Solving Two Completing Supply Chains through Algebraic Approach

Du Peng, Jinyuan Liu, Shusheng Wu

Abstract—**We applied an algebraic approach to solve a system consisting of four objective functions such that practitioners can absorb operational studies with two completed supply chains under service competition and relative price conditions. We extended the application of algebraic methods to a new environment that will arouse interested researchers to work on this research trend. Moreover, we checked a related inventory model to point out that the previously proposed solution procedure only considered the case where the numerator and denominator of the objective function both are positive real numbers. Hence, we provie a revision for their solution process to cover the overlooked four conditions.**

Index Terms—**Risk structure, Market research, Game theory, Supply chain management**

I. Introduction

In this paper, we will discuss an inventory problem consisting of two supply chains that are completed with each other to seize the market. The purpose of our study is to solve the optimal problem through algebraic methods such that those practitioners did not familiar with calculus and differential equations still can realize inventory models and their applications in the real world environment. Hafezalkotob and Makui [1] developed an inventory model with two completed supply chains under service level constraint. The first supply chain contains a manufacturer, M_1 and a retailer, R_1. The second supply chain contains a manufacturer, M_2 and a retailer, R_2, under a completing environment with respect to market seizing and relative price. In the following, we provide a brief literature introduction of this research topic. Tsay and Agrawal [2] studied channel dynamics with service competition and relative price. Tsay [3] examined manufacturer return policies for distribution channel partnerships by risk sensitivity. Cachon and Netessine [4] considered supply chain analysis through game theory under electric business models. Bernstein and Federgruen [5] developed several equilibrium industry models under service competition and relative price. Leng and Parlar [6] studied supply chain management policies by game theoretic applications. Allon and Federgruen [7] examined service industries under competition. Bernstein and Federgruen [8] considered supply chains coordination mechanisms with service competition and relative price. With marketing channels and supply chains, He et al. [9] presented a literature survey of stackelberg differential game models. Kogan and Tapiero [10] published a book of risk valuation, operations management, and supply chain games. Under uncertain demand and risk-averse retailers, Xiao and Yang [11] examined supply chains through service competition and relative price. Xiao and Yang [12] developed a supply chain with a single retailer and a single manufacturer under integrated competitors by information sharing and risk revelation.

Following this research trend of Tsay and Agrawal [2], Tsay [3], Cachon and Netessine [4], Bernstein and Federgruen [5], Leng & Parlar [6], Allon and Federgruen [7], Bernstein and Federgruen [8], He et al. [9], Kogan and Tapiero [10], Xiao and Yang [11], and Xiao and Yang [12], Hafezalkotob and Makui [1] constructed their new inventory system, and then solved the optimal solution by analytic method with differential equations. The main contribution of this articlw is to derive the optimal solution by non-analytic approach such that those readers who are not familiar with analytic procedure, still can realize and unilize the concept of supply chain in their future development.

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II. Notation and Assumptions

Owing to our article is an further discusison of Hafezalkotob and Makui [1], therefore, we used the same notation and assumotions as Hafezalkotob and Makui [1].

Notation

For retailer i, the service level is denoted as s_i.

The extra money provided by manufacturer i is denoted as MR_{M_i}.

The extra money provided by retailer i is denoted as MR_{R_i}.

The profit of retailer i to sold one item is denoted as m_i.

The price proposed by manufacturer i to the retailer i is denoted as w_i, under the restriction, $w_i > c_i$.

For each manufacturer, to decrease the uncertainty of the demand, the efficiency of the added extra money is denoted as τ_{M_i}, with $\tau_{M_i} \geq 0$.

For each retailer, to decrease the uncertainty of the demand, the efficiency of the added extra money is denoted as τ_{R_i}, with $\tau_{R_i} \geq 0$.

For each manufacturer, the variation with respect to his profit, the constant unconditional risk dislike is denoted as λ_{M_i}, with $\lambda_{M_i} > 0$.

For each retailer, the variation with respect to his profit, the constant unconditional risk dislike is denoted as λ_{R_i}, with $\lambda_{R_i} > 0$.
For each retailer, the sensitivity of demand with respect to his opponent service level is denoted as γ, under the condition, $\beta > \gamma > 0$.
For each retailer, the sensitivity of demand with respect to his service level is denoted as β, under the condition, $\beta > 0$.
For two items, the substitutability coefficient is denoted as d.
For manufacturer i, the unit item production cost is denoted as c_i, under the constraint, $\overline{\alpha_i} \geq c_i > 0$.
Before extra money added by retialer i, the original variance of the market is expressed as σ_{0i}^2.
With variance σ_i^2 and expected value $\overline{\alpha_i}$, for each retailer i, the stochastic market id expressed as $\widetilde{\alpha_i}$.
There are two supply chains proposed in this study that will be indicated as 1 and 2.

Assumptions

1. The strategy contains two steps. In the first step, within two supply chains, each manufacturer individually provides a planning price to its retailers during the period of the first step. Sometimes, careful manufacturer can spend extra money to reduce the fuzziness of the demand.

2. In the second step, the retailers of two supply chains makes a individual decision for its service level and retailers price. On the other hand, careful retailer also can spend extra money to reduce the fuzziness of the demand.

3. Two retailers on the same time to provide their items to the market for customers to purchase.

4. The demand function is influent by the information obtained through mutual actions in the completion policies.

5. The demand function is linearly related to service level and relative price of two retailers.

6. According to Xiao and Yang [11, 12], service level and related price are two significant factor for the demand. For trtailrt i, the price of one item is expressed as
$$p_i = w_i + m_i. \tag{2.1}$$
Based on Tsay and Agrawal [2], and Xiao and Yang [11], the retailers 1 and 2 assumed the demand as follows,
$$\widetilde{q_1} = \widetilde{a_1} - p_1 + dp_2 + \beta s_1 - \gamma s_2, \tag{2.2}$$
and
$$\widetilde{q_2} = \widetilde{a_2} - p_2 + dp_1 + \beta s_2 - \gamma s_1. \tag{2.3}$$

III. Our Solution Approach

In this section, we will discuss a related inventory model studied by Hafezalkotob and Makui [1] with two completing supply chains under service constraint and relative price.
In Hafezalkotob and Makui [1], under the condition of $B_1 > 0$ and $B_2 > 0$, with $B_1 = 2 - \left(\beta^2/\eta_1\right)$ and $B_2 = 2 - \left(\beta^2/\eta_2\right)$, they used the analytic approach to solve the following system of four maximum problems,
$$U_{M1}(w_1) = \left(\overline{a}_1 - (m_1 + w_1) + d(m_2 + w_2) + \beta s_1 - \gamma s_2\right) \times (w_1 - c_1) - \lambda_{M1}(w_1 - c_1)^2 \sigma_{01}^2, \tag{3.1}$$
$$U_{M2}(w_2) = \left(\overline{a}_2 - (m_2 + w_2) + d(m_1 + w_1) + \beta s_2 - \gamma s_1\right) \times (w_2 - c_2) - \lambda_{M2}(w_2 - c_2)^2 \sigma_{02}^2, \tag{3.2}$$
$$U_{R1}(m_1, s_1) = m_1\left(\overline{a}_1 - (m_1 + w_1) + d(m_2 + w_2) + \beta s_1 - \gamma s_2\right) - \frac{\eta_1 s_1^2}{2} - \lambda_{R1} m_1^2 \sigma_{01}^2, \tag{3.3}$$
and
$$U_{R2}(m_2, s_2) = m_2\left(\overline{a}_2 - (m_2 + w_2) + d(m_1 + w_1) + \beta s_2 - \gamma s_1\right) - \frac{\eta_2 s_2^2}{2} - \lambda_{R2} m_2^2 \sigma_{02}^2, \tag{3.4}$$
simultaneously. We will apply an algebraic method to find the maximum solution for the above system.

To simplify the expression, we assume that for $i = 1,2$,
$$t_i = w_i - c_i, \tag{3.5}$$
$$H_1 = \overline{a}_1 - c_1 + dc_2, \tag{3.6}$$
and
$$H_2 = \overline{a}_2 - c_2 + dc_1. \tag{3.7}$$
From Equation (3.3), we complete the square of s_1 to imply that
$$U_{R1}(m_1, s_1) = m_1\left(H_1 - (m_1 + t_1) + d(m_2 + t_2) - \gamma s_2\right) - \frac{\eta_1}{2}\left(s_1 - \frac{m_1\beta}{\eta_1}\right)^2 + \frac{m_1^2\beta^2}{2\eta_1} - \lambda_{R1} m_1^2 \sigma_{01}^2, \tag{3.8}$$

From the coefficient of $\left(s_1 - \frac{m_1\beta}{\eta_1}\right)^2$ is $-\frac{\eta_1}{2} < 0$, to achieve the maximum value, we should have
$$s_1 = \frac{m_1\beta}{\eta_1}. \tag{3.9}$$

By the same argument, we rewrite Equation (3.4) as
$$U_{R2}(m_2, s_2) = m_2\left(H_2 - (m_2 + t_2) + d(m_1 + t_1) - \gamma s_1\right) - \frac{\eta_2}{2}\left(s_2 - \frac{m_2\beta}{\eta_2}\right)^2 + \frac{m_2^2\beta^2}{2\eta_2} - \lambda_{R2} m_2^2 \sigma_{02}^2, \tag{3.10}$$

From the coefficient of $\left(s_2 - \frac{m_2\beta}{\eta_2}\right)^2$ is $-\frac{\eta_2}{2} < 0$, to attain the maximum value, we should have
$$s_2 = \frac{m_2\beta}{\eta_2}. \tag{3.11}$$

We plug the results of Equations (3.9) and (3.11) into Equations (3.8) and (3.10) to yield that
$$U_{R1}(m_1) = m_1\left(H_1 - (m_1 + t_1) + d(m_2 + t_2) - \frac{\gamma\beta}{\eta_2} m_2\right) + \frac{m_1^2\beta^2}{2\eta_1} - \lambda_{R1} m_1^2 \sigma_{01}^2, \tag{3.12}$$
and
$$U_{R2}(m_2) = m_2\left(H_2 - (m_2 + t_2) + d(m_1 + t_1) - \frac{\gamma\beta}{\eta_1} m_1\right)$$

$$+\frac{m_2^2\beta^2}{2\eta_2}-\lambda_{R2}m_2^2\sigma_{02}^2. \tag{3.13}$$

We convert Equation (3.12) in the descending order of m_1 to derive that

$$U_{R1}(m_1)=-\left(1+\lambda_{R1}\sigma_{01}^2-\frac{\beta^2}{2\eta_1}\right)m_1^2$$
$$m_1\left(H_1-t_1+d(m_2+t_2)-\frac{\gamma\beta}{\eta_2}m_2\right). \tag{3.14}$$

Under the condition $\left(1+\lambda_{R1}\sigma_{01}^2-\left(\beta^2/2\eta_1\right)\right)>0$, to attain the maximum value, we know that

$$m_1=\Delta_1/\Delta_2, \tag{3.15}$$

where

$$\Delta_1=H_1-t_1+d(m_2+t_2)-\frac{\gamma\beta}{\eta_2}m_2, \tag{3.16}$$

and

$$\Delta_2=2\left(1+\lambda_{R1}\sigma_{01}^2-\frac{\beta^2}{2\eta_1}\right), \tag{3.17}$$

are two abbreviations to simplify the expressions.

By the same argument, we convert Equation (3.13) in the descending order of m_2 to derive that

$$U_{R2}(m_2)=-\left(1+\lambda_{R2}\sigma_{02}^2-\frac{\beta^2}{2\eta_2}\right)m_2^2$$
$$+m_2\left(H_2-t_2+d(m_1+t_1)-\frac{\gamma\beta}{\eta_1}m_1\right). \tag{3.18}$$

Under the condition $\left(1+\lambda_{R2}\sigma_{02}^2-\left(\beta^2/2\eta_2\right)\right)>0$, to achieve the maximum value, we find that

$$m_2=\Delta_3/\Delta_4, \tag{3.19}$$

where

$$\Delta_3=H_2-t_2+d(m_1+t_1)-\frac{\gamma\beta}{\eta_1}m_1, \tag{3.20}$$

and

$$\Delta_4=2\left(1+\lambda_{R2}\sigma_{02}^2-\frac{\beta^2}{2\eta_2}\right), \tag{3.21}$$

are two abbreviations to simplify the expressions.

We plug the results of Equations (3.9) and (3.11) into Equation (3.1) to derive that

$$U_{M1}(t_1)=t_1\left(H_1-(m_1+t_1)+d(m_2+t_2)\right)$$
$$+t_1\left(\frac{\beta^2}{\eta_1}m_1-\frac{\gamma\beta}{\eta_2}m_2\right)-\lambda_{M1}t_1^2\sigma_{01}^2, \tag{3.22}$$

and then rewrite Equation (3.22) in the descending order of t_1

$$U_{M1}(t_1)=t_1\left(d(m_2+t_2)+\frac{\beta^2}{\eta_1}m_1-\frac{\gamma\beta}{\eta_2}m_2\right)$$
$$-\left(1+\lambda_{M1}\sigma_{01}^2\right)t_1^2+t_1(H_1-m_1). \tag{3.23}$$

Based on Equation (3.23), because the coefficient of t_1^2 is $-\left(1+\lambda_{M1}\sigma_{01}^2\right)<0$ so we know that

$$t_1=\Delta_5/2\left(1+\lambda_{M1}\sigma_{01}^2\right), \tag{3.24}$$

where

$$\Delta_5=H_1-m_1+d(m_2+t_2)+\frac{\beta^2}{\eta_1}m_1-\frac{\gamma\beta}{\eta_2}m_2, \tag{3.25}$$

is an abbreviation to simplify the expressions.

Similarly, we plug the results of Equations (3.9) and (3.11) into Equation (3.2) to obtain that

$$U_{M2}(t_2)=t_2\Delta_6-\lambda_{M2}t_2^2\sigma_{02}^2, \tag{3.26}$$

where

$$\Delta_6=H_2-(m_2+t_2)+d(m_1+t_1)+\frac{\beta^2m_2}{\eta_2}-\frac{\gamma\beta m_1}{\eta_1}, \tag{3.27}$$

is an abbreviation to simplify the expressions. And then we rewrite Equation (3.26) in the descending order of t_2

$$U_{M2}(t_2)=t_2\left(d(m_1+t_1)+\frac{\beta^2}{\eta_2}m_2-\frac{\gamma\beta}{\eta_1}m_1\right)$$
$$-\left(1+\lambda_{M2}\sigma_{02}^2\right)t_2^2+t_2(H_2-m_2). \tag{3.28}$$

Based on Equation (3.28), because the coefficient of t_2^2 is $-\left(1+\lambda_{M2}\sigma_{02}^2\right)<0$ so we know that

$$t_2=\Delta_7/2\left(1+\lambda_{M2}\sigma_{02}^2\right), \tag{3.29}$$

where

$$\Delta_7=H_2-m_2+d(m_1+t_1)+\frac{\beta^2}{\eta_2}m_2-\frac{\gamma\beta}{\eta_1}m_1, \tag{3.30}$$

is an abbreviation to simplify the expressions.

From Equations (3.15), (3.19), (3.24) and (3.29), we have a linear system of four variables,

$$-t_1+dt_2-2\left(1+\lambda_{R1}\sigma_{01}^2-\frac{\beta^2}{2\eta_1}\right)m_1$$
$$+\left(d-\frac{\gamma\beta}{\eta_2}\right)m_2=-H_1, \tag{3.31}$$

$$\left(d-\frac{\gamma\beta}{\eta_1}\right)m_1-2\left(1+\lambda_{R2}\sigma_{02}^2-\frac{\beta^2}{2\eta_2}\right)m_2$$
$$+dt_1-t_2=-H_2, \tag{3.32}$$

$$-dt_2+\left(1-\frac{\beta^2}{\eta_1}\right)m_1+\left(\frac{\gamma\beta}{\eta_2}-d\right)m_2$$
$$+2\left(1+\lambda_{M1}\sigma_{01}^2\right)t_1=H_1, \tag{3.33}$$

and

$$-dt_1+\left(\frac{\gamma\beta}{\eta_1}-d\right)m_1+\left(1-\frac{\beta^2}{\eta_2}\right)m_2$$
$$+2\left(1+\lambda_{M2}\sigma_{02}^2\right)t_2=H_2. \tag{3.34}$$

We add Equations (3.31) with (3.33) to yield that

$$t_1=\frac{1+2\lambda_{R1}\sigma_{01}^2}{1+2\lambda_{M1}\sigma_{01}^2}m_1, \tag{3.35}$$

and then we also add Equations (3.32) with (3.34) to obtain that

$$t_2 = \frac{1+2\lambda_{R2}\sigma_{02}^2}{1+2\lambda_{M2}\sigma_{02}^2} m_2, \tag{3.36}$$

We plug our results of Equations (3.35) and (3.36) into Equations (3.33) and (3.34) to imply that

$$C_1 m_1 + C_2 m_2 = H_1, \tag{3.37}$$

and

$$C_3 m_1 + C_4 m_2 = H_2, \tag{3.38}$$

with four abbreviations,

$$C_1 = 1 - \frac{\beta^2}{\eta_1} + 2\left(1+\lambda_{M1}\sigma_{01}^2\right)\frac{1+2\lambda_{R1}\sigma_{01}^2}{1+2\lambda_{M1}\sigma_{01}^2}, \tag{3.39}$$

$$C_2 = \frac{\gamma\beta}{\eta_2} - 2d\left[\frac{1+\left(\lambda_{R2}+\lambda_{M2}\right)\sigma_{02}^2}{1+2\lambda_{M2}\sigma_{02}^2}\right], \tag{3.40}$$

$$C_3 = \frac{\gamma\beta}{\eta_1} - 2d\left[\frac{1+\left(\lambda_{R1}+\lambda_{M1}\right)\sigma_{01}^2}{1+2\lambda_{M1}\sigma_{01}^2}\right], \tag{3.41}$$

and

$$C_4 = 1 - \frac{\beta^2}{\eta_2} + 2\left(1+\lambda_{M2}\sigma_{02}^2\right)\frac{1+2\lambda_{R2}\sigma_{02}^2}{1+2\lambda_{M2}\sigma_{02}^2}. \tag{3.42}$$

From Equations (3.37) and (3.38), we find that

$$m_1 = \frac{C_4 H_1 - C_2 H_2}{C_1 C_4 - C_2 C_3}, \tag{3.43}$$

and

$$m_2 = \frac{C_1 H_2 - C_3 H_1}{C_1 C_4 - C_2 C_3}. \tag{3.44}$$

Our findings of Equations (3.43), (3.44), (3.35), (3.36), (3.9) and (3.11) are the same as that proposed by Hafezalkotob and Makui [1] with analytic approach.

IV. A Related Inventory Model

In this section, we will discuss a related inventory model studied by Omar et al. [13] to provide a patchwork for their paper. Up to now, there are four papers that have cited Omar et al. [13] in their articles: Wee et al. [14], Wee et al. [15], Chung and Cárdenas-Barrón [16], and Cárdenas-Barrón [17]. Wee et al. [14] and Wee et al. [15] only cited Omar et al. [13] in their introduction without any specific discussion. Cárdenas-Barrón [17] mentioned that the algebraic development of Omar et al. [13] is complex. We remainder the readers that Omar et al. [13] only used one algebraic method of completing a square that will be further discussed in later section. We will demonstrate that the algebraic development of Omar et al. [13] is incomplete but that is not complex. Chung and Cárdenas-Barrón [16] pointed out that Omar et al. [13] is not of all complete among a list of five papers. However, Chung and Cárdenas-Barrón [16] did not provide an explanation what is the shortcoming of Omar et al. [13]. Cárdenas-Barrón [17] claimed the algebraic process of Omar et al. [13] is complicate. We remainder the researchers that Omar et al. [13] applied one skill as complete the prefect square to solve their inventory systems which cannot be classified to a complicate procedure. We will point out that the solution skill of Omar et al. [13] is unfinished and then it is not a complicated approach. Based on above literature review, we will provide a detailed examination to show the doubtful result of Omar et al. [13], and then present the improvements.

V. Recap the Solution Skill

Omar et al. [13] mentioned that to derive the inferior value of $(a/x)+bx$ is $2\sqrt{ab}$ which happens when $x=\sqrt{a/b}$, with the constant parameters, a, and b.

We remind the researchers that Omar et al. [13] neglected the positive sign or the negative sign of a, and b and then their derivation process is doubtful. We will use the next example with crisp coefficient to present a detailed explanation.

Under the condition c and d are both positive numbers, we construct an auxiliary function,

$$\mathrm{p(x)} = \mathrm{dx} + (\mathrm{c/x}), \tag{5.1}$$

with the domain of positive x.
We obtain that

$$\mathrm{p(x)} = \left(\sqrt{\mathrm{dx}} - \sqrt{(\mathrm{c/x})}\right)^2 + 2\sqrt{\mathrm{cd}}. \tag{5.2}$$

Based on the above derivations, we show that the inferior value is $2\sqrt{\mathrm{cd}}$ and the minimum point satisfies

$$\sqrt{\mathrm{dx}} = \sqrt{(\mathrm{c/x})}, \tag{5.3}$$

and then we simplify the above expression to find that

$$\mathrm{x} = \sqrt{(\mathrm{c/d})}. \tag{5.4}$$

We remind the researchers the above example is corresponding to the situation solved by Omar et al. [13].

The rest scenarios are completely neglected by Omar et al. [13]. Consequently, we will provide a patch work for Omar et al. [13] to help research realize the whole scope of the completing square skill.

We will divide into scenario (i) with $\mathrm{c}>0$ and $\mathrm{d}=0$; scenario (ii) with $\mathrm{c}>0$ and $\mathrm{d}<0$; scenario (iii) with $\mathrm{c}=0$, and scenario (iv) with $\mathrm{c}<0$.

With respect to scenario (i) with $\mathrm{c}>0$, and $\mathrm{d}=0$, when x approaches to zero, we know that $\mathrm{p(x)}$ will go to zero to imply the inferior value of $\mathrm{p(x)}$ is zero. Hence, $\mathrm{p(x)}$ cannot attain its minimum value for the domain of positive x.

With respect to scenario (ii) with $\mathrm{c}>0$ and $\mathrm{d}<0$, when x approaches to infinite, we know that $\mathrm{p(x)}$ will go to negative infinite to imply the inferior value of $\mathrm{p(x)}$ is negative infinite. Hence, $\mathrm{p(x)}$ cannot attain its minimum value for the domain of positive x.

With respect to scenario (iii) with $c = 0$, when x approaches to zero, we know that $p(x)$ will go to zero to imply that the inferior value of $p(x)$ is zero. Hence, $p(x)$ cannot attain its minimum value for the domain of positive x.

With respect to scenario (iv) with $c < 0$, when x approaches to zero, we know that $p(x)$ will go to negative infinite to imply the inferior value of $p(x)$ is negative infinite. Hence, $p(x)$ cannot attain its minimum value for the domain of positive x.

Based on our detailed explanations, we go over the main points in the following theorem.

Theorem 1. We consider the minimum value of $p(x) = dx + (c/x)$, with the domain of positive x, such that if c and d are both positive numbers, and then $2\sqrt{cd}$ is the minimum value where $x^* = \sqrt{(c/d)}$ is the minimum point. With respect to the rest scenarios, we proved that the minimum value problem did not have solutions.

Based on our above derivations, we show a patch work for Omar et al. [13] which have been presented by this paper.

VI. A Related Discussion with Logarithm Function

In this section, we will find a lower bound and un upper bound for logarithm function from a geometric point of view without referring to calculus.

We will prove that (a) $\ln(1+x) - x < 0$, for $x > 0$, (b) $\ln(1-x) + x < 0$, for $0 < x < 1$.

First, we recall the definition of a logarithm function, for $x > 0$,

$$\ln x = \int_1^x (1/t)\,dt. \tag{6.1}$$

Based on Equation (6.1), we derive that that $\ln(1+t)$, with $t > 0$ which represents the area of the region, denoted as Γ_1, among four curves: (i) $x = 1$, (ii) $x = 1+t$, (iii) $y = 0$, and (iv) $y = 1/x$.

From $(1/x) \le 1$ for $1 \le x \le 1+t$, we know that Γ_1 is bounded by the rectangular among four line segments: (i) $x = 1$, (ii) $x = 1+t$, (iii) $y = 0$, and (iv) $y = 1$ such that

$$\ln(1+t) < t, \tag{6.2}$$

with $t > 0$ is verified.

Similarly, we consider the second function, $-\ln(1-t)$ with $0 < t < 1$, represents the area of the region, denoted as Γ_2, among four curves: (i) $x = 1-t$,(ii) $x = 1$, (iii) $y = 0$, and (iv) $y = 1/x$.

From $(1/x) \ge 1$ for $1-t \le x \le 1$, we derive that a rectangular among four line segments: (i) $x = 1-t$, (ii) $x = 1$, (iii) $y = 0$, and (iv) $y = 1$ is bounded by Γ_2 such that

$$t < -\ln(1-t) \tag{6.3}$$

with $0 < t < 1$ is proved.

For later discussion, we first review the following definition of the Mean Value Theorem of Integration.

Mean Value Theorem of Integration

For a continuous function $f(x)$, defined on a compact interval, denoted as $a \le x \le b$ with $a < b$, then there is a point c with $a \le c \le b$, satisfying

$$\int_a^b f(t)\,dt = (b-a) f(c). \tag{6.4}$$

Next, we recall two related theorems: (a) the Extreme Value Theorem, and (b) the Intermediate Value Theorem for continuous functions.

The Extreme Value Theorem

For a continuous function $f(x)$, defined on $a \le x \le b$ with $a < b$, then there are two points x_1 and x_2 with $x_1, x_2 \in [a,b]$ such that x_1 is the minimum point to attain the minimum value, m and x_2 is the maximum point to attain the maximum value, M as

$$f(x_1) = m \le f(x) \le M = f(x_2), \tag{6.5}$$

for any $x \in [a,b]$.

The Intermediate Value Theorem

For a continuous function $f(x)$, defined on $a \le x \le b$ with $a < b$, with the minimum value, m and the maximum value, M, given any number L with $m \le L \le M$, then there is a point, say c with $c \in [a,b]$ satisfying

$$f(c) = L. \tag{6.6}$$

If we accept the Extreme Value Theorem and the Intermediate Value Theorem for the moment, then we know that

$$m \le f(t) \le M, \tag{6.7}$$

for $a \le x \le b$. Based on Equation (6.7), we derive that

$$m(b-a) \le \int_a^b f(t)\,dt \le M(b-a). \tag{6.8}$$

According to Equation (6.8), it yields that

$$m \le \int_a^b f(t)\,dt \Big/ (b-a) \le M. \tag{6.9}$$

Recall the Intermediate Value Theorem, we know that there is a point, say c, satisfying

$$\int_a^b f(t)\,dt \Big/ (b-a) = f(c). \tag{6.10}$$

Hence, we derive that

$$\int_a^b f(t)\,dt = (b-a) f(c), \tag{6.11}$$

which is the Mean Value Theorem of Integration.

From the above discussion, how to use algebraic method to prove (a) the Extreme Value Theorem, and (b) the Intermediate Value Theorem for continuous functions will be an interesting question for the future research.

VII. Direction for Further Studies

In this section, we will refer to several recently published articles to help researchers recognize possibly hot topics for their future academic examination. According to cyber attacks, Bhukya et al. [18] considered guard electric automobile control models for incorporating the internet of things. Based on full state conditions and nonlinear models, Liu et al. [19] applied prescribed presentation for adaptive event triggered management. Referring to unstructured natural conditions, Wen et al. [20] studied tea bud recognition under a inconsequential system. Related to capacity mistake and first order autoregressive time possessions, Kurnia et al. [21] found out a pretend unit level system with tiny region prediction. Through seizure prediction, Ge et al. [22] gained multiple mixture concentration processes to temporal and space spectral hierarchy. Under proposal, Yang and Liu [23] derived consistency and arrangement with self supervised hyper graphical transformer. With respect to discontinuous commencement mappings, Wang and Yan [24] developed neural networks to synchronize inertial delay. Owing to diseases and diverse patient environment, Poningsih et al. [25] employed outpatient services facility administration to optimize models. In complex convolution calculation, Bhat et al. [26] examined cubic theta mappings for some Eisenstein identities. To learn swarm intelligent optimization procedures, Shang-Guan et al. [27] constructed drudging procedure for neural network with spongy feeler of multi layer discernment. To realize nonlinear programming performances, Martinez et al. [28] adopted numerical studies and a sixth dimensional boundary value dilemma to obtain analytic answers. For customer elevation net programming, Qi et al. [29] used element fill in blank questions to acquire a blank element choice procedure. We provide the following possible directions for future research: (1) By algebraic method to prove the Mean Value Theorem of Integration, (2) Transform intuitionistic fuzzy sets by crisp value, and (3) Do not apply compute similarity measures and then result in complicate computation, instead, directly compare μ_A and ν_A, from $\mu_A - \Delta$ and $\nu_A - \Omega$ to decide pattern, where Δ and Ω are synthesized results from all alternatives.

VIII. An Improvment for Inventory Models

In this section, we will provide a revision for a relative inventory system. Cárdenas-Barrón [30] applied algebraic method with respect to the Cauchy-Bunyakovsky-Schwarz inequality and the arithmetic-geometric mean inequality to deal with economic ordering quantity and economic production quantity models without referring to differential equations and calculus such that those researchers without background with respect to calculus and differential equations can understand inventory models. In this section, we provide a further simplification of his approach such that more practitioners can realize his important findings and then apply algebraic methods in their research.

IX. Assumptions and Notation

To be compatible with Cárdenas-Barrón [30], we took the same notation as his except two new expressions to convert an economic production quantity model to an economic ordering quantity model.

$d =$ demand rate per time unit,

$A =$ ordering cost per order,

$h =$ per unit holding cost per unit time,

$v =$ per unit backorder cost per unit time,

$p =$ production rate per unit time,

$Q =$ order quantity,

$B =$ backorders level.

The two new expressions proposed by our article:

$A_0 = A(1 - d/p)$,

$Q_0 = Q(1 - d/p)$.

The Cauchy-Bunyakovsky-Schwarz inequality is defined as follows. If two vectors, A and B with

$$A = (a_1, a_2, \dots, a_n), \quad (9.1)$$

with $a_j \ge 0$, for $j = 1,2,\dots,n$, and

$$B = (b_1, b_2, \dots, b_n), \quad (9.2)$$

with $b_j \ge 0$, for $j = 1,2,\dots,n$.

The Cauchy-Bunyakovsky-Schwarz inequality claimed that

$$\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \ge \sum_{j=1}^{n} a_j b_j. \quad (9.3)$$

Moreover, the above inequality becomes equality, if and only if, there is a number, denoted as c, which satisfies

$$\left(a_j / b_j\right) = c, \quad (9.4)$$

for $j = 1,2,\dots,n$.

On the other hand, the arithmetic-geometric mean inequality mentioned that if $a \ge 0$, and $b \ge 0$, then

$$[(a + b)/2] \ge \sqrt{ab}. \quad (9.5)$$

X. Review of Previous Results

For an economic ordering quantity model with backorders, we recall the excellent approach proposed by Cárdenas-Barrón [30]. The total inventory cost, $TC(Q,B)$, is denoted as

$$TC(Q,B) = \frac{Ad}{Q} + \frac{h(Q-B)^2}{2Q} + \frac{vB^2}{2Q}. \quad (10.1)$$

In the following, we provide an outline of his sophistic derivations. For the detailed derivations, please refer to equations (2-11) of Cárdenas-Barrón [30].

Cárdenas-Barrón [30] assume that

$$a_1 = \sqrt{v/(v+h)}, \quad (10.2)$$

$$a_2 = \sqrt{h/(v+h)}, \quad (10.3)$$

$$b_1 = \sqrt{h}(Q - B)/Q, \quad (10.4)$$

and

$$b_2 = \sqrt{v}B/Q. \quad (10.5)$$

It follows that

$$a_1^2 + a_2^2 = 1, \quad (10.6)$$

and

$$b_1^2 + b_2^2 = \frac{h(Q-B)^2}{Q^2} + \frac{vB^2}{Q^2}. \quad (10.7)$$

Cárdenas-Barrón [30] inserted Equations (10.6) and (10.7) in to Equation (10.1) to obtain that

$$TC(Q,B) = (Ad/Q) + \frac{Q}{2}(a_1^2 + a_2^2)(b_1^2 + b_2^2). \quad (10.8)$$

and then he applied the Cauchy-Bunyakovsky-Schwarz inequality to show that

$$TC(Q,B) \ge (Ad/Q)$$

$$+\frac{Q}{2}\left(\frac{\sqrt{hv}}{\sqrt{v+h}}\left(\frac{Q-B}{Q}\right)+\frac{\sqrt{hv}}{\sqrt{v+h}}\left(\frac{B}{Q}\right)\right)^2. \quad (10.9)$$

The results of Equation (10.9) can be further simplified as

$$TC(Q,B) \geq (Ad/Q) + hvQ/2(v+h). \quad (10.10)$$

Moreover, the inequality of Equation (10.10) becomes equality, that is, attaina the minimum, when

$$(b_1/a_1) = (b_2/a_2), \quad (10.11)$$

to yield that the minimum solution occurs at

$$\left[\sqrt{h}(Q-B)/\sqrt{v}Q\right] = \left[\sqrt{v}B/\sqrt{h}Q\right]. \quad (10.12)$$

Hence, Cárdenas-Barrón [30] derived that

$$hQ^* = (h+v)B^*. \quad (10.13)$$

Cárdenas-Barrón [30] applied the arithmetic-geometric mean inequality to Equation (10.10), and then it shows that

$$\frac{Ad}{Q^*} = \frac{hvQ^*}{2(h+v)} \quad (10.14)$$

and

$$TC(Q^*, B^*) = \sqrt{\frac{2Adhv}{h+v}}. \quad (10.15)$$

For an economic production quantity model, the total cost is expressed as

$$TC(Q,B) = \frac{Ad}{Q} + \frac{vB^2}{2Q(1-d/p)}$$

$$+\frac{h(Q(1-d/p)-B)^2}{2Q(1-d/p)}. \quad (10.16)$$

In Cárdenas-Barrón [30], he repeated his approach to derive similar results in equations (13-22) of his paper. The purpose of our revisions is to provide a simplification for his solution procedure of the economic production quantity model.

XI. Our Revisions

We adopt two new notation:

$$A_0 = A(1-d/p), \quad (11.1)$$

and

$$Q_0 = Q(1-d/p), \quad (11.2)$$

then we convert Equation (10.16) as

$$TC(Q,B) = \frac{A_0 d}{Q_0} + \frac{h(Q_0 - B)^2}{2Q_0} + \frac{vB^2}{2Q_0}. \quad (11.3)$$

If we overlook the subscript, then Equation (11.3) is identical to Equation (10.1). Hence, we find the minimum point

$$hQ_0^* = (h+v)B^*, \quad (11.4)$$

$$\frac{A_0 d}{Q_0^*} = \frac{hvQ_0^*}{2(h+v)}, \quad (11.5)$$

and

$$TC(Q_0^*, B^*) = \sqrt{\frac{2A_0 dhv}{h+v}}. \quad (11.6)$$

Our findings of Equations (11.4-11.6) are the same results as equations (20-22) of Cárdenas-Barrón [30].

XII. Conclusion

We provide an alternative process to derive the optimal solution that will motivate further studies to apply non-analytic method to deal with operational research questions. We demonstrate that after a sophisticated arrangement, some complicated systems still can solved by pure algebraic approach that will help practitioners join the examination of supply chain inventory models. On the other hand, we present a patch work for the inventory model splved by an algebraic method by Omar et al. [13]. We pointed out there are four other conditiond which are overlooked by Omar et al. [13], and then we show our improvements. We apply two new expressions to find the relation between economic ordering quantity and economic production quantity models such that the derivation proposed by Cárdenas-Barrón [30] can be directly used to the solution approach for the economic production quantity model. Consequently, his tedious solution approach for the economic production quantity model can be deleted. Our findings will help readers understand the power of Cárdenas-Barrón's algebraic procedure of the economic ordering quantity model.

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