Relation Between the *RSD* and Other Topological Indices

K. P. Narayankar, P. K. Pandith Giri Mohan Das, and B. M. Geetha

***Abstract*—This article explores a recently introduced graph invariant called reciprocal status distance (*RSD*). For a connected graph G, *RSD* is defined as the sum, weighted by vertex status, of the reciprocal of distances between all pairs of vertices $\{u, v\}$ in G. Mathematically, *RSD(G)* is given by the expression:$RSD(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}$.**

Our primary objective is to explore the extremal properties of reciprocal status distance. Initially, we identify, among all nontrivial connected graphs of a specified order, those graphs exhibiting the maximum and minimum reciprocal status distances, respectively. Subsequently, we characterize the nontrivial connected graph with a given order, size, and the maximum reciprocal status distance, along with the unicyclic graph and cactus displaying the maximum reciprocal status distance, respectively. Finally, we established the lower and upper bounds for the reciprocal status distance in terms of various graph invariants, including the Wiener index, Harary index, degree distance, the first status coindex distance sum, first status connectivity and coindex, the first Zagreb index and Zagreb coindex, reformulated Zagreb index and the forgotten topological index.

***Index Terms*—Distance, status, reciprocal status-distance index, Wiener index, Harary index, first Zagreb Index, forgotten topological index, the first reformulated Zagreb index, reciprocal degree distance, and status connectivity index.**

I. Introduction

TOPOLOGICAL indices and molecular structures play a crucial role in graph theory, acting as essential bridges to various real-world applications. In mathematical chemistry, a subfield of theoretical and computational chemistry, the focus shifts from quantum mechanics to employing mathematical methodologies. Topological indices function as numerical measures that encapsulate a wide range of physicochemical characteristics of chemical compounds, providing valuable insights into their molecular architectures.

These indices are indispensable for analyzing the properties of chemical compounds. They encompass factors such as boiling point, melting point, temperature, pressure, heat of evaporation, chemical reactivity, and biological activity. The integration of graph theory with chemistry has a profound impact across multiple disciplines. Notably, topological indices are extensively utilized in computational chemistry and the pharmaceutical industry, playing a crucial role in drug development, toxicology, risk assessment, and drug design [24]. A recent addition to the set of distance-based graph invariants is the Reciprocal Status-Distance (*RSD*) Index. This innovative index has garnered significant attention from researchers exploring the Quantitative Structure-Activity Relationship (QSAR) and Quantitative Structure-Property Relationship (QSPR) aspects of chemical graphs. Its effectiveness in characterizing the properties of paraffin hydrocarbons has been demonstrated, surpassing traditional indices like the degree distance index and reciprocal degree distance index in correlating with specific hydrocarbon attributes [25]. Such groundbreaking discoveries have inspired the authors to further intensify their research efforts.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Here,n denotes the order of G (the number of vertices), and m denotes the size of G (the number of edges). The *degree* of a vertex v in G, denoted as $d_G(v)$, represents the number of edges incident to v. The *distance* between two vertices u and v in G denoted as $d_G(u, v)$, is the length of the shortest path connecting them. The *eccentricity* of a vertex v in G denoted as $e(v)$, is the maximum distance from v to any other vertex in the graph. The *diameter* of G denoted as $diam(G)$, is the maximum eccentricity among all vertices in G, while the *radius* of G, denoted as $rad(G)$, is the minimum eccentricity. In cases where the context is clear, $diam(G)$ can be represented by D. The *girth* of a graph G, denoted as $g(G)$, is the length of the shortest cycle in G. If G contains no cycles, its girth $g(G)$ is considered infinite. The *status* or vertex transmission of a vertex u in G, denoted as $\sigma_G(u)$, is defined as

$$\sigma_G(u) = \sum_{v\in V(G)} d_G(u, v)$$

If every vertex in the graph G has a constant vertex transmission equal to k, then such a graph is referred to as a k-transmission-regular (or simply transmission-regular) graph [28]. For further graph-theoretic terminology, readers are directed to the references provided in books [8], [22].

The Wiener number $W(G)$, often referred to as the Wiener index in the fields of chemical and mathematical chemistry, is one of the earliest and most extensively studied distance-based graph invariants associated with a connected graph G. The Wiener number $W(G)$ is defined as the sum of distances between all possible pairs of vertices in a connected graph G [14], [34]. Formally, this can be expressed as:

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v).$$

Similarly the Harary index of G is defined [31] as,

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u, v)}.$$

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One of the most significant graph indices is the first Zagreb index, introduced by I. Gutman and Trinajstic. It is defined as [15]:

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)].$$

The Zagreb indices have been employed in the structure-property modeling of chemicals [16], [36]. Recent advancements and findings related to the Zagreb indices can be found in references [6], [17], [18], [29]. Doslic introduced the first Zagreb coindex, defined as [35]:

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)].$$

Milicevic et al. [4] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees:

$$EM_1(G) = \sum_{e\in E(G)} deg_G(e)^2.$$

The forgotten topological index is defined as: [7]

$$\begin{aligned} F = F(G) &= \sum_{v\in V(G)} deg_G(v)^3 \\ &= \sum_{uv\in E(G)} [deg_G(u)^2 + deg_G(v)^2]. \end{aligned}$$

Independently, Dobrynin and Kochetova [1] and Gutman [21] introduced a variant of the Wiener index known as the degree distance or Schultz molecular topological index. The degree distance for a connected graph G is defined as follows:

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} (d_G(u) + d_G(v))d_G(u,v).$$

Hongbo Hua and Shenggui Zhang [11] introduced a novel graph invariant called the *reciprocal degree distance*. This can be viewed as a degree-weighted version of the Harary index and is defined as:

$$H_A(G) = RDD(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{d_G(u) + d_G(v)}{d_G(u,v)}.$$

First status connectivity and coindex $S_1(G)$ [10] and $\overline{S}_1(G)$ [13] of a connected graph are defined as:

$$S_1(G) = \sum_{uv\in E(G)} [\sigma_G(u) + \sigma_G(v)],$$

and

$$\overline{S}_1(G) = \sum_{uv\notin E(G)} [\sigma_G(u) + \sigma_G(v)].$$

The first status coindex distance sum of the graph was introduced by Afework T. K. et al. [3] and is defined as:

$$S_1^d(G) = \sum_{uv\notin E(G)} [\sigma_G(u) + \sigma_G(v)]d_G(u,v).$$

The present author introduced the *reciprocal status-distance index (RSD)* [32], defined as:

$$RSD(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{\sigma_G(u) + \sigma_G(v)}{d_G(u,v)}, \qquad (1)$$

Let

$$\overline{\sigma}_G(u) = \sum_{\{u,v\}\subseteq V(G), u\neq v} \frac{1}{d_G(u,v)}.$$

Then equation (1) can be restated as,

$$RSD(G) = \sum_{u\in V(G)} \sigma_G(u)\overline{\sigma}_G(u). \qquad (2)$$

For any vertex $u \in V(G), d(G,u,k)$ it signifies the vertex count of G, say v, where $d_G(u,v) = k$. consider that $d(G,u,k) = 0$, when $k > diam(G)$. furthermore, it apparent that, $d(G,u,1) = deg_G(u)$. Evidently $\sum_{k\geq 1} d(G,u,k) = |V(G) - 1|$. The equation (2), $(RSD(G))$ can be rewritten as,

$$RSD(G) = \sum_{u\in V(G)} \sigma_G(u) \sum_{k\geq 1} \frac{1}{k} d(G,u,k). \qquad (3)$$

To learn more about status-based indices, readers can refer to [3], [26]. Additionally, for distance-based indices, one can explore references [2], [12], [19], [20], [23], [40] for the Wiener index and references [27], [30], [33], [39] for the degree distance.

This paper explores the correlation between the Reciprocal Status-Distance (*RSD*) Index and several well-established topological indices. The structure of the paper is outlined as follows: Section II provides an analysis of the *RSD* index within a graph, investigating both lower and upper bounds in relation to various graph invariants. Section III introduces a specific variant of the Cactus graph, denoted as G_n^k . Within this section, a lemma is presented that provides an upper bound for its *RSD*.Section IV presents the K_n^p graph. This section elucidates a graph constructed by attaching p pendant edges to a vertex of K_{n-p} and further discusses its structure and properties.

II. Association with other graph parameters

In this section, we compute the *RSD* index of the graph along with other relevant graph parameters. Direct insights derived from the definition of the *RSD* index are presented. Furthermore, the section investigates both lower and upper bounds of the Reciprocal Status-Distance index in relation to various graph invariants. These include the reciprocal degree distance, Harary index, first Zagreb index, first status distance sum, coindex, first status connectivity index, and coindex.

Proposition II.1. *Let G be a nontrivial connected graph. Then*

$$RSD(G) \leq DD(G) + S_1^d(G)$$

equality holds if and only if $G \cong K_n$.

Proof: For any two vertices $u, v \in V(G)$, by definition $\frac{1}{d_G(u,v)} \leq d_G(u,v)$ equality satisfies if and only if

$d_G(u,v)=1$

$$\begin{aligned}RSD(G) &= \sum_{\{u,v\}\subseteq V(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}\\ &\leq \sum_{uv\in E(G)} [\sigma_G(u)+\sigma_G(v)]\\ &+ \sum_{uv\notin E(G)} [\sigma_G(u)+\sigma_G(v)]d_G(u,v)\\ &\leq \sum_{uv\in E(G)} [\sigma_G(u)+\sigma_G(v)] + S_1^d(G)\\ &= \sum_{u\in V(G)} d_G(u)\sigma_G(u) + S_1^d(G)\\ &= DD(G)+S_1^d(G).\end{aligned}$$

Hence, $RSD(G)\leq DD(G)+S_1^d(G)$ with equality if and only if, for any two vertices u and v in G, $d_G(u,v)=1$, implying that G is isomorphic to K_n. ■

Proposition II.2. *Let G be a nontrivial connected graph. Then*

$$RSD(G)\leq S_1(G)+\overline{S_1}(G)$$

equality holds if and only if $G\cong K_n$.

Proof: For each pair of vertices $u,v\in V(G)$, by definition $\frac{1}{d_G(u,v)}\leq 1$, the equality between the two vertices u and v is satisfied if and only if $d_G(u,v)=1$. so,

$$\begin{aligned}RSD(G) &\leq \sum_{\{u,v\}\subseteq V(G)} [\sigma_G(u)+\sigma_G(v)]\\ &= \sum_{uv\in E(G)} [\sigma_G(u)+\sigma_G(v)]\\ &\quad + \sum_{uv\notin E(G)} [\sigma_G(u)+\sigma_G(v)]\\ &\leq S_1(G)+\overline{S_1}(G).\end{aligned}$$

Equality is obtained for any two vertices u and v if and only if $d_G(u,v)=1$, that is, $G\cong K_n$. ■

Let $\Delta(G)$ and $\delta(G)$ be the maximum and minimum degree in a graph G, respectively.

Theorem II.3. *Let G be a connected graph with n vertices and m edges. Let diam(G)=D. Then,*
$RSD(G)\geq(3n-2)m-2m\Delta+\frac{1}{D}\big[4(n-1)\bar{m}-\overline{M}_1(G)\big]$.
Inequality holds good if and only if $D\leq 2$.

Proof: For a graph G

$$\begin{aligned}RSD(G) &= \sum_{\{u,v\}\subseteq V(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}\\ &= \sum_{uv\in E(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}\\ &+ \sum_{uv\notin E(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}\end{aligned}\tag{4}$$

If $uv\in E(G)$, then for every $y\in V(G)$

$$d(v,y)-1\leq d(u,y)\leq d(v,y)+1.\tag{5}$$

So that,

$$\sigma_G(u)+\sigma_G(v)\geq 2\sigma_G(u)-n+2\tag{6}$$

For any vertex u of G, there are $deg_G(u)$ vertices which are at a distance one from u, and the remaining $(n-1-deg_G(u))$ vertices are at a distance at least 2. Thus,

$$\begin{aligned}\sigma_G(u) &\geq deg_G(u)+2(n-1-deg_G(u))\\ &= 2(n-1)-deg_G(u).\end{aligned}\tag{7}$$

Using 6 and 7 in 4, we obtain

$$\begin{aligned}RSD(G) &\geq \sum_{uv\in E(G)} 2\sigma_G(u)-n+2\\ &+ \sum_{uv\notin E(G)} \frac{4(n-1)-(deg_G(u)+deg_G(v))}{D}\\ &\geq \sum_{uv\in E(G)} \big[2(2n-2-deg_G(u)-n+2)\big]\\ &+ \frac{1}{D}\big[4(n-1)\bar{m}-\overline{M}_1(G)\big]\\ &\geq (3n-2)m-2m\Delta+\frac{1}{D}\big[4(n-1)\bar{m}-\overline{M}_1(G)\big].\end{aligned}\tag{8}$$

Where, $\bar{m}=\binom{n}{2}-m$ ■

Theorem II.4. *Let G be a connected graph with n vertices and m edges. Let $diam(G)=D$. Then,*
$RSD(G)\leq m\big[2D(n-1)-n+2\big]-2m(D-1)\delta$
$+D\bar{m}(n-1)-\frac{1}{2}(D-1)\overline{M}_1(G)$.
Inequality holds good if and only if $D\leq 2$.

Proof: For a graph G

$$\begin{aligned}RSD(G) &= \sum_{uv\in E(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}\\ &\quad + \sum_{uv\notin E(G)} \frac{\sigma_G(u)+\sigma_G(v)}{d_G(u,v)}.\end{aligned}\tag{9}$$

$$\text{We have}, \sigma_G(u)+\sigma_G(v)\leq 2\sigma_G(u)-n+2\tag{10}$$

For any vertex u of G, there are $deg_G(u)$ vertices which are at a distance one from u, and the remaining $(n-1-deg_G(u))$ vertices are at a distance at most D. Thus,

$$\begin{aligned}\sigma_G(u) &\geq deg_G(u)+D(n-1-deg_G(u))\\ &= D(n-1)-(D-1)deg_G(u).\end{aligned}\tag{11}$$

Using 10 and 11 in 9, we obtain

$$\begin{aligned}RSD(G) &\leq \sum_{uv\in E(G)} 2\sigma_G(u)-n+2\\ &+ \sum_{uv\notin E(G)} \frac{2D(n-1)-(D-1)(deg_G(u)+deg_G(v))}{2}\\ &\leq \sum_{uv\in E(G)} \big[2(D(n-1)-(D-1)deg_G(u))-n+2)\big]\\ &+ D\bar{m}(n-1)-\frac{1}{2}(D-1)\overline{M}_1(G)\\ &\leq m\big[2D(n-1)-n+2\big]-2m(D-1)\delta\\ &+ D\bar{m}(n-1)-\frac{1}{2}(D-1)\overline{M}_1(G)\end{aligned}\tag{12}$$

Where, $\bar{m} = \binom{n}{2} - m$ ∎

Let $\Psi(G)$ and $\psi(G)$ be the maximum and minimum status in a graph G, respectively. Then the following Theorem (II.5) is valid.

Theorem II.5. *Let G be a nontrivial connected graph and $diam(G) \leq 2$. Then,*

$$2\psi(G)H(G) \leq RSD(G) \leq 2\Psi(G)H(G)$$

If and only if the graph G is a status regular graph, the equality $\Psi(G) = \psi(G)$ is valid.

Proof: Clearly, we have $\psi(G) \leq \sigma_G(w) \leq \Psi(G)$. So,

$$\begin{aligned}2\psi(G) \sum_{\{u,v\}\in V(G)} \frac{1}{d_G(u,v)} &\leq RSD(G) \\ &\leq 2\Psi(G) \sum_{\{u,v\}\in V(G)} \frac{1}{d_G(u,v)}.\end{aligned}$$

Thus,

$$2\psi(G)H(G) \leq RSD(G) \leq 2\Psi(G)H(G).$$

Thus, the proof is concluded. ∎

Theorem II.6. *Let G be a nontrivial connected graph. Then,*

$$RSD(G) \leq \frac{(M_1(G) + \overline{M}_1(G)).(S_1(G) + \overline{S}_1(G))}{RDD(G)}.$$

The equality holds only when G is isomorphic to the complete graph K_n.

Proof: The following result is obtained by taking into account the degree distance and reciprocal status distance:

$$\begin{aligned}RDD(G)\cdot RSD(G) &= \left(\sum_{\{u,v\}\subseteq V(G)} \frac{deg_G(u) + deg_G(v)}{d_G(u,v)}\right) \\ &\quad \left(\sum_{\{u,v\}\subseteq V(G)} \frac{\sigma_G(u) + \sigma_G(v)}{d_G(u,v)}\right) \\ &\leq \left(\sum_{\{u,v\}\subseteq V(G)} (deg_G(u) + deg_G(v))\right) \\ &\quad \left(\sum_{\{u,v\}\subseteq V(G)} (\sigma_G(u) + \sigma_G(v)\right) \\ &\leq [M_1(G) + \overline{M}_1(G)][S_1(G) + \overline{S}_1(G)].\end{aligned}$$

Therefore,

$$RSD(G) \leq \frac{[M_1(G) + \overline{M}_1(G)][(S_1(G) + \overline{S}_1(G)]}{RDD(G)}.$$

Equality holds if and only if $d_G(u,v)$ is a constant, $G \cong K_n$. ∎

Next, we categorize connected graphs with n vertices and m edges and extremal RSD.

Theorem II.7. *Let G be a connected graph of order $n \geq 2$ and size $m \geq 1$. Then*

$$\begin{aligned}\frac{2(n-1)}{D}W(G) + \frac{D-1}{D}DD(G) &\leq RSD(G) \\ &\leq (n-1)W(G) \\ &\quad + \frac{DD(G)}{2}\end{aligned}$$

with either equality if and only if $D \leq 2$, where D is the diameter of G.

Proof: To verify that the right-hand side inequality holds for every vertex u in G. Thus,

$$\begin{aligned}\overline{\sigma}_G(u) &= deg_G(u) + \sum_{u\in V(G)\setminus\{N_G(w)\}} \frac{1}{deg_G(u,v)} \\ &\leq deg_G(u) + \frac{1}{2}[n - deg_G(u) - 1] \\ &= \frac{1}{2}[n - 1 + deg_G(u)].\end{aligned}$$

The equality holds if and only if $diam(G) \leq 2$. The aforementioned inequality from Equation (2) can be utilized to draw an immediate conclusion.

$$\begin{aligned}RSD(G) &= \sum_{u\in V(G)} \sigma_G(u)\overline{\sigma}_G(u) \\ &\leq \sum_{u\in V(G)} \sigma_G(u)\frac{[n-1+deg_G(u)]}{2} \\ &= (n-1)W(G) + \sum_{u\in V(G)} \frac{\sigma_G(u)deg_G(u)}{2} \quad (13) \\ &\leq (n-1)W(G) + \frac{DD(G)}{2} \quad (14)\end{aligned}$$

with equality if and only if the $diam(G) \leq 2$.
Let us now examine the inequality on the left. For each vertex u in G

$$\begin{aligned}\overline{\sigma}_G(u) &= deg_G(u) + \sum_{v\in V(G)/N_G[u]} \frac{1}{d_G(u,v)} \\ &\geq deg_G(u) + \frac{[n - deg_G(u) - 1]}{D} \\ &\geq \frac{n + (D-1)deg_G(u) - 1}{D} \quad (15)\end{aligned}$$

where the equality holds if and only if $diam(G) \leq 2$.
Using (15) in eq(2) we get

$$\begin{aligned}RSD(G) &\geq \sum_{u\in V(G)} \sigma_G(u)\left(\frac{(n + (D-1)deg_G(u) - 1)}{D}\right) \\ &\geq \sum_{u\in V(G)} \sigma_G(u)\left(\frac{n-1}{D}\right) \\ &\quad + \sum_{u\in V(G)} \sigma_G(u)deg_G(u)\left(\frac{D-1}{D}\right) \quad (16)\end{aligned}$$

and equality holds if and only if $daim(G) \leq 2$.

$$RSD(G) \geq \frac{2(n-1)}{D}W(G) + \frac{D-1}{D}DD(G)$$

equality holds if and only if $diam(G) \leq 2$. This concludes the proof. ∎

Theorem II.8. *Let G be a graph with $|v(G)| = n$, $|E(G)| = m$ and $g(G) > 4$. Then,*

$$\begin{aligned}RSD(G) \leq& \frac{2}{3}(n-1)W(G) + \frac{1}{2}DD(G) \\ &+ \frac{1}{6}\sum_{u\in V(G)} \sigma_G(u) \sum_{uv\in E(G)} deg_G(v)\end{aligned}$$

with equality if and only if $diam(G) \leq 3$.

Proof: From the definition (I), we have

$$\begin{aligned}
&RSD(G)\\
&= \sum_{u\in V(G)} \sigma_G(u) \sum_{k\geq 1} \frac{1}{k} d(G,u,k)\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[d(G,u,1) + \frac{1}{2} d(G,u,2)\\
&\qquad + \sum_{k\geq 3} \frac{1}{k} d(G,u,k) \Bigg].
\end{aligned}$$

Alternatively, given this equality and notations,

$$\begin{aligned}
&RSD(G)\\
&\leq \sum_{u\in V(G)} \sigma_G(u) \Bigg[deg(u) + \frac{1}{2} \sum_{uv\in E(G)} (deg_G(v)-1)\\
&\qquad + \frac{1}{3} \sum_{k\geq 3} d(G,u,k) \Bigg]\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[deq_G(u) + \frac{1}{2} \sum_{uv\in E(G)} (deg_G(v)-1)\\
&\qquad + \frac{1}{3} \Bigg(n-1-deg_G(u) - \sum_{uv\in E(G)} (deg_G(v)-1) \Bigg) \Bigg]\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[\frac{1}{3}(n-1) + \frac{2}{3} deg_G(u)\\
&\qquad + \frac{1}{6} \sum_{uv\in E(G)} (deg_G(v)-1) \Bigg]\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[\frac{1}{3}(n-1) + \frac{2}{3} deg_G(u) - \frac{1}{6} deg_G(u)\\
&\qquad + \frac{1}{6} \sum_{uv\in E(G)} deg_G(v) \Bigg]\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[\frac{1}{3}(n-1) + \frac{1}{2} deg_G(u)\\
&\qquad + \frac{1}{6} \sum_{uv\in E(G)} deg_G(v) \Bigg]\\
&= \frac{2}{3}(n-1)W(G) + \frac{1}{2} DD(G)\\
&\qquad + \frac{1}{6} \sum_{u\in V(G)} \sigma_G(u) \sum_{uv\in E(G)} deg_G(v).
\end{aligned}$$

Equality exists if and only if $diam(G) \leq 3$. ■

Let us define $\beta = \max\{\sigma_G(v), \sigma_G(w) \mid d_G(v,w) = 3\}$.

Theorem II.9. *Let* G *be a graph with* $|v(G)| = n$, $|E(G)| = m$ *and* $g(G) > 6$. *Then,*

$$\begin{aligned}
RSD(G) \leq& \frac{1}{2}(n-1-m)W(G) + \frac{3}{4} DD(G)\\
&+ \frac{1}{6}\beta[M_2(G) - M_1(G) + m]\\
&+ \frac{1}{4} \sum_{u\in V(G)} \sigma_G(u) \sum_{uv\in E(G)} deg_G(v).
\end{aligned}$$

The equality holds if and only if $diam(G) \leq 4$ *and* $\{\sigma_G(v), \sigma_G(w) \mid d_G(v,w) = 3\} = \{\beta\}$

Proof: From the definition (I) and by using the properties of $d(G,u,k)$ as defined:

$$\begin{aligned}
RSD(G) &= \sum_{u\in V(G)} \sigma_G(u) \sum_{k\geq 1} \frac{1}{k} d(G,u,k)\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[d(G,u,k) + \frac{1}{2} d(G,u,k)\\
&\qquad + \frac{1}{3} d(G,u,3) + \frac{1}{4} \sum_{k\geq 4} d(G,u,k) \Bigg].
\end{aligned}$$

Alternatively, given this equality and notations,

$$\begin{aligned}
&RSD(G)\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[deg_G(u) + \frac{1}{2} \sum_{uv\in E(G)} (deg_G(v)-1)\\
&\qquad + \frac{1}{3} d(G,u,k) + \frac{1}{4} \sum_{k\geq 4} d(G,u,k) \Bigg]\\
&\leq \sum_{u\in V(G)} \sigma_G(u) \Bigg[deg_G(u) + \frac{1}{2} \sum_{uv\in E(G)} (deg_G(v)-1)\\
&\qquad + \frac{1}{3} d(G,u,3) + \frac{1}{4} (n-1-deg_G(u)\\
&\qquad - \sum_{uv\in E(G)} (deg_G(u)-1) - d(G,u,3)) \Bigg]\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[deg_G(u) - \frac{1}{4} deg_G(u) + \frac{1}{4}(n-1)\\
&\qquad - \frac{1}{2} m + \frac{1}{4} m + \frac{1}{3} d(G,u,3) - \frac{1}{4} d(G,u,3)\\
&\qquad + \frac{1}{2} \sum_{uv\in E(G)} deg_G(v) - \frac{1}{4} \sum_{uv\in E(G)} deg_G(v) \Bigg]\\
&= \sum_{u\in V(G)} \sigma_G(u) \Bigg[\frac{3}{4} deg_G(u) + \frac{1}{4}(n-1) - \frac{1}{4} m\\
&\qquad + \frac{1}{12} d(G,u,3) + \frac{1}{4} \sum_{uv\in E(G)} deg_G(v) \Bigg]\\
&= \frac{1}{2}(n-1-m)W(G) + \frac{3}{4} DD(G)\\
&\qquad + \frac{1}{4} \sum_{u\in V(G)} \sigma_G(u) \sum_{uv\in E(G)} deg_G(v)\\
&\qquad + \frac{1}{12} \sum_{u\in V(G)} \sigma_G(u) d(G,u,3)
\end{aligned}$$

Now, substitute the property of the β as defined :

$$= \frac{1}{2}(n-1-m)W(G) + \frac{3}{4}DD(G) + \frac{1}{4}\sum_{u\in V(G)}\sigma_G(u)\sum_{uv\in E(G)} deg_G(v) + \frac{1}{12}\beta\sum_{u\in V(G)} d(G,u,3)$$

$$RSD(G) = \frac{1}{2}(n-1-m)W(G) + \frac{3}{4}DD(G) + \frac{1}{4}\sum_{u\in V(G)}\sigma_G(u)\sum_{uv\in E(G)} deg_G(v) + \frac{1}{6}\beta\sum_{vw\in E(G)}(deg_G(v)-1)(deg_G(w)-1) \tag{17}$$

where

$$\sum_{vw\in E(G)}(deg_G(v)-1)(deg_G(w)-1) = M_2(G) - M_1(G) + m \tag{18}$$

Now substitute equation 10 in equation 9. Then,

$$= \frac{1}{2}(n-1-m)W(G) + \frac{3}{4}DD(G) + \frac{1}{6}\beta[M_2(G) - M_1(G) + m] + \frac{1}{4}\sum_{u\in V(G)}\sigma_G(u)\sum_{uv\in E(G)} deg_G(v)$$

It is obvious that equality exists if and only if $\{\sigma_G(v), \sigma_G(w) \mid d_G(v,w) = 3\} = \{\beta\}$ and $diam(G) \leq 4$. ∎

Theorem II.10. *Let G be a graph with $|V(G)| = n$, $|E(G)| = m$ and $g(G) > 4$. Then,*

$$RSD(G) \leq M_1(G) + 2(n-1)\bar{m} + \frac{1}{2}EM_1(G) - (F(G) - 3M_1(G) + 4m)$$

equality holds if and only if $diam(G) \leq 2$ and $\{deg_G(v) \mid uvw$ is a path of length two in $G\}$.

Proof: By definition,

$$RSD(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{\sigma_G(u) + \sigma_G(v)}{d_G(u,v)} = \sum_{uv\in E(G)} \sigma_G(u) + \sigma_G(v) + \frac{1}{2}\sum_{uv\notin E(G)} \sigma_G(u) + \sigma_G(w) + \sum_{\{u,v\}\subseteq V(G), d_G(u,v)\geq 3} \frac{\sigma_G(u) + \sigma_G(v)}{d_G(u,v)}$$

where, $\sigma_G(u) + \sigma_G(w) = 4(n-1) - (deg_G(u) + deg_G(w))$ using this in above equation and obtained as follows,

$$RSD(G) \leq M_1(G) + 2(n-1)\bar{m} - \frac{1}{2}\sum_{e\sim f, e=uv, f=vw} deg_G(u) + deg_G(w),$$

where for $e = uv$,

$$\begin{aligned} deg_G(e) &= deg_G(u) + deg_G(v) - 2 \\ &\leq M_1(G) + 2(n-1)\overline{m} + \frac{1}{2}\Big[\sum_{e\sim f, e=uv, f=vw}(deg_G(e) - (deg_G(v) - 2)) + (deg_G(f) - (deg_G(v) - 2))\Big] \\ &\leq M_1(G) + 2(n-1)\overline{m} + \frac{1}{2}\sum_{e\sim f, e=uv, f=vw}(deg_G(e) + deg_G(f)) - \sum_{e\sim f, e=uv, f=vw}(deg_G(v) - 2) \\ &\leq M_1(G) + 2(n-1)\overline{m} + \frac{1}{2}EM_1(G) - 2\sum_{v\in V(G)}\binom{deg_G(v)}{2}(deg_G(v) - 2) \\ &\leq M_1(G) + 2(n-1)\overline{m} + \frac{1}{2}EM_1(G) - 2\sum_{v\in V(G)}\left(\frac{deg_G^3(v) - 3deg_G^2(v) + 2deg_G(v)}{2}\right) \\ &\leq M_1(G) + 2(n-1)\overline{m} + \frac{1}{2}EM_1(G) - (F(G) - 3M_1(G) + 4m). \end{aligned}$$

Equality holds good if and only if $diam(G) \leq 2$ $\{deg_G(v) \mid uvw$ is a path of length two in $G\}$. Where $\overline{m} = \binom{n}{2} - m$. ∎

Theorem II.11. *Let G be a connected graph of order n of $diam(G) \leq 2$. Then,*

$$RSD(G) \leq n(n-1)^2 - \frac{1}{2}n(n-1)(D-1)(1 + \frac{1}{D}(n-1)) + \frac{(D-1)^2}{4D}n(n-1)^2$$

with equality if and only if n is odd and G is a $\frac{n-1}{2}$-regular graph.

Proof: By definition,

$$RSD(G) = \sum_{u\in V(G)} \sigma_G(u)\overline{\sigma}_G(u) \tag{19}$$

Since $diam(G) \leq 2$, for each $u \in V(G)$,

$$\sigma_G(u) = D(n-1) - (D-1)deg_G(u)$$

and

$$\overline{\sigma}_G(u) = deg_G(u) + \frac{1}{D}(n-1-deg_G(u)) = \frac{1}{D}(n-1) + (1 - \frac{1}{D})deg_G(u).$$

Therefore, (19) becomes,

$$\begin{aligned}RSD(G) = \sum_{v\in V(G)} &((n-1)^2 \\ &- (D-1)(1+\frac{1}{D}(n-1))deg_G(u)) \\ &+ \frac{(D-1)^2}{D}deg_G^2(u)\end{aligned}$$

after simplifying,

$$\begin{aligned}&\leq n(n-1)^2 - (D-1)(1+\frac{1}{D}(n-1))\sum_{u\in V(G)} deg_G(u) \\ &\quad + \frac{(D-1)^2}{D}\sum_{u\in V(G)} deg_G^2(u) \\ &\leq n(n-1)^2 - \frac{1}{2}n(n-1)(D-1)(1+\frac{1}{D}(n-1)) \\ &\quad + \frac{(D-1)^2}{4D}n(n-1)^2.\end{aligned}$$

■

III. Characteristics and Bounds of k-Cycle Cactus Graphs

In this section, we delve into the study of cactus graphs. A cactus is defined as a connected graph where any two simple cycles share at most one vertex. Put simply, a cactus graph can be constructed by connecting simple cycles at a finite number of vertices without introducing any new cycles. If a cactus graph has no cycles, it is essentially a tree; whereas if it possesses exactly one cycle, it is termed a unicyclic graph.

For $0 \leq k \leq \frac{n-1}{2}$, let G_n^k be an n-vertex k-cycle cactus derived from the n-vertex star by adding k independent edges among $n-1$ pendent vertices [11]. The subsequent lemma offers a precise upper bound for the *RSD* of the $k-$ cycle cactus.

Lemma III.1. *[38] Let G be an n-vertex k-cycle cactus with $0 \leq k \leq \frac{n-1}{2}$. Then, $M_1(G) \leq n^2 - n + 6k$, where equality holds if and only if $G \cong G_n^k$.*

Theorem III.2. *Let G be an n-vertex k-cycle cactus of diametetr D with $0 \leq k \leq \frac{n-1}{2}$. Then,*

$$RSD(G) \leq n(n-1)^2 + (n-1)(n+k-1) - \frac{n^2-n+6k}{2},$$

where equality holds if and only if $G \cong G_n^k$ and is of diameter two.

Proof: Recall that G has $n+k-1$ edges. By the definition of RSD and Lemma (III.1)

$$RSD(G) = \sum_{u\in V(G)} \sigma_G(u)\overline{\sigma}_G(u) \tag{20}$$

where $\sigma_G(u) \leq 2(n-1) - deg_G(u)$,

$\overline{\sigma}_G(u) \leq \frac{1}{2}(n-1) + \frac{1}{2}deg_G(u)$ with equality iff $D \leq 2$.

By substituting the value of $\sigma_G(u)$ and $\overline{\sigma}_G(u)$ in equation (20). Hence,

$$\begin{aligned}RSD(G) \leq &\sum_{u\in V(G)} \Big[(2(n-1) - deg_G(u))(\frac{1}{2}(n-1) \\ &+ \frac{1}{2}deg_G(u))\Big] \\ &= \sum_{u\in V(G)} (n-1)^2 + \frac{1}{2}(n-1)deg_G(u) - \\ &\frac{1}{2}deg_G^2(u) \\ &\leq n(n-1)^2 + m(n-1) - \frac{1}{2}M_1(G) \\ &\leq n(n-1)^2 + m(n-1) - \frac{1}{2}M_1(G_n^k) \\ &= n(n-1)^2 + (n-1)(n+k-1) - \frac{n^2-n+6k}{2}.\end{aligned}$$

The equality is true if and only if $G \cong G_n^k$ and is of diameter at most two. ■

According to Theorem III 7, yields immediate results for the *RSD* of trees and unicyclic graphs.

Corollary III.3. *Consider a tree T with $n \geq 2$ vertices. Then,*

$$RSD(T) \leq n(n-1)^2 + (n-1)^2 - \frac{n^2-n}{2}.$$

Equality occurs if and only if T is isomorphic to the star graph S_n

Corollary III.4. *Suppose G is a unicyclic graph with $n \geq 3$ vertices. Then,*

$$RSD(G) \leq n(n-1)^2 + n(n-1) - \frac{n^2-n+6}{2}.$$

IV. Graph Construction: K_n^p Representation

In this section, the representation K_n^p is introduced, indicating the graph formed by attaching p pendant edges to a vertex of K_{n-p}. To proceed with the discussion, the following results will be utilized.

Lemma IV.1. *[11] Let G be an n-vertex connected graph with p pendent vertices. Then,*

$$\begin{aligned}M_1(G) \leq &n^3 - (3p-1)n^2 + (3p^2+6p+1)n \\ &- p^3 - 3p^2 - 2p - 1.\end{aligned}$$

With equality if and only if $G \cong K_n^p$

Theorem IV.2. *Let G be a connected graph with n vertices and p pendent vertices. Then,*

$$\begin{aligned}RSD(G) \leq &n(n-1)^2 + (n-1)\big[\frac{(n-p)^2+3p-n}{2}\big] \\ &- \frac{1}{2}[n^3 - (3p-1)n^2 + (3p^2+6p+1)n \\ &- p^3 - 3p^2 - 2p - 1].\end{aligned}$$

The equality holds if G is isomorphic to K_n^p.

Proof: Let G^* be a connected graph with n vertices where p vertices $u_1, u_2, ..., u_p$ are pendent such that

$G[V(G^*)-\{u_1,u_2,...,u_p\}]$, which represents the sub graph of G^* induced by the vertices in $V(G^*)-\{u_1,u_2,...,u_p\}$, forms a clique within G^*. We may conclude that G^* has $p+\binom{n-p}{2}=p+\frac{(n-p)(n-p-1)}{2}$ edges, based on Corollary (III) and Lemma (IV.1). Hence,

$$\begin{aligned}RSD(G^*) &\leq n(n-1)^2\\ &\quad + (n-1)m - \frac{M_1(G^*)}{2}\\ &= n(n-1)^2 + (n-1)\Big[\frac{(n-p)^2+3p-n}{2}\Big]\\ &\quad - \frac{M_1(G^*)}{2}\\ &\leq n(n-1)^2\\ &\quad + (n-1)\Big[\frac{(n-p)^2+3p-n}{2}\Big]\\ &\quad - \frac{K_n^p}{2}.\end{aligned}$$

The equivalence is true if G^* has a diameter of at most 2. Observe that K_n^p has diameter 2. So,

$$\begin{aligned}RSD(G) \leq & n(n-1)^2 + (n-1)\Big[\frac{(n-p)^2+3p-n}{2}\Big]\\ & - \frac{1}{2}[n^3-(3p-1)n^2+(3p^2+6p+1)n\\ & - p^3-3p^2-2p-1]\end{aligned}$$

with equality if and only if $G \cong K_n^p$. This concludes the proof. ∎

Hence, both lower and upper bounds for the reciprocal status distance have been determined using a variety of graph invariants. These include the degree distance, Harary index, first Zagreb index, forgotten topological index, first reformulated Zagreb index, reciprocal degree distance, first status distance sum, and first status connectivity index and coindex.

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