Numerical Simulation of Generalized FitzHugh-Nagumo Equation by Shifted Chebyshev Spectral Collocation Method

Seema Sharma and Nidhi Prabhakar

***Abstract*—The present study focuses on numerical simulation of generalized FitzHugh-Nagumo equation employing shifted Chebyshev spectral collocation method (SCSCM). The collocation method makes use of shifted Chebyshev polynomials for approximation of spatial variable and its derivatives, whereas, Chebyshev-Gauss-Lobatto points are employed for collocation purpose. The approximation of FitzHugh-Nagumo equation gives rise to a system of nonlinear ordinary differential equations (ODEs). The solution of this system of ODEs has been obtained by using Runge-Kutta scheme of order 4. The obtained numerical solutions are shown in graphical and tabular form. The convergence of SCSCM has been demonstrated as well. Further, to verify the efficiency and accuracy of the present method, the absolute, maximum absolute, root mean square and relative errors have been calculated for some examples of FitzHugh-Nagumo equation. The comparison of present numerical solutions with exact and approximate solutions obtained by other methods reveals that present method provides better accuracy.**

***Index Terms*—FitzHugh-Nagumo equation, Shifted Chebyshev polynomials, Runge-Kutta scheme, Partial differential equation.**

I. Introduction

THE study of nonlinear partial differential equations (PDEs) has gained wide popularity among researchers as they are used to model complex problems that arise in several scientific disciplines, particularly in chemical physics, plasma wave and fluid mechanics ([1], [2], [3], [4], [5]). It is sometimes difficult to obtain the exact solutions of these models. Thus, various numerical methods ([6], [7], [8], [9], [10]) have been established for obtaining numerical solutions of nonlinear PDEs, to mention a few.

Consider a class of one dimensional nonlinear PDE [11]

$$y_t + \alpha(t)y_x - \beta(t)y_{xx} + \gamma(t)(\xi(y))_x - \psi(t)\zeta(y) = 0, \quad x \in [a, b] \text{and } t \in [0, T], \tag{1}$$

where, $y = y(x, t)$ denotes the unknown function, $\xi(y)$ and $\zeta(y)$ are nonlinear functions, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\psi(t)$ are functions of t, and a and b are arbitrary constants.

Equation (1) is known as nonlinear FitzHugh-Nagumo (F-N) equation, if $\xi(y) = 0$ and $\zeta(y) = y(y - \rho)(1 - y)$. Thus, a generalized F-N equation with linear dispersion term and time dependent coefficients is defined as

$$y_t + \alpha(t)y_x - \beta(t)y_{xx} - \psi(t)y(y - \rho)(1 - y) = 0, \tag{2}$$

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subject to initial condition

$$y(x, 0) = g(x), \qquad x \in [a, b], \tag{3}$$

and Dirichlet boundary conditions

$$\left.\begin{aligned} y(a, t) = f_1(t) \\ y(b, t) = f_2(t) \end{aligned}\right\}, \qquad t \in [0, T], \tag{4}$$

where, ρ is constant.

F-N equation, which was derived by FitzHugh [12] and Nagumo et al. [13], includes diffusion and nonlinearity due to the term $y(y - \rho)(1 - y)$. This equation is an essential nonlinear reaction-diffusion equation that is frequently applied in the field of population genetics and to represent the propagation of nerve impulses ([14], [15]). F-N equation has been solved and analyzed by different physicists as well as mathematicians because of its significance in mathematical physics. Kawahara and Tanaka [16] employed formal perturbation method and obtained exact solutions of F-N equation. Nucci and Clarkson [17] have used nonclassical method to obtain some new exact solutions of F-N equation. Wazwaz and Gorguis [18] obtained exact solution of Fisher type equation employing adomian decomposition method. Wazwaz [19] solved nonlinear parabolic equations using tanh-coth method and obtained its solitons and kink solutions. Abbasbandy [20] employed homotopy analysis method for soliton solutions of F-N equation. Hariharan and Kannan [21] obtained the numerical solution of F-N equation utilizing Haar wavelet approach. Triki and Wazwaz [22] derived soliton solutions for generalized F-N equation using tanh method and solitary wave ansatz. Bhrawy [23] obtained numerical solutions of generalized F-N equation using Jacobi-Gauss-Lobatto collocation (J-GL-C) method. Jiwari et al. [24] investigated the numerical solutions of F-N equation implementing partial differential quadrature method (PDQM). Exact solutions of fractional F-N equation were obtained by Cevikel et al. [25] using conformable derivatives. Chin [26] designed a scheme based on non-standard finite difference scheme and Galerkin method to analyze the effect on the solution of F-N equation by external parameter.

In the past few decades, the spectral collocation method has achieved substantial popularity in solving different nonlinear PDEs numerically ([27], [28], [29], [30], [31]). This method has been chosen because it has a higher convergence rate ([32], [33], [34], [35]) and offers excellent accuracy even with small number of collocation points. In the polynomials family, Chebyshev polynomials are one of the important orthogonal polynomials. The formulation of Chebyshev polynomials based spectral collocation method requires less human effort and maintains good accuracy for numerical solutions due to which Chebyshev spectral collocation method has gained popularity among researchers. Various analysts used this method to solve different kind of linear and nonlinear differential equations. Linear and nonlinear PDEs are solved using Chebyshev finite difference approximation by Elbarbary and El-Kady [36]. Khater et al. [37] solved Burgers' type equations by implementing Chebyshev spectral collocation method. Zarebnia and Jalili [38] obtained the solution of Fokker Planck equation using Chebyshev spectral collocation method. Sharma et al. [39] solved Lane-Emden equation employing Chebyshev operational matrix method. Jaiswal et al. [40] applied shifted Chebyshev polynomials to estimate approximate solution of Burgers equation and Huxley equation. Ahmad et al. [41] applied local meshless method with explicit Euler method (EEM) with Runge-Kutta scheme of order 4 to solve time dependent PDEs. Agbavon and Appadu [42] solved FitzHugh-Nagumo equation using four versions of nonstandard finite difference (NSFD) scheme. Ramos et al. [43] solved distinct nonlinear PDEs by employing hybrid block method together with modified cubic B-spline method named as OHBCBM. Wang et al. [44] solved Emden-Fowler equation using Picard iteration and Chebyshev collocation method. Raji et al. [45] applied Chebyshev collocation method for solving eighth order BVPs.

Although numerous work has been done for solving F-N equation but to the best of author's knowledge, generalized F-N equation has not yet been simulated using SCSCM. The objective of present paper is to employ spectral collocation method in combination with shifted Chebyshev polynomials and Runge-Kutta scheme of order 4 to solve generalized F-N equation. In solution procedure, shifted Chebyshev polynomials have been used for approximating the unknown function and its space derivatives. The approximation of F-N equation using SCSCM reduces the equation into a system of nonlinear ODEs. The efficiency and convergence of a collocation method depend upon the selection of collocation points. Here, Chebyshev-Gauss-Lobatto points are chosen as the collocation points. Further, Runge-Kutta scheme of order 4 has been employed to solve obtained system of nonlinear ODEs. To verify the accuracy and efficiency of the present method, four examples of F-N equation have been solved. The obtained numerical results are shown in graphical and tabular form. It is revealed that present results are in excellent agreement with exact solutions. Moreover, they provide very good accuracy in comparison to existing results obtained by other methods.

II. Preliminaries of Chebyshev polynomials

This section presents some basic and important properties of Chebyshev polynomials.

A. Chebyshev polynomials of first kind

The Chebyshev polynomials are given by

$$T_m(\phi) = cos(mcos^{-1}\phi), \quad \phi \in [-1, 1]. \tag{5}$$

The Chebyshev polynomials can be constructed using the recurrence relation

$$T_m(\phi) = 2\phi T_{m-1}(\phi) - T_{m-2}(\phi), \quad m = 2, 3, \ldots; \tag{6}$$

where, $T_0(\phi) = 1, T_1(\phi) = \phi$.

The Chebyshev polynomials are orthogonal and their inner products are given as

$$\langle T_m(\phi), T_n(\phi)\rangle = \int_{-1}^{1} w(\phi) T_m(\phi) T_n(\phi) d\phi = \begin{cases} \pi, & m = n = 0, \\ \pi/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases} \tag{7}$$

where, $w(\phi) = \frac{1}{\sqrt{1-\phi^2}}$ is the weight function.

B. Shifted Chebyshev polynomials of first kind

By using the transformation

$$\phi = \frac{2x - (a + b)}{b - a}, \tag{8}$$

in expression (5), the Chebyshev polynomials $T_m(\phi)$ reduces to yield shifted Chebyshev polynomials $T^*_m(x)$ with applicability range $[a, b]$ expressed as

$$T^*_m(x) = T_m\left(\frac{2x - (a + b)}{b - a}\right). \tag{9}$$

The shifted Chebyshev polynomials can be constructed using the recurrence relation

$$T^*_m(x) = 2\left(\frac{2x - (a + b)}{b - a}\right) T^*_{m-1}(x) - T^*_{m-2}(x), \quad m = 2, 3, \ldots \tag{10}$$

with $T^*_0(x) = 1, T^*_1(x) = \frac{2x-(a+b)}{b-a}$.

The shifted Chebyshev polynomials are orthogonal and their inner products are given as

$$\langle T^*_m(x), T^*_n(x)\rangle = \int_{a}^{b} w(x) T^*_m(x) T^*_n(x) dx = \begin{cases} \pi, & m = n = 0, \\ \pi/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases} \tag{11}$$

where, $w(x) = \frac{1}{\sqrt{(x-a)(b-x)}}$ is the weight function.

The first order derivatives of shifted Chebyshev polynomials are given by

$$T^{*\prime}_m(x) = 2m\beta \sum_{j=0,\ (j+m)\ odd}^{m-1} d_j T^*_j(x), \tag{12}$$

where, $\beta = \frac{2}{b-a}$ and

$$d_j = \begin{cases} 1, & 1 \leq j \leq N - 1 \\ \frac{1}{2}, & j = 0, N \end{cases}. \tag{13}$$

Let function $u(x) \in L^2[a, b]$ be expressed in the form of an infinite series of shifted Chebyshev polynomials $T^*_m(x)$ [31], i.e.

$$u(x) = {\sum_{m=0}^{\infty}}' c_m T^*_m(x). \tag{14}$$

Here, prime denotes that the first term in summation is halved. The unknown coefficients c_m can be obtained using the relation

$$c_m = \frac{2}{\pi}\int_{a}^{b} w(x) u(x) T^*_m(x) dx, \quad m = 0, 1, 2, \ldots. \tag{15}$$

III. Solution of generalized FitzHugh-Nagumo equation by SCSCM

Consider the F-N equation

$$y_t + \alpha(t)y_x - \beta(t)y_{xx} - \psi(t)y(y-\rho)(1-y) = 0, \quad (16)$$

subject to initial condition

$$y(x,0) = g(x), \qquad x \in [a,b], \quad (17)$$

and Dirichlet boundary conditions

$$\left.\begin{aligned} y(a,t) &= f_1(t) \\ y(b,t) &= f_2(t) \end{aligned}\right\}, \qquad t \in [0,T]. \quad (18)$$

According to shifted Chebyshev spectral collocation method [37], the approximation of unknown function $y(x,t)$ by considering finite number of terms in series (14) gives

$$y(x,t) = \sum_{m=0}^{N}{}'' c_m T_m^*(x), \quad (19)$$

where, double prime in summation denotes that first and last terms are halved. Here, the unknown coefficients c_m are defined by

$$c_m = \frac{2}{N}\sum_{k=0}^{N}{}'' T_m^*(x_k)y(x_k,t), \quad (20)$$

which is an approximation to integral (15). This approximation of c_m yields excellent solutions, even for small N, when it is not easy to calculate c_m by integral (15).

Further, Chebyshev-Gauss-Lobatto points have been chosen as the collocation points x_k which are given as

$$x_k = \frac{1}{2}\left((a+b) + (a-b)cos\left(\frac{\pi k}{N}\right)\right), \quad k = 0,1,\ldots,N. \quad (21)$$

The derivative $y_x(x,t)$ can be obtained by differentiating equation (19) as

$$\begin{aligned} y_x(x,t) &= \sum_{m=0}^{N}{}'' c_m T_m^{*\prime}(x) \\ &= \sum_{k=0}^{N}{}'' \left(\frac{2}{N}\sum_{m=0}^{N}{}'' T_m^{*\prime}(x)T_m^*(x_k)\right) y(x_k,t). \end{aligned}$$

Now, the derivative at collocation point x_j is given by

$$\begin{aligned} y_x(x_j,t) &= \sum_{k=0}^{N}{}'' \left(\frac{2}{N}\sum_{m=0}^{N}{}'' T_m^{*\prime}(x_j)T_m^*(x_k)\right) y(x_k,t) \\ &= \sum_{k=0}^{N}[P_x]_{jk}y(x_k,t), \end{aligned} \quad (22)$$

where,

$$[P_x]_{jk} = \frac{2d_j}{N}\sum_{m=0}^{N}{}'' T_m^{*\prime}(x_j)T_m^*(x_k), \quad j,k = 0,1,\ldots,N,$$

and $T_m^{*\prime}(x_j)$ and d_j are given by equation (12) and (13), respectively.

Similarly, the second order derivative $y_{xx}(x_j,t)$ at collocation point x_j can be approximated by differentiating equation (22) as

$$\begin{aligned} y_{xx}(x_j,t) &= \sum_{k=0}^{N}[P_x]_{jk}y_x(x_k,t) \\ &= \sum_{k=0}^{N}[P_x]_{jk}\left(\sum_{l=0}^{N}[P_x]_{kl}y(x_l,t)\right) \\ &= \sum_{l=0}^{N}\left(\sum_{k=0}^{N}[P_x]_{jk}[P_x]_{kl}\right)y(x_l,t) \\ &= \sum_{l=0}^{N}[Q_x]_{jl}y(x_l,t), \end{aligned} \quad (23)$$

where, $[Q_x]_{jl} = \sum_{k=0}^{N}[P_x]_{jk}[P_x]_{kl}, \ \ j,l = 0,1,\ldots,N.$

Using boundary conditions (18), the expressions (22) and (23) can be rewritten as

$$y_x(x_j,t) = D_j(t) + \sum_{l=1}^{N-1}[P_x]_{jl}y(x_l,t), \quad (24)$$

$$y_{xx}(x_j,t) = D_j^*(t) + \sum_{l=1}^{N-1}[Q_x]_{jl}y(x_l,t), \quad (25)$$

where,

$$D_j(t) = [P_x]_{j0}f_1(t) + [P_x]_{jN}f_2(t),$$
$$D_j^*(t) = [Q_x]_{j0}f_1(t) + [Q_x]_{jN}f_2(t).$$

Now, discretizing the equation (16) at internal collocation points $x_j; j = 1,2,\ldots,N-1$, it becomes

$$\begin{aligned} y_t(x_j,t) + \alpha(t)y_x(x_j,t) - \beta(t)y_{xx}(x_j,t) - \psi(t)y(x_j,t) \\ (y(x_j,t)-\rho)(1-y(x_j,t)) = 0, \\ j = 1,2,\ldots,N-1. \end{aligned} \quad (26)$$

Substituting expression (24) and (25) into equation (26) and representing $y(x_j,t)$ and $y_t(x_j,t)$ by $y_j(t)$ and $\dot{y}_j(t)$ respectively, leads to

$$\begin{aligned} \dot{y}_j(t) + \alpha(t)\sum_{l=1}^{N-1}[P_x]_{jl}y_l(t) - \beta(t)\sum_{l=1}^{N-1}[Q_x]_{jl}y_l(t) \\ +\alpha(t)D_j(t) - \beta(t)D_j^*(t) - \psi(t)y_j(t) \\ (y_j(t)-\rho)(1-y_j(t)) = 0, \\ j = 1,2,\ldots,N-1, \end{aligned} \quad (27)$$

along with the initial conditions

$$y_j(0) = g(x_j), \quad j = 1,2,\ldots,N-1. \quad (28)$$

The system of ODEs (27) and initial conditions (28) can be expressed as

$$\left.\begin{aligned} \dot{y}(t) &= F(t,y(t)), \\ \text{and } \ y(0) &= y_0, \end{aligned}\right\} \quad (29)$$

where,

$$y(t) = [y_1(t), y_2(t), \ldots, y_{N-1}(t)]^T,$$

$$\dot{y}(t) = [\dot{y}_1(t), \dot{y}_2(t), \dots, \dot{y}_{N-1}(t)]^T,$$
$$y_0 = [y_1(0), y_2(0), \dots, y_{N-1}(0)]^T,$$
$$F(t, y(t)) = [F_1(t, y(t)), F_2(t, y(t)), \dots, F_{N-1}(t, y(t))]^T.$$

and

$$F_j(t, y(t)) = -\alpha(t) \sum_{l=1}^{N-1} [P_x]_{jl} y_l(t) + \beta(t) \sum_{l=1}^{N-1} [Q_x]_{jl} y_l(t)$$
$$-\alpha(t) D_j(t) + \beta(t) D_j^*(t) + \psi(t) y_j(t)(y_j(t) - \rho)(1 - y_j(t)),$$
$$j = 1, 2, \dots, N-1.$$

The system of equations (29) is a system of simultaneous ODEs of first order. The solution of this system $y(t_{i+1})$ at $(i+1)^{th}$ time level, can be obtained using fourth order Runge-Kutta scheme, when the solution $y(t_i)$ at i^{th} time level is known, for $i = 0, 1, 2, \dots$. Runge-Kutta scheme is an explicit scheme, which provides very accurate solutions. The solution $y(t_{i+1})$ of system of ODEs (29) employing fourth order Runge-Kutta scheme is given as

$$y(t_{i+1}) = y(t_i) + \frac{\Delta t}{6} [F(t_i, y(t_i)) + 2F\left(t_i + \frac{\Delta t}{2}, y^{(1)}\right) + 2F\left(t_i + \frac{\Delta t}{2}, y^{(2)}\right) + F(t_i + \Delta t, y^{(3)})], \tag{30}$$

where,

$$y^{(1)} = y(t_i) + \frac{1}{2}\Delta t F(t_i, y(t_i)),$$
$$y^{(2)} = y(t_i) + \frac{1}{2}\Delta t F\left(t_i + \frac{\Delta t}{2}, y^{(1)}\right),$$
$$\text{and } y^{(3)} = y(t_i) + \Delta t F\left(t_i + \frac{\Delta t}{2}, y^{(2)}\right).$$

IV. Error and convergence analysis

In order to examine the convergence of SCSCM, the following convergence theorems are discussed.

Theorem 1 The polynomials $2^{-(2m-1)} T_m^*(x)$ have the smallest norm among all m^{th} degree monic polynomials defined on the interval $[a, b]$ i.e.,

$$\|2^{-(2m-1)} T_m^*(x)\| = 2^{-(2m-1)}$$

Proof: It can be proved following Chebyshev's theorem (See ref. [35]).

Theorem 2 If $y(x) \in L^2[a, b]$ is approximated in the form of a series of shifted Chebyshev polynomials. Then this series is strongly convergent.

Proof: Let $y(x)$ be approximated in the form of a series of shifted Chebyshev polynomials $T_m^*(x)$ as

$$y(x) = \sum_{m=0}^{\infty} c_m T_m^*(x), \tag{31}$$

where,

$$T_m^*(x) = a_m ((x-a)(b-x))^{\frac{1}{2}} \frac{d^m}{dx^m} ((x-a)(b-x))^{m-\frac{1}{2}}. \tag{32}$$

The polynomials $T_m^*(x)$ are orthogonal w.r.t. the weight function

$$w(x) = ((x-a)(b-x))^{-\frac{1}{2}}.$$

In equation (31), the coefficients c_m are given by

$$c_m = \frac{\int_a^b ((x-a)(b-x))^{-\frac{1}{2}} y(x) T_m^*(x) dx}{\int_a^b ((x-a)(b-x))^{-\frac{1}{2}} T_m^*(x) T_m^*(x) dx}, \tag{33}$$

Substituting value of $T_m^*(x)$ from equation (32) and integrating the numerator and denominator, the coefficient

$$c_m = \frac{\int_a^b ((x-a)(b-x))^{m-\frac{1}{2}} y^{(m)}(x) dx}{m! \beta_m \int_a^b ((x-a)(b-x))^{m-\frac{1}{2}} dx}$$
$$= \frac{\int_a^b w^m(x) y^{(m)}(x) dx}{m! \beta_m \int_a^b w^m(x) dx}, \tag{34}$$

where, $\beta_m = 2^{2m-1}$.

The value of coefficient c_m is not more than a weighted mean with non-negative weight function, therefore

$$c_m = \frac{y^{(m)}(\varphi)}{m! \beta_m}, \qquad (a \le \varphi \le b) \tag{35}$$

Now, writing equation (31) as

$$y(x) = \sum_{m=0}^{N-1} c_m T_m^*(x) + E_N, \tag{36}$$

where,

$$E_N = \sum_{m=N}^{\infty} c_m T_m^*(x). \tag{37}$$

Now, $\sup_{a \le x \le b} |T_m^*(x)| = 1$. Therefore, using Chebyshev truncation theorem, the bound on error

$$|E_N| \le \sum_{m=N}^{\infty} |c_m| \approx |c_N|. \tag{38}$$

Now, substitution of equation (35) in equation (38) yields

$$|E_N| \le \left| \frac{y^{(N)}(\varphi)}{(N)! \beta_N} \right| = \left| \frac{y^{(N)}(\varphi)}{(N)! 2^{2N-1}} \right|, \tag{39}$$

which shows that, $|E_N| \to 0$ as $N \to \infty$. Therefore, the accuracy in approximation using shifted Chebyshev polynomials gets improved as the value of N is increased. This shows that the series for $y(x)$ is strongly convergent.

V. Numerical examples

To examine the accuracy and efficiency of the SCSCM with Runge-Kutta scheme of order 4 for finding numerical solutions of generalized F-N equation, four examples are considered and solved. The computational work has been carried out using MATLAB software. The obtained approximate results have been compared with approximate results obtained by other numerical methods and exact results. To examine the accuracy of SCSCM, different error norms and order of convergence (R) have been computed using following expressions:

$$L_\infty = \max_i |y_i^* - y_i|,$$

$$L_2 = \sqrt{\left(\sum_{i=1}^{N} |y_i^* - y_i|^2 \right)},$$

$$\text{RMS} = \sqrt{\left(\left(\sum_{i=1}^{N} |y_i^* - y_i|^2\right)/N\right)},$$

$$\text{R} = \frac{log\left[\frac{L_\infty(m_2)}{L_\infty(m_1)}\right]}{log\left(\frac{m_1}{m_2}\right)},$$

where, y_i^* and y_i represent exact and numerical solutions respectively; m_1 and m_2 are successive approximations of solutions.

Example 1 Consider standard F-N equation with $\alpha(t) = 0$ and $\beta(t) = \psi(t) = 1$ ([23], [24]), i.e.

$$y_t = y_{xx} + y(y-\rho)(1-y),$$

with boundary conditions

$$y(a,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{a}{2\sqrt{2}} - \frac{2\rho-1}{4}t\right)\right),$$

$$y(b,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{b}{2\sqrt{2}} - \frac{2\rho-1}{4}t\right)\right),$$

and initial condition

$$y(x,0) = \frac{1}{2}\left(1 + \tanh\left(\frac{x}{2\sqrt{2}}\right)\right).$$

The exact solution is given by

$$y(x,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{x}{2\sqrt{2}} - \frac{2\rho-1}{4}t\right)\right).$$

The numerical solutions are obtained using SCSCM taking $\triangle t$ = 0.001 and are given in tabular and graphical form for two domains (i) $[a, b]$ = [-10,10] and (ii) $[a, b]$ = [-1,1]. The error norms and order of convergence in numerical results for ρ = 0.75 and t = 1.0 are presented in Table I for domain [-10,10] and Table II for domain [-1,1]. It is observed that the results improve by increasing the number of collocation points N, since L_∞ error norm decreases and order of convergence increases by increasing N. For domain [-10,10], the errors reduce to the order of 10^{-7} for N = 30, while, for domain [-1,1], they reduce to the order of 10^{-9} for N = 8. In Table III, the comparison of error norms in numerical results by present method and existing results by PDQM [24] at different time levels t for $[a, b]$ = [-10,10] and ρ = 0.75 are depicted. The results are also compared with those given by EEM [41] and OHBCBM [43] and are presented in Table IV. Table V displays the comparison of L_∞ error norms in present numerical results and results obtained by J-GL-C method [23] for $[a, b]$ = [-1,1] and ρ = 0.75. From tables III, IV and V, it is revealed that the present method provides lesser error in numerical results as compared to PDQM, EEM, OHBCBM and J-GL-C method. The numerical and exact solutions for $[a, b]$ = [-10,10] have been depicted in figures 1 and 2 for ρ = 0.75 and ρ = -2.0, respectively. The graphs of numerical and exact solutions are almost same which demonstrates the accuracy of the present method.

TABLE I: Error norms and order of convergence in numerical results of example 1 in the domain [-10, 10] for t = 1.0 and ρ = 0.75

N	L_2	RMS	L_∞	Order of Convergence R
5	1.39e-02	6.22e-03	9.01e-03	-
10	5.62e-03	1.78e-03	3.68e-03	1.29
15	9.35e-04	2.41e-04	7.76e-04	3.84
20	9.57e-05	2.14e-05	5.90e-05	8.96
25	1.02e-05	2.04e-06	6.67e-06	9.76
30	9.89e-07	1.81e-07	5.61e-07	13.58

TABLE II: Error norms and order of convergence in numerical results of example 1 in the domain [-1, 1] for t = 1.0 and ρ = 0.75

N	L_2	RMS	L_∞	Order of Convergence R
4	2.87e-05	1.44e-05	2.13e-05	-
6	2.16e-07	8.82e-08	1.70e-07	11.91
8	2.01e-09	7.09e-10	1.47e-09	16.52

TABLE III: Comparison of error norms in numerical results of example 1 in the domain [-10, 10] for ρ = 0.75 and N = 30

t	L_∞		RMS		L_2	
	PDQM [24]	Present method	PDQM [24]	Present method	PDQM [24]	Present method
0.2	4.74e-05	3.03e-07	1.59e-05	1.38e-07	2.30e-06	7.55e-07
0.5	1.23e-04	4.54e-07	3.84e-05	1.73e-07	5.57e-06	9.46e-07
1.0	2.63e-04	5.61e-07	8.19e-05	1.81e-07	1.19e-05	9.89e-07
1.5	4.21e-04	6.51e-07	1.34e-04	1.84e-07	1.94e-05	1.01e-06
2.0	6.00e-04	6.03e-07	1.94e-04	1.83e-07	2.82e-05	1.00e-06
3.0	1.03e-03	4.31e-07	3.43e-04	1.70e-07	4.97e-05	9.33e-07
5.0	2.30e-03	5.15e-07	7.86e-04	1.58e-07	1.14e-04	8.66e-07

TABLE IV: Comparison of error norms in numerical results of example 1 in the domain [-10, 10] for ρ = 0.75 and N = 30 by present method with EEM and OHBCBM

t	L_∞			RMS		
	EEM [41]	OHBCBM [43]	Present method	EEM [41]	OHBCBM [43]	Pesent method
0.2	1.89e-05	1.89e-05	3.03e-07	2.20e-07	7.46e-06	1.38e-07
0.5	4.15e-05	4.15e-05	4.54e-07	1.57e-06	1.64e-05	1.73e-07
1.0	6.99e-05	6.97e-05	5.61e-07	7.14e-06	2.74e-05	1.81e-07
1.5	9.17e-05	9.12e-05	6.51e-07	1.73e-05	3.53e-05	1.84e-07
2.0	1.10e-04	1.08e-04	6.03e-07	3.19e-05	4.13e-05	1.83e-07
3.0	1.39e-04	1.36e-04	4.31e-07	7.29e-05	4.97e-05	1.70e-07
5.0	1.90e-04	1.80e-04	5.15e-07	1.88e-04	6.13e-05	1.58e-07

TABLE V: Comparison of L_∞ error norms in numerical results of example 1 in the domain [-1, 1] for ρ = 0.75 and t = 1.0

N	J-GL-C method [23]	Present Method
4	5.22e-05	2.13e-05
8	4.31e-07	1.47e-09
12	2.76e-08	5.74e-11

Example 2 Consider F-N equation, with $\alpha(t) = 0$ and $\beta(t) = \psi(t) = 1$ ([16]), i.e.

$$y_t = y_{xx} + y(y-\rho)(1-y),$$

with initial condition

$$y(x,0) = \frac{1}{2}\left(1+\rho+(1-\rho)\tanh\left(\pm\frac{(1-\rho)x}{2\sqrt{2}}\right)\right),$$

and boundary conditions

$$y(-10,t) = y(10,t) = \frac{1}{2}(1+\rho+(1-\rho)\tanh(\pm\frac{(1-\rho)5}{\sqrt{2}} + \frac{(1-\rho^2)}{4}t)).$$

The exact solution is

$$y(x,t) = \frac{1}{2}(1+\rho+(1-\rho)\tanh(\pm\frac{(1-\rho)x}{2\sqrt{2}} + \frac{(1-\rho^2)}{4}t)).$$

The numerical solutions have been obtained employing SCSCM taking $\triangle t$ = 0.001 and are shown in tabular and graphical form. The order of convergence and error norms in approximate solutions at t = 1.0 for different number of collocation points N have been shown in tables VI and VII for ρ = 0.75 and ρ = 0.5, respectively. It is clear from tables VI and VII that accuracy in numerical solutions improves as value of N is increased. The present method is reliable and accurate, because the error norms for N = 20 are reduced to the order of 10^{-14} and 10^{-9} for ρ = 0.75 and ρ = 0.5, respectively and order of convergence increases by increasing the value of N. Table VIII depicts the error norms in numerical solutions for several values of t taking N = 20 and ρ = 0.75. Further, numerical and exact solutions for ρ = 0.75 and ρ = 0.5 are depicted in figures 3 and 4, respectively. A good agreement of numerical and exact solutions is revealed.

TABLE VI: Error norms and order of convergence in numerical solutions of example 2 at t = 1.0 and ρ = 0.75

N	L_2	RMS	L_∞	Order of Convergence$R$
5	1.43e-05	6.38e-06	1.02e-05	-
10	5.88e-08	1.86e-08	4.24e-08	7.91
15	6.18e-11	1.59e-11	3.23e-11	17.71
20	1.05e-13	2.35e-14	8.50e-14	20.64

TABLE VII: Error norms and order of convergence in numerical solutions of example 2 for t = 1.0 and ρ = 0.5

N	L_2	RMS	L_∞	Order of Convergence$R$
5	1.08e-03	4.81e-04	8.63e-04	-
10	4.36e-05	1.38e-05	2.20e-05	5.29
15	8.34e-07	2.15e-07	4.30e-07	9.70
20	1.27e-08	2.85e-09	6.61e-09	14.52

TABLE VIII: Error norms in numerical solutions of example 2 taking N = 20 and ρ = 0.75

t	L_∞	RMS	L_2
0.2	8.76e-14	2.61e-14	1.17e-13
0.5	9.15e-14	2.66e-14	1.19e-13
1.0	8.50e-14	2.35e-14	1.05e-13
1.5	6.84e-14	1.96e-14	8.76e-14
2.0	5.01e-14	1.66e-14	7.45e-14
3.0	3.76e-14	1.33e-14	5.96e-14
5.0	6.84e-14	1.80e-14	8.06e-14

Example 3 Consider the generalized nonlinear F-N equation with $\alpha(t) = \beta(t) = \cos(t)$ and $\psi(t)$ = 2cos(t) ([11], [23], [24]), i.e.

$$y_t + cos(t)y_x - cos(t)y_{xx} - 2cos(t)y(y-\rho)(1-y) = 0,$$

with boundary conditions

$$y(a,t) = \frac{\rho}{2}\left(1+\tanh\left(\frac{\rho a}{2} - \frac{(3\rho-\rho^2)\sin t}{2}\right)\right),$$

$$y(b,t) = \frac{\rho}{2}\left(1+\tanh\left(\frac{\rho b}{2} - \frac{(3\rho-\rho^2)\sin t}{2}\right)\right),$$

and initial condition

$$y(x,0) = \frac{\rho}{2}\left(1+\tanh\left(\frac{\rho x}{2}\right)\right).$$

The exact solution is

$$y(x,t) = \frac{\rho}{2}\left(1+\tanh\left(\frac{\rho x}{2} - \frac{(3\rho-\rho^2)\sin t}{2}\right)\right).$$

The numerical solutions are obtained for two domains (i) $[a,b]$ = [-10,10] and (ii) $[a,b]$ = [-1,1] taking $\triangle t$ = 0.001 and are demonstrated in tabular and graphical form. Tables IX and X present the order of convergence and error norms for t = 1.0 and ρ = 0.75, respectively, for $[a,b]$ = [-10,10] and $[a,b]$ = [-1,1]. It is clear that SCSCM provides better and better accuracy if number of collocation points is increased. SCSCM performs very well as order of convergence increases with an increase in the value of N. For domain [-10, 10], the errors reduce to the order of 10^{-8} for N = 40, while, for domain [-1,1], they reduce to the order of 10^{-9} for N = 8. Table XI shows the comparative study of error norms in numerical results given by present method and PDQM [24] for several values of t and $[a,b]$ = [-10,10] taking ρ = 0.75 and N = 40. From Table XI, it is revealed that present method yields more accurate results than PDQM. Table XII presents the comparison of L_∞ and L_2 error norms for present method and improved pseudospectral method [11] for $[a,b]$ = [-10,10] and reveals that errors provided by present method are less than those given by improved pseudospectral method. Table XIII demonstrates the comparison of L_∞ error norms in numerical results computed by present method and J-GL-C method [23] for $[a,b]$ = [-1,1]. It is revealed that present method yields more accurate results than J-GL-C method. The numerical and exact solutions for $[a,b]$ = [-10,10] are depicted in figures 5 and 6 for ρ = 0.5 and ρ = 0.75, respectively. The numerical results are in quite good agreement with exact results.

TABLE IX: Error norms and order of convergence in numerical results of example 3 in the domain [-10, 10] for ρ = 0.75 and t = 1.0

N	L_2	RMS	L_∞	Order of Convergence$R$
5	2.62e-02	1.17e-02	1.80e-02	-
10	1.04e-02	3.29e-03	8.52e-03	1.08
15	1.37e-03	3.55e-04	8.81e-04	5.60
20	1.43e-04	3.19e-05	6.93e-05	8.83
30	1.38e-06	2.50e-07	4.96e-07	12.19
40	3.59e-08	5.67e-09	1.44e-08	12.29

TABLE X: Error norms and order of convergence in numerical results of example 3 in the domain [-1, 1] for ρ = 0.75 and t = 1.0

N	L_2	RMS	L_∞	Order of Convergence$R$
4	9.39e-05	4.69e-05	6.83e-05	-
6	1.06e-06	4.31e-07	6.93e-07	11.32
8	7.66e-09	2.71e-09	5.57e-09	16.77

TABLE XI: Comparison of error norms in numerical results of example 3 in the domain [-10, 10] for ρ = 0.75 and N = 40

t	L_∞		RMS		L_2	
	PDQM [24]	Present method	PDQM [24]	Present method	PDQM [24]	Present method
0.2	1.24e-05	1.21e-08	4.57e-05	5.15e-09	1.31e-06	3.26e-08
0.5	5.20e-04	1.69e-08	5.64e-05	6.60e-09	6.10e-06	4.18e-08
1.0	6.33e-04	1.44e-08	8.17e-05	5.67e-09	2.12e-05	3.59e-08
1.5	8.54e-04	7.12e-09	2.37e-04	3.20e-09	3.23e-05	2.02e-08

TABLE XII: Comparison of L_∞ and L_2 error norms in numerical results of example 3 in the domain [-10, 10] for ρ = 0.75 and N = 45

t	Present method		Pseudospectral method [11]	
	L_∞	L_2	L_∞	L_2
0.2	2.38e-09	6.05e-09	7.48e-07	7.55e-07
0.5	1.96e-09	5.078e-09	5.58e-07	5.49e-07
1.0	1.75e-09	4.63e-09	4.90e-07	8.69e-06
1.5	8.61e-10	2.48e-09	9.24e-06	7.28e-06

TABLE XIII: Comparison of L_∞ error norms in numerical results of example 3 in the domain [-1,1] for ρ = 1 and t = 1.0

N	J-GL-C method [23]	Present Method
4	4.48e-04	6.83e-05
8	3.20e-07	5.57e-09
12	2.05e-08	1.10e-09

Example 4 Consider the stiff case of standard F-N equation with $\alpha(t) = 0$ and $\beta(t) = 1$ ([42]), i.e.

$$y_t = y_{xx} + \psi y(y - \rho)(1 - y),$$

with initial condition

$$y(x,0) = \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{\psi}}{2\sqrt{2}}x\right)\right),$$

and boundary conditions

$$y(-10,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{\psi}}{2\sqrt{2}}(-10 - ct)\right)\right),$$

$$y(10,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{\psi}}{2\sqrt{2}}(10 - ct)\right)\right).$$

The exact solution is

$$y(x,t) = \frac{1}{2}\left(1 - \tanh\left(\frac{\sqrt{\psi}}{2\sqrt{2}}(x - ct)\right)\right),$$

where, $c = -\sqrt{\frac{\psi}{2}}(2\rho - 1)$.

The numerical solutions have been obtained employing SCSCM for $t = 0.5$, $\rho = 0.2$ and different values of ψ taking $\triangle t$ = 0.001 and are shown in tabular and graphical form. The error norms and order of convergence in numerical solutions for different number of collocation points N have been shown in Table XIV for ρ = 0.2 and ψ = 2. It demonstrates that accuracy in numerical solutions improves as value of N is increased. The present method is reliable and accurate, because the error norms for N = 30 are reduced to the order of 10^{-5} and order of convergence increases by an increase in the value of N. The comparison of L_∞ error norms in present numerical results and results obtained using four NSFD schemes [42] for different values of ψ has been displayed in Table XV taking N = 30. It is revealed that present method yields comparatively more accurate results than NSFD shemes [42]. The numerical and exact solutions for the three cases of ψ have been depicted in figures 7, 8 and 9 for $t = 0.5$, $\rho = 0.2$ and $N = 30$. The graphs of exact and numerical results are almost same which demonstrates the accuracy of the method. As the value of ψ is increased, the stiffness of the profile increases which makes the problem more challenging for present method.

TABLE XIV: Error norms and order of convergence in numerical results of example 4 for $\rho = 0.2$, $t = 0.5$ and $\psi = 2$

N	L_2	RMS	L_∞	Order of Convergence$R$
5	9.6474e-03	4.3144e-03	7.1208e-03	-
10	5.1824e-03	3.7392e-03	6.5318e-03	0.1246
15	4.9046e-03	1.2664e-03	2.3102e-03	2.5633
20	1.1341e-03	2.5359e-04	6.5788e-04	4.3662
25	2.6936e-04	5.3871e-05	2.1636e-04	4.9837
30	4.9818e-05	9.0954e-06	3.0191e-05	10.8018

TABLE XV: Comparison of L_∞ error norms in numerical results of example 4 for $\rho = 0.2$, $t = 0.5$ and $N = 30$

ψ	NSFD[42]				Present method
	Scheme 1	Scheme 2	Scheme 3	Scheme 4	
0.5	2.6767e-03	1.1164e-05	4.4063e-06	4.1322e-06	2.0616e-09
1	7.4715e-03	5.0919e-5	2.5881e-05	2.4227e-05	4.7312e-07
2	2.0671e-02	2.3146e-04	1.6408e-04	1.5759e-04	3.0191e-05

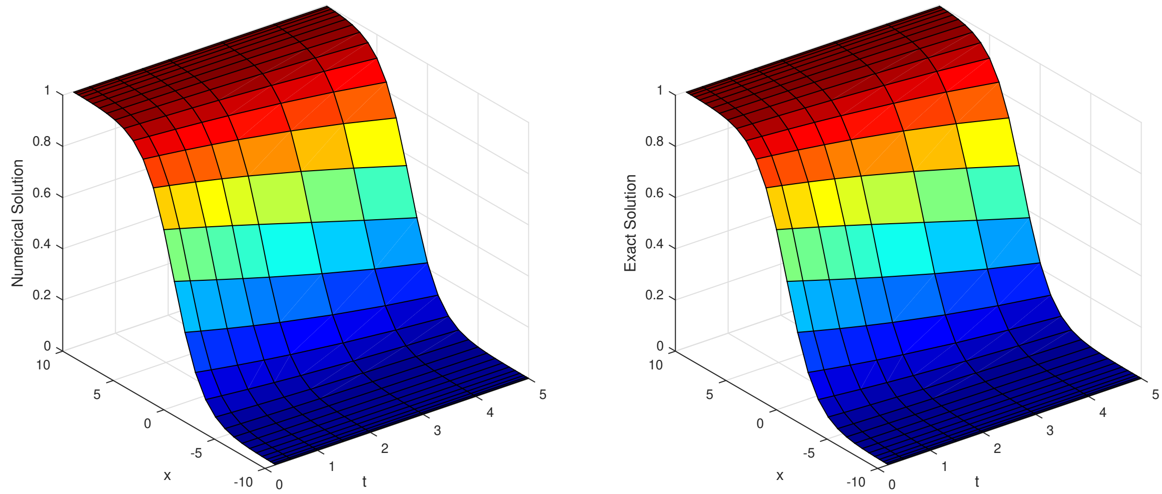

Fig. 1: Numerical and exact solutions of example 1 in the domain [-10, 10] for N = 30 and ρ = 0.75

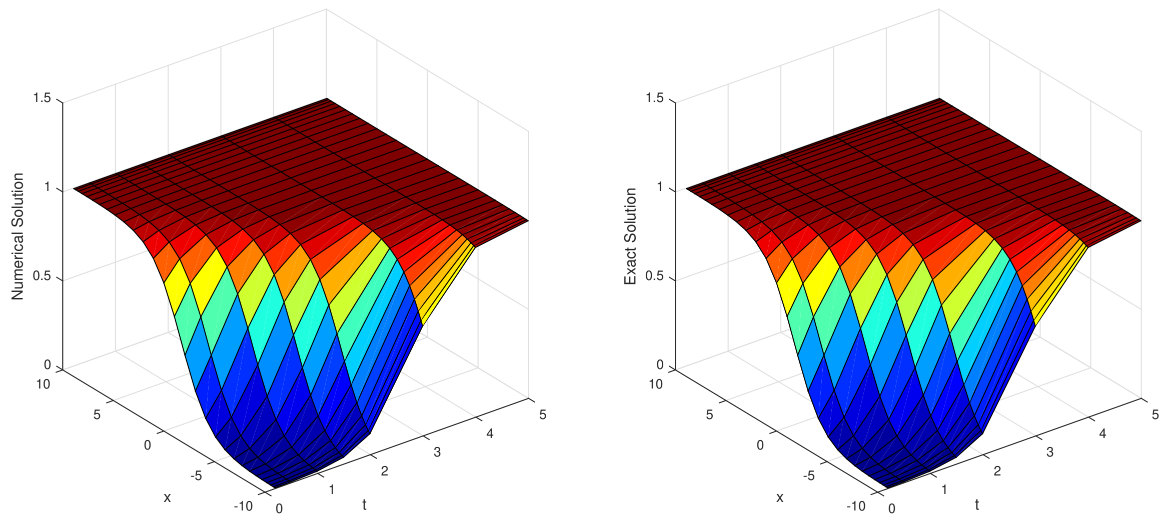

Fig. 2: Numerical and exact solutions of example 1 in the domain [-10, 10] for N = 30 and ρ = -2.0

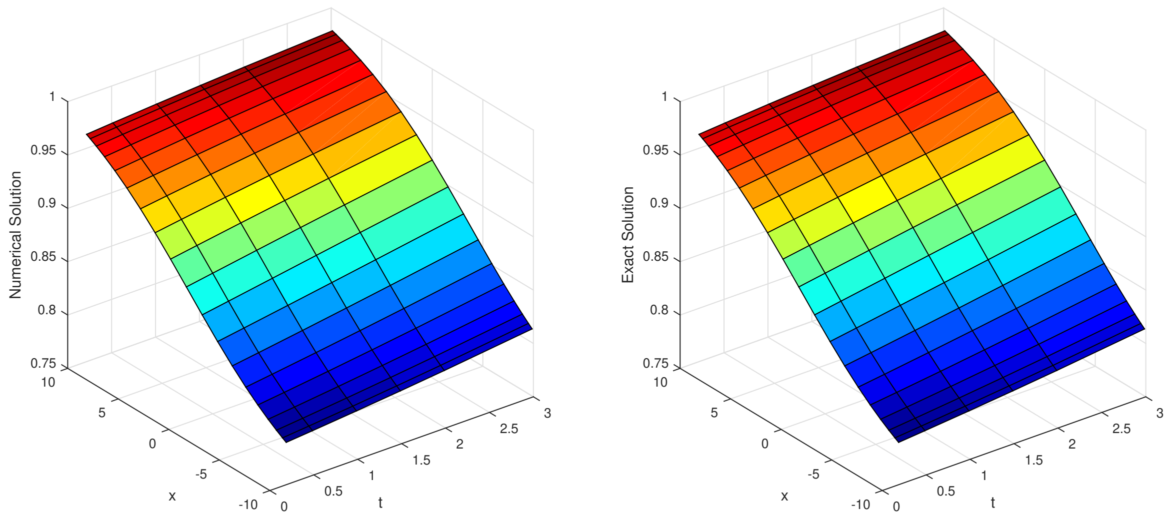

Fig. 3: Numerical and exact solutions of example 2 for N = 20 and ρ = 0.75

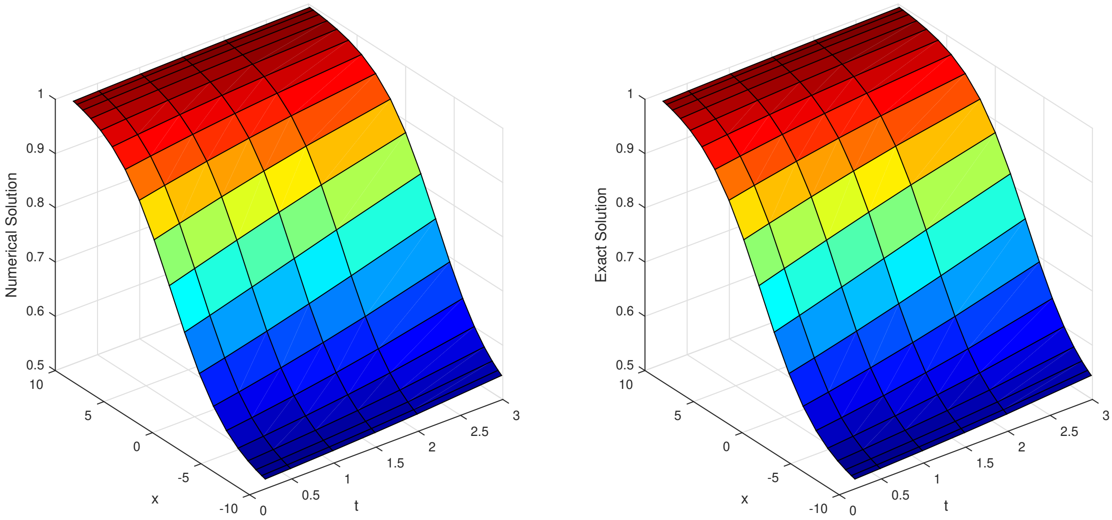

Fig. 4: Numerical and exact solutions of example 2 for $N = 20$ and $\rho = 0.5$

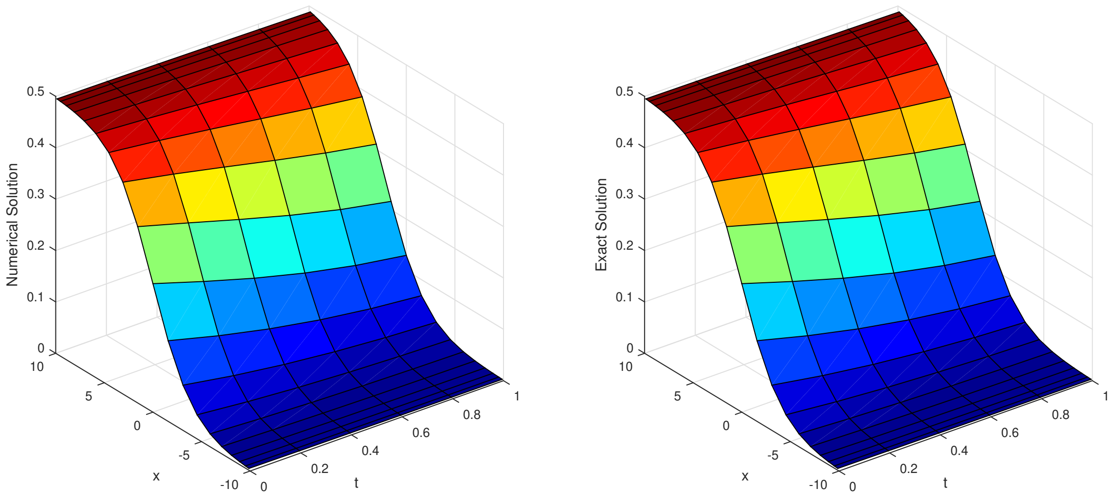

Fig. 5: Numerical and exact solutions of example 3 in the domain [-10, 10] for $\rho = 0.5$ and $N = 20$

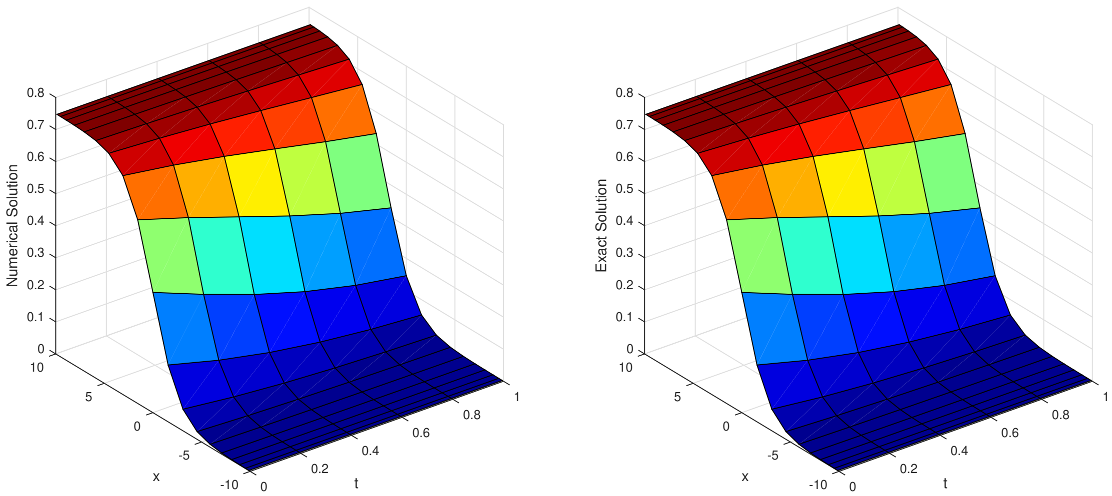

Fig. 6: Numerical and exact solutions of example 3 in the domain [-10, 10] for $\rho = 0.75$ and $N = 20$

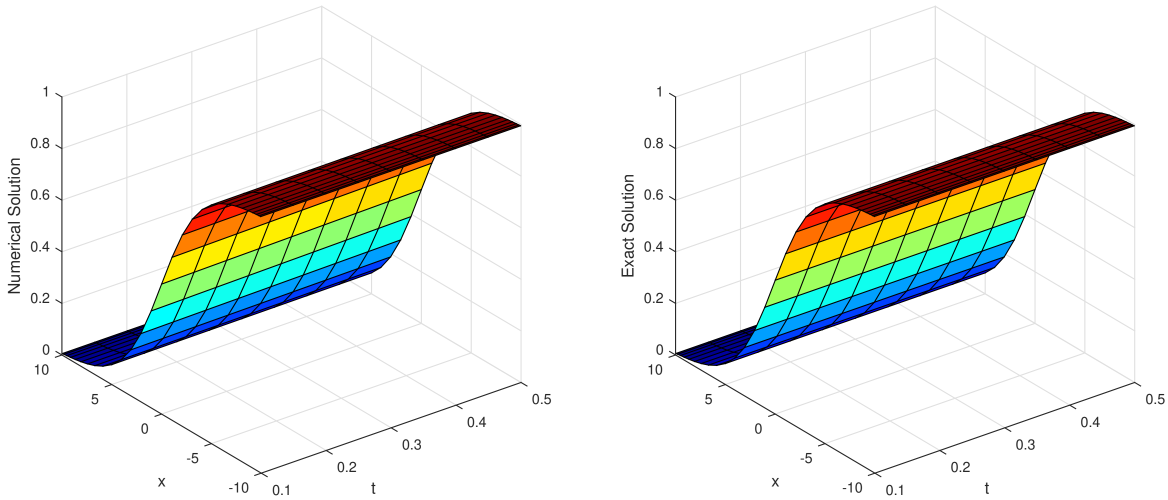

Fig. 7: Numerical and exact solutions of example 4 for $\psi = 0.5$, $\rho = 0.2$ and $N = 30$

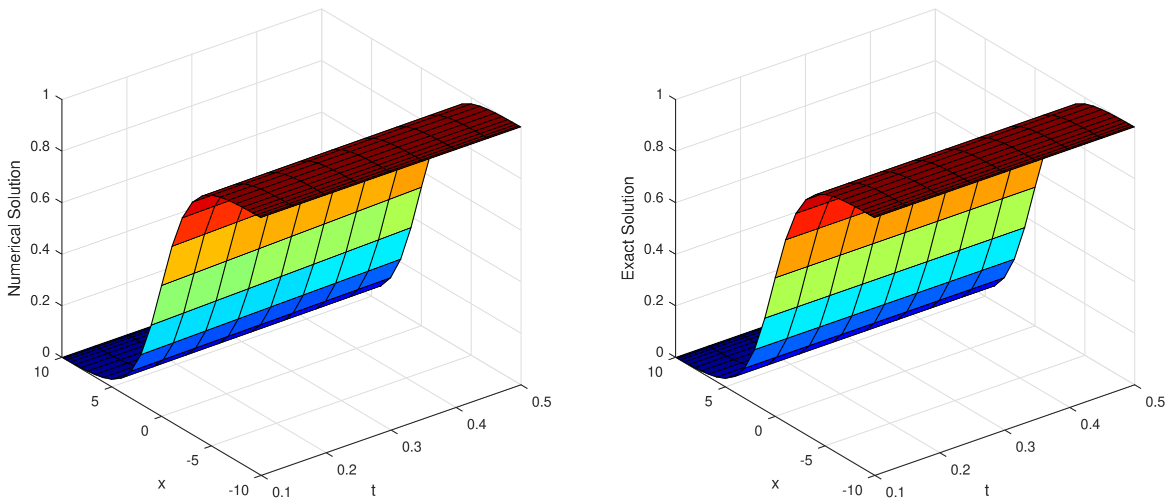

Fig. 8: Numerical and exact solutions of example 4 for $\psi = 1$, $\rho = 0.2$ and $N = 30$

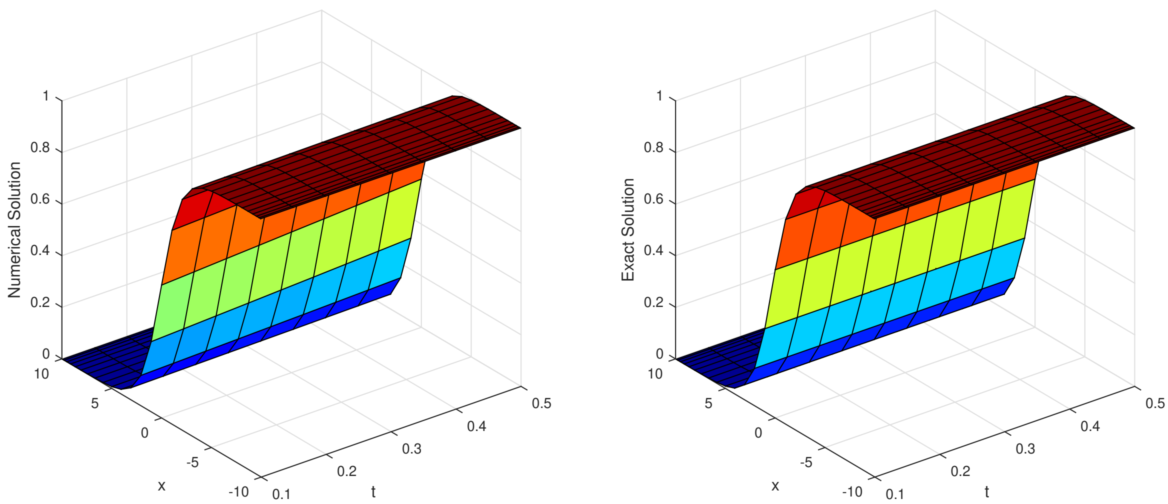

Fig. 9: Numerical and exact solutions of example 4 for $\psi = 2$, $\rho = 0.2$ and $N = 30$

VI. Conclusion

In this paper, the shifted Chebyshev spectral collocation method with Runge-Kutta scheme of order 4 has been employed to solve generalized nonlinear F-N equation. In the solution process, the SCSCM transforms the nonlinear F-N equation into a system of nonlinear ODEs. Subsequently, this system is solved using Runge-Kutta scheme of order 4. The convergence analysis of SCSCM has been demonstrated. To examine the accuracy and efficiency of the present method, four examples of generalized F-N equation have been considered and L_∞, L_2, RMS error norms and order of convergence in numerical solutions of these examples for different number of collocation points N are computed. From the obtained numerical solutions and error norms, it is observed that order of convergence increases and error norms decrease by increasing the value of N and the present method yields most accurate results as compared to the numerical results obtained by other existing methods. Therefore, the present method is more reliable for numerical solution of one dimensional generalized nonlinear F-N equation and can be extended for solving coupled nonlinear F-N equation.

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