Study on a New Fractional Logistic Model via Deformable Fractional Derivative

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Abstract—**In this paper, conducted within the purview of deformable fractional calculus, we explore a distinct class of fractional logistic differential equation models endowed with variable coefficient intrinsic growth rates. By using the intrinsic properties of deformable fractional calculus and employing precise analytical techniques, we have successfully deduced the analytical solution of the model. To further elucidate our main findings, we present detailed examples and perform corresponding numerical simulations. Remarkably, in contrast to the prevailing literature, the fractional logistic differential equation model examined herein incorporates a novel feature of variable coefficient intrinsic growth rates. Therefore, this work not only extends previous research results but also enriches the literature in the related field.**

Index Terms—**Fractional logistic equation, Deformable fractional derivative, Variable coefficient intrinsic growth rate, Analytical solution**

I. Introduction

In recent years, the logistic equation has garnered significant scholarly attention due to its extensive applicability in various fields such as biomedicine, economics, optical network data security, and notably, in modeling the dynamics of the COVID-19 pandemic [1]. The logistic equation is especially relevant in scenarios where the aggregate growth rates diminish with increasing population size. Fundamentally, the per capita growth rate within a population model is characterized as a diminishing function of the population size, represented by $\lambda - ax$. This principle was originally posited by Verhulst in 1838 and is succinctly expressed through the subsequent logistic differential equation

$$x'(t) = x(t)(\lambda - \alpha x(t)). \quad (1)$$

Upon further examination, Pearl and Reed (1920) reformulated equation (1) as follows

$$x'(t) = rx(t)\Big[1 - \frac{x(t)}{K}\Big], \quad (2)$$

where r is called the intrinsic growth rate, and K is represented the carrying capacity of the population [2]. If the population size at time $t = 0$ is x_0, that is, $x(0) = x_0$, then equation (2) has the following analytical solution

$$x(t) = \frac{Kx_0}{x_0 + (K - x_0)\exp(-rt)}. \quad (3)$$

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On the other hand, due to the extensive applications of fractional differential equations in various disciplines, including physics, biomedicine, finance, non-Newtonian fluid mechanics, control theory, and signal processing [3–9], etc. Based on the powerful theoretical and practical background of fractional calculus, some scholars have cultivated a profound interest in the study of fractional logistic equations, yielding many intriguing findings [1, 10–19]. In 2007, El-Sayed et al. [10] discussed the initial value problem of the fractional logistic differential equation as follows

$$D^\alpha[x(t)] = \rho x(t)(1 - x(t)), \quad t > 0, \quad x(0) = x_0, \quad (4)$$

where $\alpha \in (0, 1]$, $x_0 > 0$, D^α represents the Caputo fractional derivative. The author discussed the stability, existence, uniqueness, and numerical solution of equation (4).

In 2015, West [11] investigated the following initial value problem of a fractional logistic differential equation

$$D_t^\alpha[u(t)] = k^\alpha u(t)[1 - u(t)], \quad t > 0, \quad u(0) = u_0, \quad (5)$$

where $\alpha \in (0, 1]$, $k > 0$, $u_0 > 0$, D_t^α represents the Caputo fractional derivative. The author obtained the solution of equation (5) by using Laplace transform and matrix methods.

It is noteworthy that the definitions of fractional differential operators, including Riemann-Liouville, Caputo, and Hadamard, are characterized by singular kernels, which engender a high computational cost and diminished efficiency in the fractional differential equations formulated by these operators, particularly in computations over extended durations. Consequently, a faction of scholars advocates for local fractional calculus approaches such as Caputo-Fabrizio, Conformable, Deformable, and fractal calculus. Of late, some researchers have delved into the analytical solution conundrum of local fractional logistic differential equations [1, 12].

In 2020, Abreu-Blaya, et al. [12] studied the following initial value problem of fractional logistic differential equation involving local fractional derivative

$$G_T^\alpha\left[y(t)\right] = ay(t)\left[K - y(t)\right], \quad y(t_0) = y_0, \quad (6)$$

where $\alpha \in (0, 1]$, $a, K > 0$ and G_T^α represents the conformable fractional derivative. The author obtained the analytical solution of equation (6) in the following form

$$y(t) = \frac{Ky_0}{(K - y_0)\exp\left(-aKJ_{T,t_0}^\alpha(1)(t)\right) + y_0}.$$

In 2021, Nieto [1] considered the initial value problem of the following fractional logistic differential equation via local fractional derivative

$$\mathbf{D}^\alpha\left[x(t)\right] = x(t)[1 - x(t)], \quad t > 0, \quad x(0) = x_0, \quad (7)$$

where $\alpha \in (0,1]$, $x_0 > 0$, $\mathbf{D}^\alpha$ is Caputo-Fabrizio fractional derivative. The author obtained the analytical solution of equation (7) in implicit form given as follows

$$\frac{x(t)-x^2(t)}{[1-x(t)]^{2/\alpha}} = \frac{x_0 - x_0^2}{(1-x_0)^{2/\alpha}} \exp(t).$$

Motivated by the above literature, we observe that most studies on fractional logistic differential equations assume a constant intrinsic growth rate of the population. Therefore, in this paper, we explore a different type of fractional logistic differential equation that involves a non-singular kernel (local fractional derivative), where the intrinsic growth rate is a function of time t. Specifically, we consider the following initial value problem of a fractional logistic differential equation

$$\mathrm{D}^\alpha x(t) = r(t)x(t)\Big[1 - \frac{x(t)}{K}\Big], \quad t>0, \quad x(0)=x_0, \quad (8)$$

where $\alpha \in (0,1]$, D^α represents deformable fractional derivative, $r(t)$ is a continuous function, $x_0 > 0$. Through analysis, we get the analytical solution of equation (8) in the following explicit form

$$x(t) = \frac{Kx_0 \exp(h(t))}{x_0 \exp(h(t)) + K - x_0 + \dfrac{x_0\beta}{\alpha}\displaystyle\int_0^t \exp(h(s))ds}, \quad (9)$$

where $\alpha + \beta = 1$, $h(t) = \displaystyle\int_0^t \frac{r(s)-\beta}{\alpha} ds$. The fractional logistic model (8) contains the following problems as its special case

- If $\alpha \to 1$, then the equation (8) degenerates into the following classical logistic differential equation initial value problem
$$x'(t) = r(t)x(t)\Big[1 - \frac{x(t)}{K}\Big], \quad t>0, \quad x(t)=x_0. \quad (10)$$
It follows from (9) that the logistic differential equation (10) possesses an analytical solution in the following explicit form
$$x(t) = \frac{Kx_0}{x_0 + (K-x_0)\exp\Big(-\displaystyle\int_0^t r(s)ds\Big)}.$$
- If $\alpha \to 1$, and setting $r(t) \equiv r$ (r is a constant), then equation (8) degenerates into the initial value problem of the logistic differential equation (2), and the equation (9) can degenerate into (3).

Finally, by utilizing the python, we plotted numerical simulation images for equation (9) with different values of α. The novelty of this paper is manifested as follows:

(1) In population ecology, considering variable intrinsic growth rates is essential due to the phenomenon known as the Allee effect. In natural environments, resource availability can fluctuate, such as due to seasonal changes or environmental pollution. If we only consider constant intrinsic growth rates, we cannot adequately reflect the impact of these changes on populations. Environmental conditions may also vary across different regions or over time. Variable intrinsic growth rates allow us to more accurately describe population growth under diverse environmental conditions. Furthermore, interactions and changes within ecosystems influence population dynamics. Therefore, considering variable intrinsic growth rates is crucial for understanding population growth and adaptability within ecosystems. Hence, the research presented in this paper holds significant importance.

(2) This paper extends the fractional logistic model with constant intrinsic growth rates considered in existing literature [1, 10–19] to a variable-coefficient fractional logistic model. By analysis, we obtain an analytical solution. Firstly, this paper generalizes the existing models, forming a new class of problems. Secondly, this extension has a certain biological significance, beyond mere mathematical research. Therefore, this paper is novel.

II. Preliminaries

In this section, we recall some fundamental definitions of deformable fractional calculus and related results, for more details we refer readers to [20] and references therein.

Definition 1. The deformable fractional integral of order α for a function $x \in C([a,b],\mathbf{R})$ is defined as

$$\mathrm{I}_a^\alpha x(t) = \frac{1}{\alpha}\exp\Big(-\frac{\beta}{\alpha}t\Big)\int_a^t \exp\Big(\frac{\beta}{\alpha}s\Big)x(s)ds, \quad t\in[a,b],$$

where $\alpha+\beta=1$, $\alpha\in(0,1]$.

Definition 2. The deformable fractional derivative of order α for a function $x:[a,b]\to\mathbf{R}$ is defined as

$$\mathrm{D}^\alpha x(t) = \lim_{\varepsilon\to 0}\frac{(1+\varepsilon\beta)x(t+\varepsilon\alpha)-x(t)}{\varepsilon}, \quad t\in(a,b),$$

if the right limit exists, where $\alpha+\beta=1$, $\alpha\in(0,1]$, we say that x is α-differentiable at t.

Remark 1. If $\alpha=1$, $\beta=0$, the deformable fractional derivative of the function will degenerate into the classical first-order derivative.

Lemma 1. Let x be a real valued function on $[a,b]$,

(i) If $x\in C[a,b]$, then $\mathrm{I}_a^\alpha(x)$ is α-derivative on (a,b) and
$$\mathrm{D}^\alpha(\mathrm{I}_a^\alpha x)(t) = x(t).$$

(ii) If x is α-differentiable, then
$$\mathrm{I}_a^\alpha(\mathrm{D}^\alpha)x(t) = x(t) - \exp\Big(\frac{\beta}{\alpha}(a-t)\Big)x(a).$$

III. Main results

In this section, we define Banach space $X = C^1[a,b]$. If $x\in X$ is the solution of equation (8), then we have:

$$\mathrm{I}_0^\alpha \mathrm{D}_0^\alpha x(t) = \mathrm{I}_0^\alpha r(t)x(t)\Big[1-\frac{x(t)}{K}\Big].$$

By using Lemma 1 (ii), one has

$$\begin{aligned} & x(t) - \exp\Big(-\frac{\beta}{\alpha}t\Big)x(0) \quad (11)\\ &= \frac{1}{\alpha}\exp\Big(-\frac{\beta}{\alpha}t\Big)\int_0^t \exp\Big(\frac{\beta}{\alpha}s\Big)r(s)x(s)\Big[1-\frac{x(s)}{K}\Big]ds. \end{aligned}$$

Multiplying both sides of equation (11) by $\exp(\beta t/\alpha)$, we obtain

$$\begin{aligned} & \exp\Big(\frac{\beta}{\alpha}t\Big)x(t) - x(0) \quad (12)\\ &= \frac{1}{\alpha}\int_0^t \exp\Big(\frac{\beta}{\alpha}s\Big)r(s)x(s)\Big[1-\frac{x(s)}{K}\Big]ds. \end{aligned}$$

Taking derivative to the both sides of equation (12) with respect to t, we derive

$$\frac{\beta}{\alpha}\exp\Big(\frac{\beta}{\alpha}t\Big)x(t)+\exp\Big(\frac{\beta}{\alpha}t\Big)x'(t) = \frac{1}{\alpha}\exp\Big(\frac{\beta}{\alpha}t\Big)r(t)x(t)\Big[1-\frac{x(t)}{K}\Big].$$

It then follows that

$$\frac{\beta}{\alpha}x(t)+x'(t)=\frac{1}{\alpha}r(t)x(t)\Big[1-\frac{x(t)}{K}\Big]. \quad (13)$$

Rearranging the equation (13) gives

$$x'(t)+\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]x(t)=-\frac{1}{\alpha K}r(t)x^2(t). \quad (14)$$

Let $y(t)=x^{-1}(t)$, then

$$y'(t)=-x^{-2}(t)x'(t).$$

Equation (14) becomes a linear equation

$$y'(t)-\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]y(t)=\frac{1}{\alpha K}r(t). \quad (15)$$

Considering the homogeneous equation corresponding to equation (15) by

$$y'(t)-\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]y(t)=0. \quad (16)$$

Solving equation (16) yields

$$y(t)=\tilde{C}\exp\Big\{\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\},\quad \tilde{C}\in\mathbb{R}.$$

Using the method of variation of constants, let

$$y(t)=\tilde{C}(t)\exp\Big\{\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\}$$

be the solution to equation (15), then

$$y'(t)=\tilde{C}'(t)\exp\Big\{\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\} + \tilde{C}(t)\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]\exp\Big\{\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\}.$$

Substituting $y(t)$ and $y'(t)$ into equation (15), we obtain

$$\tilde{C}'(t)\exp\Big\{\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\}=\frac{1}{\alpha K}r(t). \quad (17)$$

Solving equation (17), we get

$$\tilde{C}(t)=\int\frac{1}{\alpha K}r(t)\exp\Big\{-\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\}dt+C,\ \ C\in\mathbb{R}.$$

Therefore, we obtain the solution to equation (14) as follows

$$\frac{1}{x(t)}=\exp\Big\{\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\} \times\Big[\int\frac{1}{\alpha K}r(t)\exp\Big\{-\int\Big[\frac{\beta}{\alpha}-\frac{1}{\alpha}r(t)\Big]dt\Big\}dt+C\Big]. \quad (18)$$

Substituting the initial condition $x(0)=x_0$ into equation (18), we obtain

$$x(t)=\frac{Kx_0\exp(h(t))}{x_0\exp(h(t))+K-x_0+\dfrac{x_0\beta}{\alpha}\displaystyle\int_0^t\exp(h(s))ds}. \quad (19)$$

Therefore, we have obtained the solution for fractional logistic equation (8).

Corollary 1. Consider the following fractional logistic differential equation

$$\begin{cases} \mathrm{D}^{\alpha}x(t)=tx(t)[1-2x(t)], & t>0,\\ x(0)=1/2, \end{cases} \quad (20)$$

then example (20) has analytic solution

$$x(t)=\frac{\alpha\exp(g(t))}{\alpha\exp(g(t))+\alpha+(1-\alpha)\displaystyle\int_0^t\exp(g(s))ds}, \quad (21)$$

where $g(t)=\dfrac{t^2-2(1-\alpha)t}{2\alpha}$.

Proof. Corresponding to problem (8), let

$$r(t)=t,\quad K=1,\quad x_0=\frac{1}{2}.$$

Then, by using (19) the problem (20) has a solution (21).

Corollary 2. Consider the following fractional logistic differential equation

$$\begin{cases} \mathrm{D}^{\alpha}x(t)=2^{\alpha}x(t)[1-x(t)], & t>0,\\ x(0)=1/2, \end{cases} \quad (22)$$

then (19) has analytic equation

$$x(t)=\frac{\exp(p(t))}{\exp(p(t))+1+\dfrac{1-\alpha}{2^{\alpha}-1+\alpha}(\exp(p(t))-1)}, \quad (23)$$

where $p(t)=\dfrac{2^{\alpha}-1+\alpha}{\alpha}t$.

Proof. Corresponding to equation (8), let

$$r(t)=2^{\alpha},\quad K=1,\quad x_0=\frac{1}{2}.$$

Hence, by applying (19) the problem (22) has a solution (23).

Corollary 3. Consider the following fractional logistic differential equation

$$\begin{cases} \mathrm{D}^{\alpha}x(t)=x(t)[1-x(t)], & t>0,\\ x(0)=1/2, \end{cases} \quad (24)$$

then (24) has analytic equation

$$x(t)=\frac{\exp(t)}{\exp(t)+1+\dfrac{1-\alpha}{\alpha}(\exp(t)-1)}, \quad (25)$$

Proof. Corresponding to equation (8), let

$$r(t)=1,\quad K=1,\quad x_0=\frac{1}{2}.$$

Hence, by applying (19) the problem (24) has a solution (25).

IV. Numerical simulation

We now plot the solutions of (20), (22) and (24) for different values of α.

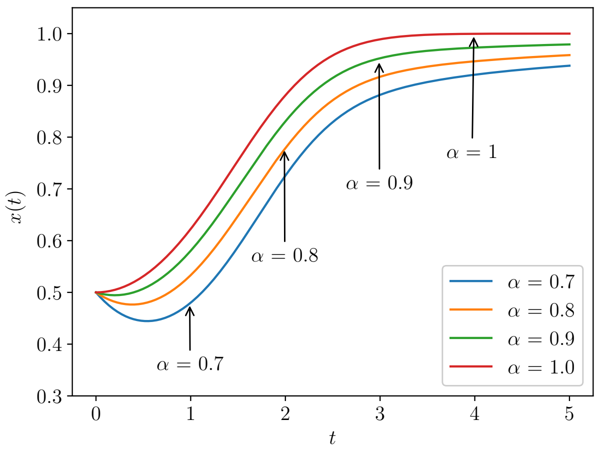

Fig. 1. Explain the solutions of the logistic differential equations (20).

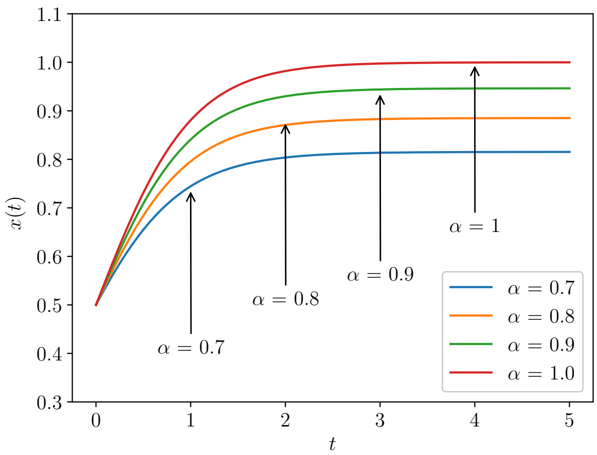

Fig. 2. Explain the solutions of the logistic differential equations (22).

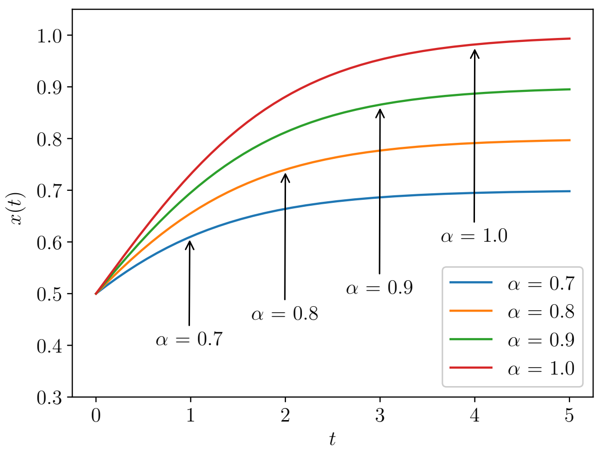

Fig. 3. Explain the solutions of the logistic differential equations (24).

V. CONCLUSION

This study discussed a class of deformable fractional logistic differential equations characterized by time-varying intrinsic growth rates. Employing analytical methodologies, we derive an analytical solution and furnish numerical simulations to corroborate our findings. Our result extends the existing literature on fractional logistic differential equations with constant intrinsic growth rates to include models with variable coefficient intrinsic growth rates. Furthermore, as $\alpha \to 1$ and $r(t) \equiv r$ (a constant), our findings converge to the results of the classical integer-order logistic differential equation. Consequently, our research has significant implications for the field. Taking into account the background of the logistic model in practical applications as well as that of fractional calculus, we can explore following fractional logistic model for mosquito population suppression based on Wolbachia-infected mosquitoes in the future

$$\begin{cases} \mathbf{D}^{\alpha}x(t) = ax(t) - (\mu + \xi x(t))x(t), \quad t > 0, \\ x(0) = x_0, \end{cases}$$

where $\alpha \in (0,1]$, $x_0 > 0$, $a, \mu, \xi \in \mathbf{R}^+$, $\mathbf{D}^{\alpha}$ is Caputo-Fabrizio fractional derivative.

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APPENDIX

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad
from matplotlib import font_manager


plt.rc('text',usetex=True)
plt.rcParams['text.latex.preamble'] =
    r'\usepackage{amsmath}'
plt.rcParams['pgf.preamble'] =
    r'\usepackage{amsmath}'


def integrand(s, alpha):
  return np.exp((s ** 2 - 2 * (1 - alpha) *
      s) / (2 * alpha))


def function_1(t, alpha):
  numerator = alpha * np.exp((t ** 2 - 2 *
      (1 - alpha) * t) / (2 * alpha))
  integral, _ = quad(integrand, 0, t,
      args=(alpha,))
  denominator = alpha * np.exp((t ** 2 - 2 *
      (1 - alpha) * t) / (2 * alpha)) +
      alpha + (1 - alpha) * integral

  return numerator / denominator


def function_2(t, alpha):
  numerator = np.exp(((2 ** alpha) - 1 +
      alpha) / (alpha * t))
  denominator = np.exp(((2 ** alpha) - 1 +
      alpha) / alpha * t) + 1 + ((1 - alpha)
      / (2 ** alpha - 1 + alpha)) * (
           np.exp(((2 ** alpha) - 1 + alpha)
               / alpha * t) - 1)

  return numerator / denominator


def function_3(t, alpha):
  numerator = (np.exp(t))
  denominator =
      (np.exp(t)+1+((1-alpha)/alpha)*(np.exp(t)-1))

  return return numerator / denominator
```