find the criterion,

$$\frac{ic}{2} + c\frac{e^r(e^r-1-r)}{e^r-1} > \frac{S}{D}, \tag{5.28}$$

to assure the interior minimum solution without referring to $\frac{d^2}{dQ^2}TC(Q)$.

VI. Our Proposed Approximated Closed-form Optimal Solution

In this section, we will derive another closed-form approximated optimal solution that is more accurate than that of Çalışkan [8].

We refer to Rachamadugu [11], Chung and Lin [12], Chung [13], Wan and Chu [10], Çalışkan [9], and Çalışkan [4], to recall that researchers used a lower bound of e^{-x}, as

$$e^{-x} > \frac{2-x}{2+x}, \tag{6.1}$$

or an upper bound of e^x, as

$$\frac{2+x}{2-x} > e^x, \tag{6.2}$$

to present a reasonable motivation for the substitution, for small values of x,

$$e^x \approx \frac{2+x}{2-x}. \tag{6.3}$$

We recall Equation (4.2), for the third term on the right-hand side, to rewrite it as

$$\frac{1-e^{-rQ/D}-(rQ/D)e^{-rQ/D}}{\left(1-e^{-rQ/D}\right)^2} = \frac{e^{rQ/D}\left(e^{rQ/D}-1-rQ/D\right)}{\left(e^{rQ/D}-1\right)^2}. \tag{6.4}$$

Using Equation (6.3), we estimate

$$e^{rQ/D} - 1 - rQ/D \approx \frac{(rQ/D)^2}{2-rQ/D}, \tag{6.5}$$

and

$$e^{rQ/D} - 1 \approx \frac{2rQ/D}{2-rQ/D}. \tag{6.6}$$

According to Equations (6.3), (6.5) and (6.6), we estimate Equation (6.4) as

$$\frac{1-e^{-rQ/D}-(rQ/D)e^{-rQ/D}}{\left(1-e^{-rQ/D}\right)^2} \approx \frac{1}{2} + \frac{rQ}{4D}. \tag{6.7}$$

By Equation (6.7), we derive a simplified approximation for $\frac{d}{dQ}TC(Q) = 0$ as

$$\frac{cr(e^r-1)}{4D}Q^3 + \frac{c(i+e^r-1)}{2}Q^2 - DS = 0. \tag{6.8}$$

The left-hand side of Equation (6.8) is a cubic polynomial. We recall the algebraic formula discussed in Chang and Schonfeld [14], Yang et al. [15], and Tung et al. [16], and the trigonometric representation in Çalışkan [1]. Hence, we can write the zeros for Equation (6.8) as follows from the following Equation (6.9) to Equation (6.24).

For the general equation for a cubic polynomial

$$Y^3 + a_1Y^2 + a_2Y + a_3 = 0. \tag{6.9}$$

Researchers assumed that

$$P = \frac{3a_2 - a_1^2}{9}, \tag{6.10}$$

$$R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}, \tag{6.11}$$

$$S = \left[R + (R^2 + P^3)^{1/2}\right]^{1/3}, \tag{6.12}$$

and

$$T = \left[R - (R^2 + P^3)^{1/2}\right]^{1/3}. \tag{6.13}$$

According to Spiegel [17], the solution for Equation (6.9) can be found if $R^2 + P^3 > 0$, then there is a real root

$$Y = S + T - \frac{a_1}{3}. \tag{6.14}$$

We find that

$$a_1 = \frac{2D(i+e^r-1)}{r(e^r-1)}, \tag{6.15}$$

$$a_2 = 0, \tag{6.16}$$

and

$$a_3 = \frac{-4D^2S}{c\,r(e^r-1)}, \tag{6.17}$$
and then it shows that
$$R = \frac{2D^2S}{c\,r(e^r-1)} - \frac{8D^3(i+e^r-1)^3}{27r^3(e^r-1)^3}, \tag{6.18}$$
and
$$P = \frac{-4D^2(i+e^r-1)^2}{9r^2(e^r-1)^2}. \tag{6.19}$$
Referring to Equations (6.12) and (6.13), we derive a closed-form optimal solution in the rational expression for our approximated inventory model, denoted as Q_α^*, then
$$Q_\alpha^* = S + T - \frac{a_1}{3}. \tag{6.20}$$

On the other hand, we refer to Çalışkan [1] for a cubic polynomial
$$N^3 - aN - b = 0, \tag{6.21}$$
having the positive real solution
$$N = 2\sqrt{\frac{a}{3}\left[\cos\left(\frac{1}{3}\arccos\sqrt{\frac{27b^2}{4a^3}}\right)\right]}. \tag{6.22}$$
We rewrite Equation (6.8), with $Q = 1/N$ to convert it as
$$N^3 - \frac{c(i+e^r-1)}{2DS}N - \frac{cr(e^r-1)}{4D^2S} = 0. \tag{6.23}$$
Hence, we obtain a closed-form approximated solution in trigonometric expression, denoted as Q_β^*, then
$$Q_\beta^* = \frac{1}{2\sqrt{\frac{c(i+e^r-1)}{6DS}}\cos\left(\frac{1}{3}\arccos\sqrt{\frac{27r^2(e^r-1)^2S}{8cD(i+e^r-1)^3}}\right)}. \tag{6.24}$$

VII. Numerical Examples

Based on Equation (6.8), we can provide a sequenced approach for our approximated inventory model as
$$Q_{n+1} = \sqrt{\frac{2DS}{c(i+e^r-1)+(cr(e^r-1)Q_n/2D)}}. \tag{7.1}$$
If the sequence (Q_n) converges, then we will accept the limit, denoted as Q_γ^*, with
$$Q_\gamma^* = \lim_{n\to\infty} Q_n. \tag{7.2}$$

If we assume that $Q_0 = 0$, then
$$Q_1 = \sqrt{\frac{2DS}{c(i+e^r-1)}}, \tag{7.3}$$
which is the closed-form solution proposed by Çalışkan [8] for his approximated inventory system.

For $S = 100$, $r = 0.05$, D=500, $i = 0$ and $c = 10$, we list the computation result for sequence (Q_n), for $n = 1, 2,3...$, with $Q_0 = 0$, in the following Table 1.

From Table 1, we derive that
$$Q_6 = Q_7 = 436.889191, \tag{7.4}$$
such that we accept the sequence solution for our approximated inventory model,
$$Q_\gamma^* = 436.889191. \tag{7.5}$$

We refer to Çalışkan [8] that obtained $Q^* = 441.64$ which is the round off result for our $Q_1 = 441.635217$.

We consider the rational results of Equation (6.20) to find that $a_1 = 2\times 10^4$, $a_2 = 0$, $a_3 = -3.900833\times 10^9$, $R = -2.943459\times 10^{11}$, $P = -4.444444\times 10^7$, $S = 3.551778\times 10^3 + 5.641748i\times 10^3$, and $T = 3.551778\times 10^3 - 5.641748i\times 10^3$. Hence, based on Equation (6.20), we obtain the closed-form optimal solution in rational expression as
$$Q_\alpha^* = S + T - \frac{a_1}{3} = 436.889191. \tag{7.6}$$

We recall the trigonometric expression of the closed-form solution mentioned in Çalışkan [1] to find
$$\frac{c(i+e^r-1)}{6DS} = 1.709037\times 10^{-6}, \tag{7.7}$$
$$\frac{27r^2(e^r-1)^2S}{8cD(i+e^r-1)^3} = 3.291328\times 10^{-3}, \tag{7.8}$$
$$\arccos\sqrt{\frac{27r^2(e^r-1)^2S}{8cD(i+e^r-1)^3}} = 1.513395, \tag{7.9}$$
and
$$\cos\left(\frac{1}{3}\arccos\sqrt{\frac{27r^2(e^r-1)^2S}{8cD(i+e^r-1)^3}}\right) = 0.875433, \tag{7.10}$$
such that based on Equation (6.24), we show that
$$Q_\beta^* = \frac{1}{2\sqrt{\frac{c(i+e^r-1)}{6DS}}\cos\left(\frac{1}{3}\arccos\sqrt{\frac{27r^2(e^r-1)^2S}{8cD(i+e^r-1)^3}}\right)}, = 436.889191. \tag{7.11}$$

We compare our results of Equations (7.5), (7.6) and (7.11) to illustrate our proposed solutions for three different methods for our approximated inventory model: (a) our sequence approach, Q_γ^*, (b) closed-form solution in rational expression, Q_α^*, and (c) closed-form solution in trigonometric expression, Q_β^*, are all identical as expected to demonstrate the validity of our derivations.

In the following Table 2, we list several computation results of Q and $\frac{d}{dQ}TC(Q)$ to illustrate our finding which satisfies $\frac{d}{dQ}TC(Q) = 0$.

Based on Table 2, we find that when Q increases from 438.44, 438.443, and 438.4431, the value of dTC(Q)/dQ gradually increases to 0^-. On the other hand, when Q decreases from 438.45, 438.444, and 438.4432, the value of dTC(Q)/dQ gradually decreases to 0^+. It is numerical evidence to support our detailed analytic proof that dTC(Q)/dQ is an increasing function.
We know that the exact value of Q^* satisfying
$$438.4431 < Q^* < 438.4432. \tag{7.12}$$
Owing to the absolute value of 3.8×10^{-8} being less than the absolute value of -8.2×10^{-8}, we accept that
$$Q^* = 438.4432. \tag{7.13}$$
We refer to Çalışkan [8] to mention that his approximated solution, based on Equation (4.8),
$$Q_C^* = 441.64. \tag{7.14}$$

Hence we compute the relative error between (a) Approximated solution of Çalışkan [8], Q_C^* and Q^*, (b) Our approximated solution, Q_α^* and Q^*, as follows
$$\frac{|Q_\alpha^*-Q^*|}{|Q_C^*-Q^*|} = \frac{438.4432-436.8892}{441.6352-438.4432} = 48.68\%. \tag{7.15}$$

Based on Equation (7.15), we can say that our approximated closed-form solution is more close to the exact optimal than the approximated closed-form solution proposed by Çalışkan [8].

Table 1. Our sequence approach of (Q_n), for $n = 1, 2, 3..., 7$.

n	1	2	3	4	5	6	7
Q_n	441.635217	436.838471	436.889734	436.889186	436.889192	436.889191	436.889191

Table 2. Some computation results with Q and $\frac{d}{dQ}TC(Q)$.

Q	438.44	438,443	438.4431	438.4432	438.444	438.45
dTC(Q)/dQ	-3.8 x 10^{-6}	-2.0 x 10^{-7}	-8.2 x 10^{-8}	3.8 x 10^{-8}	9.9 x 10^{-7}	8.2 x 10^{-6}

VIII. Direction for Future Research

We refer to a selection of published articles to highlight potentially hot points for practitioners seeking study resources. Purwani et al. [18] employ the Newton-Raphson algorithm in conjunction with the Aitken extrapolation method to approximate stock volatility. Addressing structural dynamics, Adhitya et al. [19] analyze loads and concrete structures under earthquake conditions. Assis and Coelho [20] investigate a remote learning and teaching project that employs temperature control as an educational tool. Investigating the breaking wave effect, Unyapoti and Pochai [21] develop a binary model involving wave crest and shoreline evolution. Tobar et al. [22] explore segmentation problems, employing label enhancement and base representation methods. Tang et al. [23] examine customer behavior in a supermarket during the Chinese New Year period using customer analysis techniques. Mane and Lodhi [24] study singularly perturbed equations and provide numerical solutions using a cubic approach. Zhu et al. [25] explore optimal train scheduling, taking carbon emissions into consideration. By employing machine learning techniques, Zhang et al. [26] develop a super-resolution image enhancement approach for morphologically sparse areas. Alomari and Massoun [27] utilize the Caputo fractional derivative to determine numerical solutions. Wan et al. [28] present an optimal solution for retailer warehouse operations through allocation arrangement strategies. Yang et al. [29] introduce a novel information system based on reciprocal accumulation generation operation and vector continued fractions. Based on our literature review, researchers will discover several intriguing topics to guide their future study directions.

IX. A Related Inventory Model

In this section, we review Chiu and Chiu [30] and Chiu et al. [31] for solving inventory systems by algebraic approaches. First, we recall their objective function and then rewrite it in a compact form with several abbreviations to simplify the expressions,

$$f(Q,n) = a_0 + \frac{a_1}{Q} + \frac{n}{Q}a_2 + a_3 Q + \frac{Q}{n}a_4, \qquad (9.1)$$

where we assume five abbreviations as follows,

$$a_0 = \frac{C\lambda}{1-\theta E[x]} + C_T\lambda,$$

$$+\frac{\lambda E[x]}{1-\theta E[x]}\left(\theta C_s + (1-\theta)C_R\right), \qquad (9.2)$$

$$a_1 = \frac{K\lambda}{1-\theta E[x]}, \qquad (9.3)$$

$$a_2 = \frac{K_1\lambda}{1-\theta E[x]}, \qquad (9.4)$$

$$a_3 = \frac{h}{2}\left(1-\theta E[x]\right) + \frac{h\lambda}{2P\left(1-\theta E[x]\right)} + \frac{\lambda E[x]}{2P_1\left(1-\theta E[x]\right)},$$

$$\left\{h(1-\theta)\left(2-(1+\theta)E[x]\right) + h_1 E[x](1-\theta)^2\right\},$$

$$+\frac{h_2-h}{2}\left(\frac{\lambda}{P} + \frac{\lambda}{P_1}(1-\theta)E[x]\right), \qquad (9.5)$$

and

$$a_4 = \frac{h_2-h}{2}\left(1-\theta E[x] - \frac{\lambda}{P} - \frac{\lambda}{P_1}(1-\theta)E[x]\right). \qquad (9.6)$$

In the next section, we will present our improvements.

X. Our Analytical Approach

In this section, we first provide our analytic approach to help readers to realize the optimal problem.

Based on Equation (9.1), we derive the first-order partial derivatives,

$$f_Q = a_3 + \frac{a_4}{n} - \frac{a_1 + na_2}{Q^2}, \qquad (10.1)$$

and

$$f_n = \frac{a_2}{Q} - \frac{Q}{n^2}a_4. \qquad (10.2)$$

Moreover, we obtain the second-order partial derivatives,

$$f_{QQ} = 2\frac{(a_1 + na_2)}{Q^3}, \qquad (10.3)$$

$$f_{Qn} = f_{nQ} = -\frac{a_2}{Q^2} - \frac{a_4}{n^2}, \qquad (10.4)$$

and

$$f_{nn} = 2\frac{Q}{n^3}a_4. \qquad (10.5)$$

Now, we need an extra condition to claim that

$$a_4 > 0, \qquad (10.6)$$

the reason for the extra condition of Equation (10.6) will be self-explained in the following derivations.

We compute the determinant of the Hessian matrix,

$$f_{QQ}f_{nn}-\left(f_{Qn}\right)^2=,$$

$$4\frac{a_1a_4}{Q^2n^3}+2\frac{a_2a_4}{Q^2n^2}-\left(\frac{a_2^2}{Q^4}+\frac{a_4^2}{n^4}\right). \quad (10.7)$$

However, if researches refer to Equation (10.7), then they will find that it is too difficult to prove that is positive.

We consider another approach to evaluate its quadratic form as follows,

$$\begin{pmatrix}Q & n\end{pmatrix}\begin{pmatrix}f_{QQ} & f_{Qn}\\ f_{nQ} & f_{nn}\end{pmatrix}\begin{pmatrix}Q\\ n\end{pmatrix}=\frac{2a_1}{Q}>0. \quad (10.8)$$

Based on Equation (10.8) we know that if the pair of critical solutions exists, then they are the optimal solution.

Next, we consider the optimal solution through the first partial derivation system. We recall Equation (10.1) to compute $f_Q=0$ and then we obtain that

$$Q^2=\frac{n\left(a_1+na_2\right)}{a_4+na_3}. \quad (10.9)$$

On the other hand, based on Equation (10.2) to compute $f_n=0$ and then we find that

$$n^2a_2=Q^2a_4. \quad (10.10)$$

Based on Equation (10.10), we provide a motivation of our extra condition of Equation (10.6) to show that a_4 is positive. According to $Q^2=\frac{n\left(a_1+na_2\right)}{a_4+na_3}$ of Equation (10.9) and $n^2a_2=Q^2a_4$ of Equation (10.10), it implies that

$$n^2=\frac{a_1a_4}{a_2a_3}, \quad (10.11)$$

and

$$Q^2=n^2\frac{a_2}{a_4}=\frac{a_1}{a_3}. \quad (10.12)$$

Therefore, we derive the pair of the optimal solutions in Equations (10.11) and (10.12) through analytic methods.

XI. Our Algebraic Method

In this section, we will apply algebraic methods to solve the minimum problem of Equation (9.1).
We directly compute that

$$f(Q,n)=a_0+\frac{a_1}{Q}+\frac{n}{Q}a_2+a_3Q+\frac{Q}{n}a_4,$$

$$=a_0+\frac{a_1}{Q}+a_3Q+\left(\sqrt{\frac{na_2}{Q}}-\sqrt{\frac{Qa_4}{n}}\right)^2+2\sqrt{a_2a_4},$$

$$=a_0+2\sqrt{a_2a_4}+2\sqrt{a_1a_3},$$

$$+\left(\sqrt{\frac{a_1}{Q}}-\sqrt{a_3Q}\right)^2+\left(\sqrt{\frac{na_2}{Q}}-\sqrt{\frac{Qa_4}{n}}\right)^2. \quad (11.1)$$

Based on Equation (13.1), the minimum point occurs at

$$Q=\sqrt{\frac{a_1}{a_3}}, \quad (11.2)$$

and

$$n=Q\sqrt{\frac{a_4}{a_2}}=\sqrt{\frac{a_1a_4}{a_2a_3}}, \quad (11.3)$$

where the minimum value is

$$a_0+2\sqrt{a_2a_4}+2\sqrt{a_1a_3}, \quad (11.4)$$

even before the minimum points of Equations (11.3) and (11.4) are explicitly derived.

XII. An Interesting Research Direction

In this section, we will provide a possible direction for the future research. We may mimic Tung and Deng [32] to improve Ahmed et al. [33] to present an alternative approach that will follow the proof procedure of equation (26) in Lin et al. [34]. In the Page 4297 of Ahmed et al. [33], the authors found three local minimums:

(i) interior minimum $\left(t_1^*,T^*\right)$;

(ii) boundary minimum along $t_1=T$, to find a local optimal solution ,denoted as $\widetilde{T}$, and

(iii) boundary minimum along $t_1=0$, to find another local optimal solution, denoted as $\hat{T}$.

We will predict the following two interrelationships among these three local minimums:

$$TC\left(t_1^*,T^*\right)<TC\left(\widetilde{T},\widetilde{T}\right), \quad (12.1)$$

and

$$TC\left(t_1^*,T^*\right)<TC\left(0,\hat{T}\right). \quad (12.2)$$

We are motivated by Tung and Deng [32] that in their Theorem 2, they proved that the boundary minimum is larger than the interior minimum as

$$JTEC\left(\sqrt{\frac{\alpha_1}{\alpha_2}},0,m\right)>JTEC\left(q^*,k^*,m\right). \quad (12.3)$$

We recall that Lin et al. [34] proved if the interior minimum existing, then the absolute minimum will be the interior minimum. Hence, there are two papers of Tung and Deng [32] and Lin et al. [34] have handled the similar problems to show the interior minimum is less than the boundary minimum, if the interior minimum exists.

XIII. Conclusion

There are three contributions to our paper. First, we show that the tedious discussion of the convexity property by the second derivative proposed by Çalışkan [8] is redundant since Çalışkan [8] did not solve the zeros of the first derivative. Second, we present a detailed examination of the first derivative to prove the optimal solution exists and is unique. Third, we provide another approximated inventory model to simplify the inventory system proposed by Çalışkan [8]. We developed three different methods to find the optimal solution for our approximated inventory model. A numerical example is presented to demonstrate our three approximated solutions are more accurate than that of Çalışkan [8] which is Q_1 in our sequenced solution (Q_n), with $Q_0 = 0$.

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