Stability and Bifurcation Analysis of Commensal Symbiosis System with the Allee Effect and Single Feedback Control

Lili Xu, Yalong Xue, Qifa Lin, and Fengde Chen

Abstract—**The commensal symbiosis system with the Allee effect and single feedback control is proposed and analyzed in this paper. The stability analysis of all possible equilibrium points is discussed, and the sufficient conditions for global stability of the interior equilibrium points are obtained. The occurrence of transcritical bifurcation and saddle-node bifurcation around the equilibrium points is investigated. Finally, the main results of the model are illustrated by numerical simulations.**

Index Terms—**single feedback control, Allee effect, commensalism model, transcritical bifurcation, saddle-node bifurcation**

I. Introduction

BIODIVERSITY is the basis of human survival and development, providing a wide range of necessities, a safe and reliable ecological environment, and a unique landscape culture. However, in many areas, due to human activities and the influence of the ecological environment, many species are extinct or on the verge of extinction, resulting in a severe decline in biodiversity. We have to face problems such as the loss of biodiversity and the decline of ecosystem function and stability. Therefore, we must control the ecosystem, which is reflected in the biological mathematical model by adding feedback control variables. We hope to protect endangered species through appropriate feedback control so that resources can be rationally developed, thus ensuring the sustainable development of the ecosystem.

In the natural environment, interaction between populations is very common. Commensal symbiosis is one of these interactions in which one group benefits while the other is not affected. In 1876, the Belgian zoologist Pierre Beneden put the phenomenon of commensalism in a biological context for the first time [1]. Although the phenomenon of commensal symbiosis is often observed in nature, it was not until 2003 that Sun [2] established a mathematical model of commensal symbiosis. The dynamic behavior of commensal symbiosis has become an essential topic for biologists and mathematicians. This kind of model has substantial theoretical and practical exploration value. In recent years, scholars have conducted a series of studies on the dynamic behavior of commensal symbiosis system([3]-[20],[30]-[36]).

In the context of the commensal symbiosis model, work on the influence of feedback control is relatively rare ([16]-[20]). Han [16] proposed and studied the following Lotka-Volterra commensal symbiosis model with feedback controls:

$$\begin{aligned} \frac{dx}{dt} &= x(b_1 - a_{11}x + a_{12}y - \alpha_1\mu), \\ \frac{dy}{dt} &= y(b_2 - a_{22}y - \alpha_2\nu), \\ \frac{d\mu}{dt} &= -\eta_1\mu_1 + a_1x, \\ \frac{d\nu}{dt} &= -\eta_2\mu_2 + a_2y, \end{aligned} \tag{1}$$

By constructing a suitable Lyapunov function, it is proven that the positive equilibrium of the system is globally stable.

Xu [18] studied the dynamic behavior of the commensal symbiosis system with both the Allee effect and feedback control based on Han:

$$\begin{aligned} \frac{dx}{dt} &= x(b_1 - a_{11}x - \frac{\alpha}{x+\gamma}) + a_{12}xy - c_1x\mu, \\ \frac{dy}{dt} &= y(b_2 - a_{22}y) - c_2y\nu, \\ \frac{d\mu}{dt} &= -p_1\mu + q_1x, \\ \frac{d\nu}{dt} &= -p_2\nu + q_2y, \end{aligned} \tag{2}$$

Research has shown that the Allee effect does not affect the stability of the equilibrium point of the feedback control based commensalism system. However, species with the Allee effect can reach equilibrium only when the population is large.

As we can see, the above system contains two or more feedback control variables, which means different control strategies are adopted for different species. However, strategies applicable to one species in the real world may also affect other species, meaning such strategies significantly impact both species. Let's give a few examples: spraying pesticides can effectively reduce the number of weeds, but it can also have a negative impact on the growth of crops or beneficial organisms [21]. When chemotherapy treatments are used to treat cancer patients, the number of cancer cells will rapidly decline, but at the same time, the drugs will harm healthy cells and the regulatory immune function of the body [22]. These examples demonstrate that the study of a single feedback control variable has substantial theoretical and applied usefulness. However, there is still relatively little research on population ecosystem dynamics with a single feedback control [23]-[28]. And most of them only studied

Manuscript received September 13, 2023; revised June 18, 2024. This work was supported by the Scientific Research Project of Ningde Normal University(2021Q202, 2022ZX415), the Natural Science Foundation of Fujian Province (2021J011155,FJ2021B031).

L. Xu is an associate professor at the College of Mathematics and Physics, Ningde Normal University, Ningde, China (e-mail: xll@ndnu.edu.cn).

Y. Xue is a lecturer at the College of Mathematics and Physics, Ningde Normal University, Ningde, China (e-mail: xueyalong@ndnu.edu.cn).

Q. Lin is an associate professor at the College of Mathematics and Physics, Ningde Normal University, Ningde, China (e-mail: lqfnd_118@163.com).

F. Chen is a professor at the College of Mathematics and Statistics, Fuzhou University, Fuzhou, China (e-mail:fdchen@fzu.edu.cn).

issues such as permanence, stability, and extinction of the system.

According to the aforementioned, on the basis of model (2), we change the feedback variable into a single case:

$$\begin{aligned}\frac{dx}{dt} &= x(1-x-\frac{m}{x+\alpha})+axy-bx\mu,\\ \frac{dy}{dt} &= y(1-cy)-dy\mu,\\ \frac{d\mu}{dt} &= -\mu+px+qy,\end{aligned} \tag{3}$$

where $a, b, c, d, p, q, and\alpha$ are all positive constants. $xandy$ represent the densities of the two populations at time t, respectively. μ is a feedback control variable, parameters b and d respectively describe the degree of influence of feedback control variables on populations x and y. $\frac{m}{x+\alpha}$ indicates the Allee effect.

It is the first time that a model has considered both the Allee effect and a single feedback control variable. Is a single feedback control variable different from multiple feedback control variables in terms of its effects on the stability of the system? Does bifurcation occur at the equilibrium point? The purpose of this paper is to study the dynamic behavior of system (3). In particular, we will find out the answers to the above questions.

II. Existence of the Equilibria

The equilibria of system (3) are the set of non-negative solutions to the following simultaneous equations.

$$\begin{aligned}x(1-x-\frac{m}{x+\alpha})+axy-bx\mu &= 0,\\ y(1-cy-d\mu) &= 0,\\ -\mu+px+qy &= 0.\end{aligned} \tag{4}$$

Notably, the system (3) has the boundary equilibrium point $E_0(0,0,0)$ for all parameters. As for other equilibrium points, we will discuss them as follows.

Case 1. When $x=0$, then

$$y=\frac{1}{c+dq}, \mu=\frac{q}{c+dq},$$

the system (3) has the boundary equilibrium point

$$E_1(0, \frac{1}{c+dq}, \frac{q}{c+dq}).$$

Case 2. When $y=0, \mu=0$, from equation(4), x is nonnegative root of the equation:

$$x^2+(\alpha-1)x+(m-\alpha)=0. \tag{5}$$

Let Δ_1 denote the discriminant of equation (5), then

$$\Delta_1=(1+\alpha)^2-4m,$$

and

$$\Delta_1 \geq 0 \Leftrightarrow m \leq \frac{(1+\alpha)^2}{4}.$$

Thereout, let

$$x_{10}=\frac{1-\alpha}{2}, x_{11}=\frac{1-\alpha+\sqrt{\Delta_1}}{2}, x_{12}=\frac{1-\alpha-\sqrt{\Delta_1}}{2}.$$

So, we can get the following results.

(a) If $m<\alpha$, the system (3) has boundary equilibrium point $E_{11}(x_{11},0,0)$.

(b) If $\alpha<m<\frac{(1+\alpha)^2}{4}$, and $\alpha<1$, the system (3) has two boundary equilibrium points $E_{11}(x_{11},0,0)$, and $E_{12}(x_{12},0,0)$.

(c) If $m=\frac{(1+\alpha)^2}{4}$, and $\alpha<1$, the system (3) has boundary equilibrium point $E_{10}(x_{10},0,0)$.

Case 3. When $y=0$, from equation(4), x is the nonnegative root of the equation

$$(1+bp)x^2+(\alpha bp+\alpha-1)x+m-\alpha=0. \tag{6}$$

Let

$$A_2=1+bp, B_2=\alpha bp+\alpha-1, C_2=m-\alpha,$$

Δ_2 denote the discriminant of equation (6), then

$$\Delta_2=(\alpha+\alpha bp+1)^2-4m(1+bp).$$

Make $\Delta_2=0$, we have

$$\begin{aligned}m_1 &\equiv \frac{(\alpha bp+\alpha+1)^2}{4(1+bp)}\\ &= \frac{\alpha^2(bp+1)}{4}+\frac{1}{4(bp+1)}+\frac{\alpha}{2}\\ &\geq \frac{\alpha}{2}+\frac{\alpha}{2}=\alpha.\end{aligned}$$

Thereout, let

$$x_{20}=\frac{-B_2}{2A_2}, x_{21}=\frac{-B_2+\sqrt{\Delta_2}}{2A_2}, x_{22}=\frac{-B_2-\sqrt{\Delta_2}}{2A_2}.$$

So, we can get the following results.

(a) If $m<\alpha$, the system (3) has boundary equilibrium point $E_{21}(x_{21},0,px_{21})$.

(b) If $\alpha<m<m_1$, and $\alpha<\frac{1}{1+bp}$, the system (3) has two boundary equilibrium points $E_{21}(x_{21},0,px_{21})$, $E_{22}(x_{22},0,px_{22})$.

(c) If $m=m_1$, and $\alpha<\frac{1}{1+bp}$, the system (3) has boundary equilibrium point $E_{20}(x_{20},0,px_{20})$.

Case 4. When

$$y=\frac{1-d\mu}{c},$$

and

$$\mu=\frac{cp}{c+dq}x+\frac{q}{c+dq}.$$

From equation(4), by simple calculation, x is the nonnegative root of the equation:

$$\begin{aligned}&(c+dq+adp+bcp)x^2\\ &+[(c+dq)(\alpha-1)+\alpha p(ad+bc)+bq-a]x\\ &+[(c+dq)(m-\alpha)+(bq-a)\alpha]=0.\end{aligned} \tag{7}$$

Let

$$\begin{aligned}A_3 &= c+dq+adp+bcp,\\ B_3 &= (c+dq)(\alpha-1)+\alpha p(ad+bc)+bq-a,\\ C_3 &= (c+dq)(m-\alpha)+(bq-a)\alpha,\end{aligned}$$

Δ_3 denote the discriminant of equation (7), then

$$\Delta_3=B_3^2-4A_3C_3.$$

Make $\Delta_3 = 0$, we have

$$\begin{aligned} m_2 &= \frac{B_3^2}{4A_3(c+dq)} + \frac{(a-bq)\alpha}{c+dq} + \alpha \\ &\geq \frac{(a-bq)\alpha}{c+dq} + \alpha \equiv m_0. \end{aligned}$$

In addition, $B_3 > 0$ is equivalent to

$$\alpha > \frac{a+c+dq-bq}{c+dq+adp+bcp} \equiv \alpha_0.$$

Thereout, let

$$x_{30} = \frac{-B_3}{2A_3}, x_{31} = \frac{-B_3+\sqrt{\Delta_3}}{2A_3}, x_{32} = \frac{-B_3-\sqrt{\Delta_3}}{2A_3}.$$

And

$$y_{3i} = -\frac{1}{c+dq}(dpx_{3i}-1), \mu_{3i} = \frac{1}{c+dq}(cpx_{3i}+q),$$

where $i = 0, 1, 2$.

So, we can get the following results.

(a) If $m < m_0$, the system (3) has a unique positive equilibrium point $E_{31}(x_{31}, y_{31}, \mu_{31})$.

(b) If $m_0 < m < m_2$, and $\alpha < \alpha_0$, the system (3) has two positive equilibrium points $E_{31}(x_{31}, y_{31}, \mu_{31})$ and $E_{32}(x_{32}, y_{32}, \mu_{32})$.

(c) If $m = m_2$, and $\alpha < \alpha_0$, the system (3) has a unique positive equilibrium point $E_{30}(x_{30}, y_{30}, \mu_{30})$.

III. Local Stability of Equilibria

In section 2, we established the conditions for all equilibrium points in system (3). Now we will discuss the local stability of each equilibrium point.

Theorem 3.1 In the presence of the equilibrium point of the system (3), their local stability is as follows:

(1) $E_0(0,0,0)$ is unstable.

(2) $E_1(0, \frac{1}{c+dq}, \frac{q}{c+dq})$ is locally stable if $m > m_0$ hold, and is unstable if $m < m_0$ hold .

(3) $E_{1i}(x_{1i}, 0, 0)(i = 0, 1, 2)$ is unstable.

(4) $E_{21}(x_{21}, 0, px_{21})$ is locally stable if $x_{21} > \frac{1}{dp}$ hold, and is unstable if $x_{21} < \frac{1}{dp}$ hold .

(5) $E_{22}(x_{22}, 0, px_{22})$ is unstable.

(6) $E_{3i}(x_{3i}, y_{3i}, \mu_{3i})(i = 0, 1, 2)$ is locally stable when $a < c$ and $m < (x_{3i} + \alpha)^2$ hold.

Proof. At any equilibrium point $E(x, y, \mu)$, the Jacobian matrix of the system (3) has the following structure.

$J(E) =$

$$\begin{pmatrix} (1-x-\frac{m}{x+\alpha})+ay & & \\ -b\mu+x(-1+\frac{m}{(x+\alpha)^2}) & ax & -bx \\ 0 & 1-2cy-d\mu & -dy \\ p & q & -1 \end{pmatrix}.$$

The Jacobian matrix of the system (3) at $E_0(0, 0, 0)$ is

$$J(E_0) = \begin{pmatrix} 1-\frac{m}{\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ p & q & -1 \end{pmatrix}.$$

Obviously, $\lambda_{01} = 1 - \frac{m}{\alpha}$, $\lambda_{02} = -1$, $\lambda_{03} = 1$. Note that $\lambda_{03} > 0$ is always true, so $E_0(0, 0, 0)$ is unstable.

Now, the Jacobian matrix of the system (3) at $E_1(0, \frac{1}{c+dq}, \frac{q}{c+dq})$ is

$$J(E_1) = \begin{pmatrix} (1-\frac{m}{\alpha})+\frac{a-bq}{c+dq} & 0 & 0 \\ 0 & \frac{-c}{c+dq} & \frac{-d}{c+dq} \\ p & q & -1 \end{pmatrix}.$$

By brief calculation we can get,

$\lambda_{11} = 1 - \frac{m}{\alpha} + \frac{a-bq}{c+dq}$,

$$\lambda_{12} = \frac{-(1+\frac{c}{c+dq})+\sqrt{(1+\frac{c}{c+dq})^2-4}}{2} < 0,$$

$$\lambda_{13} = \frac{-(1+\frac{c}{c+dq})-\sqrt{(1+\frac{c}{c+dq})^2-4}}{2} < 0.$$

If $m > (1 + \frac{a-bq}{c+dq})\alpha \equiv m_0$ hold , then $\lambda_{11} < 0$, E_1 is locally stable, if $m < m_0$ hold , then $\lambda_{11} > 0$, E_1 is unstable.

Further, it is possible to simplify the Jacobian matrix of the system (3) at $E_{1i}(x_{1i}, 0, 0)(i = 0, 1, 2)$ as follows.

$$J(E_{1i}) = \begin{pmatrix} x_{1i}(-1+\frac{m}{(x_{1i}+\alpha)^2}) & ax_{1i} & -bx_{1i} \\ 0 & 1 & 0 \\ p & q & -1 \end{pmatrix}.$$

The Jacobian matrix $J(E_{1i})$ have a characteristic value $\lambda_{12} = 1 > 0$, so $E_{1i}(x_{1i}, 0, 0)(i = 0, 1, 2)$ is unstable.

Next, the Jacobian matrix of the system (3) at $E_{2i}(x_{2i}, 0, px_{2i})$ is

$$J(E_{2i}) = \begin{pmatrix} x_{2i}(-1+\frac{m}{(x_{2i}+\alpha)^2}) & ax_{2i} & -bx_{2i} \\ 0 & 1-d\mu_{2i} & 0 \\ p & q & -1 \end{pmatrix}.$$

Then,

$$\begin{aligned} & \mid \lambda E_{2i} - A \mid \\ = & \begin{vmatrix} \lambda - x_{2i}(-1+\frac{m}{(x_{2i}+\alpha)^2}) & -ax_{2i} & bx_{2i} \\ 0 & \lambda-1+d\mu_{2i} & 0 \\ -p & -q & \lambda+1 \end{vmatrix} \\ = & [\lambda^2 + (1 - x_{2i}(-1+\frac{m}{(x_{2i}+\alpha)^2}))\lambda \\ & +[bpx_{2i} - x_{2i}(-1+\frac{m}{(x_{2i}+\alpha)^2})](\lambda-1+d\mu_{2i}) \end{aligned}$$

Since x_{2i} satisfies

$$1 - x_{2i} - \frac{m}{x_{2i}+\alpha} - bpx_{2i} = 0,$$

we have

$$\frac{m}{x_{2i}+\alpha} = 1 - x_{2i} - bpx_{2i}.$$

Let

$$\begin{aligned} T_1 &= 1 - x_{2i}(-1 + \frac{m}{(x_{2i}+\alpha)^2}) \\ &= \frac{1}{x_{2i}+\alpha}((2+bp)x_{2i}^2 + \alpha x_{2i} + \alpha) > 0. \\ T_2 &= bpx_{2i} - x_{2i}(-1 + \frac{m}{(x_{2i}+\alpha)^2}) \\ &= \frac{1}{x_{2i}+\alpha}(2(1+bp)x_{2i}^2 + (\alpha bp + \alpha - 1)x_{2i} \\ &= \frac{x_{2i}}{x_{2i}+\alpha}(2A_2x_{2i} + B_2), \end{aligned}$$

where $A_2 = 1 + bp, B_2 = \alpha bp + \alpha - 1$. When $x_{2i} = x_{20}$, we have $T_2 = 0$, so

$$\lambda_{21} = 0, \lambda_{22} = -T_1 < 0,$$

the system may bifurcate at E_{20}.

When $x_{2i} = x_{21}$, where $T_2 > 0$, so

$$\lambda_{21} = \frac{-T_1 + \sqrt{T_1^2 - 4T_2}}{2} < 0,$$

$$\lambda_{22} = \frac{-T_1 - \sqrt{T_1^2 - 4T_2}}{2} < 0.$$

If $dpx_{21} > 1$ is true, then we have $\lambda_{23} < 0$. Therefore, if and only if the following condition is true, E_{21} is locally asymptotically stable:

$$dpx_{21} > 1 \tag{8}$$

When $x_{2i} = x_{22}$,where $T_2 < 0$,

$$\lambda_{21} = \frac{-T_1 + \sqrt{T_1^2 - 4T_2}}{2} > 0,$$

so E_{22} is unstable.

Finally, the Jacobian matrix of the system (3) at $E_{3i}(x_{3i}, y_{3i}, \mu_{3i})(i = 0, 1, 2)$ can be simplified as follows:

$$\begin{aligned} J(E_{3i}) &= \begin{pmatrix} x_{3i}(-1 + \frac{m}{(x_{3i}+\alpha)^2}) & ax_{3i} & -bx_{3i} \\ 0 & -cy_{3i} & -dy_{3i} \\ p & q & -1 \end{pmatrix} \\ &= (h_{jk})(j, k = 1, 2, 3). \end{aligned}$$

The characteristic equation associated with $J(E_{3i})$ is given by

$$\lambda^3 + h_1\lambda^2 + h_2\lambda + h_3 = 0, \tag{9}$$

where

$$\begin{aligned} h_1 &= -(h_{11} + h_{22} + h_{33}), \\ h_2 &= h_{11}h_{22} + h_{11}h_{33} + h_{22}h_{33} - h_{13}h_{31} - h_{23}h_{32}, \\ h_3 &= -h_{11}(h_{22}h_{33} - h_{23}h_{32}) - h_{31}(h_{12}h_{23} - h_{13}h_{22}). \end{aligned}$$

According to the Routh-Hurwitz criterion, when $H_1 > 0, H_2 > 0$, and $H_3 > 0$, equation(9) has three roots with negative real parts. The straightforward calculation of H_1, H_2, H_3 is as follows.

$$\begin{aligned} H_1 &= h_1 \\ &= -(h_{11} + h_{22} + h_{33}) \\ &= x_{3i} - \frac{mx_{3i}}{(x_{3i}+\alpha)^2} + cy_{3i} + 1 \\ &= \frac{1}{x_{3i}+\alpha}[x_{3i}(2x_{3i} + b\mu_{3i} + \alpha + (c-a)y_{3i}) \\ &\quad +(cy_{3i}+1)\alpha], \\ H_2 &= h_1h_2 - h_3 = h_{11}D_1 + D_2, \end{aligned}$$

where,

$$\begin{aligned} D_1 &= h_{13}h_{31} - 2h_{22}h_{33} - h_{22}^2 - h_{33}^2 < 0, \\ D_2 &= -h_{11}^2h_{22} - h_{11}^2h_{33} - h_{22}^2h_{33} + h_{22}h_{23}h_{32} \\ &\quad -h_{22}h_{33}^2 + h_{13}h_{31}h_{33} + h_{23}h_{32}h_{33} + h_{12}h_{23}h_{31} \\ &\quad > 0. \\ H_3 &= h_3H_2. \end{aligned}$$

When $a < c$, we have $H_1 > 0$. When $h_{11} < 0$, that is $m < (x_{3i} + \alpha)^2$, we have $H_2 > 0$,and $h_3 > 0$, so $H_3 > 0$.

Based on the above analysis, according to the Hurwitz criterion, when $a < c$ and $m < (x_{3i} + \alpha)^2$ holds, $E_{3i}(i = 0, 1, 2)$ is locally asymptotically stable.

IV. Global Stability of Equilibria

In this section, we discuss the global stability of the interior equilibrium point $E_{3i}(x_{3i}, y_{3i}, \mu_{3i})(i = 0, 1, 2)$, by constructing the appropriate Lyapunov function.

Theorem 4.1 If the interior equilibrium $E^*(x^*, y^*, z^*)$ exists, assume that

$$m < (1 - \frac{a^2dp}{4bcq})\alpha^2 \tag{10}$$

hold, then $E_{3i}(x_{3i}, y_{3i}, \mu_{3i})(i = 0, 1, 2)$ is globally asymptotically stable.

Proof. Let's consider the Lyapunov function

$$\begin{aligned} V(t) &= \delta_1(x - x^* - x^*ln\frac{x}{x^*}) + \delta_2(y - y^* - y^*ln\frac{y}{y^*}) \\ &\quad +\delta_3(\mu - \mu^*)^2. \end{aligned}$$

We calculate the derivative of $V(t)$ along the positive solution E_{3i} of system (3),we have

$$\begin{aligned} \frac{dV}{dt} &= \delta_1(x - x^*)(1 - x - \frac{m}{x+\alpha} + ay - b\mu) \\ &\quad +\delta_2(y - y^*)(1 - cy - du) \\ &\quad +2\delta_3(\mu - \mu^*)(-\mu + px + qy) \\ &= \delta_1(x - x^*)(-(x - x^*) + \frac{m(x - x^*)}{(x+\alpha)(x^*+\alpha)} \\ &\quad +a(y - y^*) - b(\mu - \mu^*)) \\ &\quad -\delta_2(y - y^*)(c(y - y^*) + d(\mu - \mu^*)) \\ &\quad +2\delta_3(\mu - \mu^*)(-(\mu - \mu^*) \end{aligned}$$

$$
\begin{aligned}
&+p(x-x^*)+q(y-y^*))\\
=\;& -(1-\frac{m}{(x+\alpha)(x^*+\alpha)})\delta_1(x-x^*)^2\\
&+a\delta_1(x-x^*)(y-y^*)-c\delta_2(y-y^*)^2\\
&+(2p\delta_3-b\delta_1)(x-x^*)(\mu-\mu^*)\\
&+(2q\delta_3-d\delta_2)(y-y^*)(\mu-\mu^*)-2\delta_3(\mu-\mu^*)^2\\
\leq\;& -[(1-\frac{m}{\alpha^2})\delta_1(x-x^*)^2-a\delta_1(x-x^*)(y-y^*)\\
&-(2p\delta_3-b\delta_1)(x-x^*)(\mu-\mu^*)+c\delta_2(y-y^*)^2\\
&-(2q\delta_3-d\delta_2)(y-y^*)(\mu-\mu^*)+2\delta_3(\mu-\mu^*)^2]\\
=\;& -(x-x^*,y-y^*,\mu-\mu^*)\cdot\\
&\begin{pmatrix} (1-\frac{m}{\alpha^2})\delta_1 & -\frac{a}{2}\delta_1 & \frac{b}{2}\delta_1-p\delta_3\\ -\frac{a}{2}\delta_1 & c\delta_2 & \frac{d}{2}\delta_2-q\delta_3\\ \frac{b}{2}\delta_1-p\delta_3 & \frac{d}{2}\delta_2-q\delta_3 & 2\delta_3 \end{pmatrix}\cdot\\
&\begin{pmatrix} x-x^*\\ y-y^*\\ \mu-\mu^* \end{pmatrix}
\end{aligned}
$$

Let

$$
\begin{aligned}
&\mid B\mid\\
=\;&\begin{vmatrix} (1-\frac{m}{\alpha^2})\delta_1 & -\frac{a}{2}\delta_1 & \frac{b}{2}\delta_1-p\delta_3\\ -\frac{a}{2}\delta_1 & c\delta_2 & \frac{d}{2}\delta_2-q\delta_3\\ \frac{b}{2}\delta_1-p\delta_3 & \frac{d}{2}\delta_2-q\delta_3 & 2\delta_3 \end{vmatrix}\\
=\;& 2c(1-\frac{m}{\alpha^2})\delta_1\delta_2\delta_3-\frac{a^2}{2}\delta_1^2\delta_3\\
&-c\delta_2(\frac{b}{2}\delta_1-p\delta_3)^2\\
&-(1-\frac{m}{\alpha^2})\delta_1(\frac{d}{2}\delta_2-q\delta_3)^2\\
&-a\delta_1(\frac{d}{2}\delta_2-q\delta_3)(\frac{b}{2}\delta_1-p\delta_3).
\end{aligned}
$$

We select

$$\delta_1=\frac{2p}{b},\ \ \delta_2=\frac{2q}{d},\ \ \delta_3=1,$$

then

$$\mid B\mid=[2c(1-\frac{m}{\alpha^2})\delta_2-\frac{a^2}{2}\delta_1]\delta_1\delta_3.$$

When

$$m<(1-\frac{a^2dp}{4bcq})\alpha^2,$$

we have $\mid B\mid>0$, then B is positive definite, so $\frac{dV}{dt}\leq 0$. $\frac{dV}{dt}=0$ if and only if $x=x^*,y=y^*,\mu=\mu^*$.

Therefore, when

$$m<(1-\frac{a^2dp}{4bcq})\alpha^2$$

holds, as long as the equilibrium point $E_{3i}(x_{3i},y_{3i},z_{3i})$ $(i=0,1,2)$ exists, it is globally asymptotically stable.

This completes the proof of Theorem 4.1.

V. Bifurcation Analysis

Now, we discuss the existence of bifurcation around the equilibriums by taking the parameter m as the bifurcation parameter and keeping other parameters fixed. In this section, we use Sotomayor's bifurcation theorem to study the occurrence of bifurcation and specify the type of bifurcation.

Theorem 5.1 If $m\neq\alpha^2(1+bp)$ holds, the system (3) experiences a transcritical bifurcation at the trivial equilibrium $E_0(0,0,0)$ as the parameter m passes through the bifurcation value $m\equiv m^*=\alpha$.

Proof. Note that when $m=m^*$, it is possible to write the Jacobian matrix of system (3) at E_0 as

$$J(E_0,m^*)=\begin{pmatrix} 0 & 0 & 0\\ 0 & 1 & 0\\ p & q & -1 \end{pmatrix}.$$

Obviously, $\lambda_{01}=0,\ \lambda_{02}=-1,\ \lambda_{03}=1.$

Let

$$V_1=(v_1,v_2,v_3)^T,$$

and

$$W_1=(w_1,w_2,w_3)^T,$$

be the eigenvectors of $J(E_0,m^*)$ and $J^T(E_0,m^*)$ that correspond to the eigenvalue $\lambda_{01}=0$, respectively. By simple calculation, we get

$$V_1=(v_1,0,pv_1)^T, W_1=(w_1,0,0)^T,$$

where v_1, w_1 represents any nonzero real number.

Let

$$F=(F_1,F_2,F_3)^T,$$

where

$$
\begin{aligned}
F_1 &= x(1-x-\frac{m}{x+\alpha})+axy-bx\mu,\\
F_2 &= y(1-cy),\\
F_3 &= -\mu+\beta x.
\end{aligned}
$$

Then

$$F_m=\frac{\partial F}{\partial m}=(-\frac{x}{x+\alpha},0,0)^T,$$

we can obtain that $F_m(E_0,m^*)=(0,0,0)^T$. So, we have

$$W_1^T[F_m(E_0,m^*)]=0. \tag{11}$$

Moreover,

$$
\begin{aligned}
DF_m(E_0,m^*) &= \begin{pmatrix} -\frac{\alpha}{(x+\alpha)^2} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}_{(0,0,0)}\\
&= \begin{pmatrix} -\frac{1}{\alpha} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

we have

$$W_1^T[DF_m(E_0,m^*)V_1]=-\frac{w_1v_1}{\alpha}\neq 0. \tag{12}$$

Let

$$\begin{aligned}\gamma_1 &= V^T D^2 F_1(E_0) V \\ &= (v_1, 0, pv_1)\begin{pmatrix} -2+\frac{2m}{\alpha^2} & a & -b \\ a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ 0 \\ pv_1 \end{pmatrix} \\ &= -2(1-\frac{m}{\alpha^2}+bp)v_1^2,\end{aligned}$$

similarly, we can obtain

$$\gamma_2 = V^T D^2 F_2(E_0) V = 0,$$

$$\gamma_3 = V^T D^2 F_3(E_0) V = 0.$$

So we have

$$\begin{aligned} & W_1^T[D^2F(E_0, m^*)(V_1, V_1)] \\ = \; & (w_1, 0, 0)\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} \\ = \; & -2(1-\frac{m}{\alpha^2}+bp)v_1^2 w_1 \\ \neq \; & 0. \end{aligned} \quad (13)$$

Combining equations (11), (12), and (13), according to Sotomayor's theorem [29], when $m = \alpha$, the transcritical b-ifurcation occurs at E_0. This concludes the proof of Theorem 5.1.

Theorem 5.2 If

$$m \neq (1+\frac{p(ad+bc)}{c+dq})\alpha^2$$

holds, system (3) experiences a transcritical bifurcation at $E_1(0, \frac{1}{c+dq}, \frac{q}{c+dq})$ as the parameter m passes through the bifurcation value $m \equiv \overline{m} = m_0$.

Proof. The Jacobian matrix at E_1 with $m = \overline{m}$ is

$$J(E_1, \overline{m}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{c}{c+dq} & -\frac{d}{c+dq} \\ p & q & -1 \end{pmatrix}.$$

Obviously, $\lambda_{11} = 0, \quad \lambda_{12} = -1, \quad \lambda_{13} = -1.$

By straightforward computation, we get the corresponding eigenvector

$$V_2 = (\overline{v}_1, -\frac{dp}{c+dq}\overline{v}_1, \frac{cp}{c+dq}\overline{v}_1)^T,$$

to the eigenvector of $J(E_1, \overline{m})$, and the corresponding eigen-vector $W_2 = (\overline{w}_1, 0, 0)^T$ to the eigenvector of $J^T(E_1, \overline{m})$, that corresponds to the eigenvalue $\lambda_{11} = 0$. Where $\overline{v}_1$, and $\overline{w}_1$ represent any nonzero real number.

After a simple calculation, we obtain

$$F_m(E_1, \overline{m}) = (0,0,0)^T, W_2^T[F_m(E_1, \overline{m})] = 0.$$

Moreover,

$$\begin{aligned} & W_2^T[DF_m(E_1, \overline{m})V_2] = -\frac{\overline{v}_1\overline{w}_1}{\alpha} \neq 0, \\ & W_2^T[D^2F(E_1, \overline{m})(V_2, V_2)] \\ & = -2(1-\frac{m}{\alpha^2}+\frac{p(ad+bc)}{c+dq})\overline{v}_1^2\overline{w}_1 \neq 0. \end{aligned}$$

By Sotomayor's theorem, when $m = m_0$, the transcritical bifurcation occurs at E_1.

This completes the proof of Theorem 5.2.

Theorem 5.3 If $m = m_1$, system (3) undergoes a saddle-node bifurcation at $E_{20}(x_{20}, 0, px_{20})$.

Proof. Note that when $m \equiv \widetilde{m} = m_1$, E_{20} exist. In this connection, $\lambda_{21} = 0$. Then it is possible to write the Jacobian matrix of system (3) at E_{20} as

$$J(E_{20}, \widetilde{m}) = \begin{pmatrix} -x_{20} + \frac{\widetilde{m}x_{20}}{(x_{20}+\alpha)^2} & ax_{20} & -bx_{20} \\ 0 & 1-dpx_{20} & 0 \\ p & q & -1 \end{pmatrix}.$$

By straightforward computation, we get the corresponding eigenvector

$$V_3 = (\widetilde{v}_1, 0, p\widetilde{v}_1),$$

to the eigenvector of $J(E_{20}, \widetilde{m})$, and the corresponding eigenvector

$$W_3 = (\widetilde{w}_1, \frac{(a-bq)x_{20}}{dpx_{20}-1}\widetilde{w}_1, -bx_{20}\widetilde{w}_1)^T,$$

to the eigenvector of $J^T(E_{20}, \widetilde{m})$, that corresponds to the eigenvalue $\lambda_{21} = 0$. where $\widetilde{v}_1$, and $\widetilde{w}_1$ represent any nonzero real number.

By simple calculation, we can obtain that

$$F_m(E_{20}, \overline{m}) = (-\frac{x_{20}}{x_{20}+\alpha}, 0, 0)^T,$$

$$W_3^T[F_m(E_{20}, \widetilde{m})] = -\frac{x_{20}}{x_{20}+\alpha}\widetilde{w}_1 \neq 0,$$

$$W_3^T[DF_m(E_{20}, \widetilde{m})V_3] = -\frac{\alpha}{(x_{20}+\alpha)^2}\widetilde{v}_1\widetilde{w}_1 \neq 0,$$

$$W_3^T[D^2F(E_{20}, \widetilde{m})(V_3, V_3)] =$$

$$-2(1-\frac{m}{(x_{20}+\alpha)^2}+\frac{mx_{20}}{(x_{20}+\alpha)^3}+bp)\widetilde{v}_1^2\widetilde{w}_1 \neq 0.$$

According to Sotomayor's theorem, when $m = m_1$, the saddle-node bifurcation occurs at E_{20}.

This completes the proof of Theorem 5.3.

VI. The influence of the Allee effect and feedback control

The key question in this section is how the Allee effects and feedback control variables affect the dynamic behavior of populations. The following discussion provides the answer.

It can be seen from equation (3) that the value of parameter m reflects the strength of the Allee effect, while parameters b and d represent the influence of feedback control variables on population x and y, respectively.

We will first discuss the influence of the Allee effect on the positive equilibra.

Denote that

$$\begin{aligned} F_1(x^*, y^*, \mu^*, b, d, q) &= 1 - x^* - \frac{m}{x^*+\alpha} \\ &\quad + ay^* - b\mu^*, \\ F_2(x^*, y^*, \mu^*, b, d, q) &= 1 - cy^* - d\mu^*, \\ F_3(x^*, y^*, \mu^*, b, d, q) &= -\mu^* + px^* + qy^*. \end{aligned}$$

Then x^*, y^*, and μ^* satisfy the following equations:

$$\begin{aligned} F_1(x^*, y^*, \mu^*, b, d, q) &= 0, \\ F_2(x^*, y^*, \mu^*, b, d, q) &= 0, \\ F_3(x^*, y^*, \mu^*, b, d, q) &= 0. \end{aligned} \tag{14}$$

We have

$$\begin{aligned} J &= \frac{D(F_1, F_2, F_3)}{D(x^*, y^*, u^*)} \\ &= \begin{vmatrix} F_{1x^*} & F_{1y^*} & F_{1u^*} \\ F_{2x^*} & F_{2y^*} & F_{2u^*} \\ F_{3x^*} & F_{3y^*} & F_{3u^*} \end{vmatrix} \\ &= \begin{vmatrix} -1 + \dfrac{m}{(x^*+\alpha)^2} & a & -b \\ 0 & -c & -d \\ p & q & -1 \end{vmatrix} \\ &= (c + dq)(\frac{m}{(x^*+\alpha)^2} - 1) - (ad + bc)p. \end{aligned} \tag{15}$$

If we treat b, d, and q as variables, then x^*, y^*, and u^* can be expressed as functions of b, d, and q by equation (14): $x^* = x^*(b, d, q)$, $y^* = y^*(b, d, q)$, $u^* = u^*(b, d, q)$.
By calculating , we obtain

$$\begin{aligned} \frac{\partial x^*}{\partial m} &= -\frac{1}{J}\frac{D(F_1, F_2, F_3)}{D(m, y^*, u^*)} = -\frac{-(c+dq)\dfrac{1}{x^*+\alpha}}{J}, \\ \frac{\partial y^*}{\partial m} &= -\frac{1}{J}\frac{D(F_1, F_2, F_3)}{D(x^*, m, u^*)} = -\frac{dp\dfrac{1}{x^*+\alpha}}{J}, \\ \frac{\partial u^*}{\partial m} &= -\frac{1}{J}\frac{D(F_1, F_2, F_3)}{D(x^*, y^*, m)} = -\frac{-cp\dfrac{1}{x^*+\alpha}}{J}. \end{aligned}$$

From equation (15), it can be concluded that when

$$m < (1 + \frac{(ad+bc)p}{c+dq})(x^*+\alpha)^2 \equiv m^*,$$

we have $J < 0$, and when $m > m^*$, we have $J > 0$.
Furthermore, we can obtain, when $m < m^*$, we have

$$\frac{\partial x^*}{\partial m} < 0, \frac{\partial y^*}{\partial m} > 0, \frac{\partial u^*}{\partial m} < 0,$$

and when $m > m^*$, we have

$$\frac{\partial x^*}{\partial m} > 0, \frac{\partial y^*}{\partial m} < 0, \frac{\partial u^*}{\partial m} > 0.$$

This means that the Allee effect's influence parameter m on the system has a threshold value of m^*. When $m < m^*$, as m increases, the equilibrium point positions of population x and feedback control variable u decrease, and the equilibrium point positions of population y increase. However, when $m > m^*$, as m increases, the equilibrium point positions of population x and feedback control variable u increase, and the equilibrium point positions of population y decrease. In other words, the effect of parameter m on the equilibrium point of the system is that with the increase of m, x^* and u^* first decrease and then increase, and y^* first increase and then decrease.

We then discuss the influence of the feedback control on the positive equilibrium.

By calculation, we can conclude,

$$\begin{aligned} \frac{\partial x^*}{\partial b} &= -\frac{-(c+dq)u^*}{J}, \quad \frac{\partial y^*}{\partial b} = -\frac{dpu^*}{J}, \\ \frac{\partial u^*}{\partial b} &= -\frac{-cpu^*}{J}, \quad \frac{\partial x^*}{\partial d} = -\frac{-(bq-a)u^*}{J}, \\ \frac{\partial y^*}{\partial d} &= -\frac{-u^*(bp + 1 - \dfrac{m}{(x^*+\alpha)^2})}{J}, \\ \frac{\partial u^*}{\partial d} &= -\frac{-apu^* - qu^*(1 - \dfrac{m}{(x^*+\alpha)^2})}{J}. \end{aligned}$$

Therefore, when $m < m^*$, $J < 0$, we have

$$\frac{\partial x^*}{\partial b} < 0, \frac{\partial y^*}{\partial b} > 0, \frac{\partial u^*}{\partial b} < 0,$$

and when $m > m^*$, $J > 0$, we have

$$\frac{\partial x^*}{\partial b} > 0, \frac{\partial y^*}{\partial b} < 0, \frac{\partial u^*}{\partial b} > 0.$$

Similar to the influence of the Allee effect on the equilibrium point, the feedback control of population x affects parameter b. The influence on the equilibrium point of the system is also that with the increase of b, x^* and u^* first decrease and then increase, and y^* first increase and then decrease.

The symbols of $\frac{\partial x^*}{\partial d}, \frac{\partial y^*}{\partial d}$, and $\frac{\partial u^*}{\partial d}$ are more complicated, and we cannot find a uniform threshold for m that makes the symbols of them determinate. In other words, the feedback control of population y affects the parameter d, which has a more complex impact on the system equilibrium point.

From the above discussion, it can be seen that both the Allee effect and feedback control variables significantly influence the system's equilibrium point.

VII. Numeric simulations

In this section, we conducted numerical simulations on the global dynamics of system (3). By changing the value of m, we verified the influence of parameter m on the system. Now let's consider the following example.

Example 7.1 Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - \frac{m}{x+1}) + 0.8xy - 0.5\mu x, \\ \frac{dy}{dt} &= y(1 - 0.6y) - 0.2\mu y, \\ \frac{d\mu}{dt} &= -\mu + 0.1x + 0.3y, \end{aligned} \tag{16}$$

In this system, we take the following set of hypothetical parameter values:

$$a = 0.8, b = 0.5, c = 0.6, d = 0.2, p = 0.1, q = 0.3, \alpha = 1.$$

Then we have $m_0 = 1.9848$, $m_1 = 1.0006$, $m_2 = 2.180582$, and $\alpha_0 = 1.8555$.

For $m = 0.9$,

$$m < 0.964 = (1 - \frac{a^2dp}{4bcq})\alpha^2,$$

and $m < m_0$ hold, then it follows from Theorem 4 that $E_{31}(1.5219, 1.4690.0.5929)$ is globally asymptotically stable.

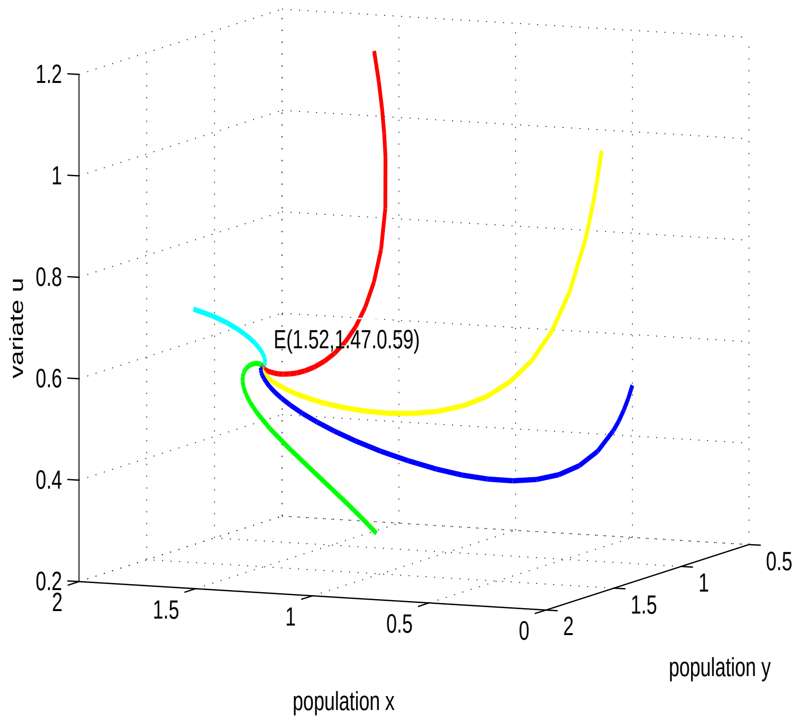

Fig. 1. Dynamic behaviors of system(16) with the initial condition $(x(0), y(0), \mu(0)) = (0.5, 0.5, 0.5)$, $(0.9, 1.7, 0.3)$, $(1.8, 1.5, 0.7)$,$(0.4, 0.9, 1)$ and $(1.2, 1.2, 1.2)$ respectively.

As we know, the Allee effect becomes stronger from weak with the increase of m. Next, let us understand the impact of changing the system parameter m and confirm our analysis results. We keep the other parameters in Example 6.1 and their initial values unchanged, changing only the value of the parameter m.

(1) For $m = 0.9$, we have $m < m_0$. The system (16) has a unique positive equilibrium point $E_{31}(1.5219, 1.4690.0.5929)$, it is globally asymptotically stable. See Figure 2.

(2) For $m = 2.12$, we have $m_0 < m < m_2$, and $\alpha < \alpha_0$. The system (16) has two positive equilibrium points $E_{31}(0.6657, 1.4950, 0.5151)$ and $E_{32}(0.1898, 1.5094, 0.4718)$. See Figure 3.

(3) For $m = 2.180582$, we have $m = m_2$, and $\alpha < \alpha_0$. The system (16) has a unique positive equilibrium point $E_{30}(0.4277, 1.5022, 0.4934)$. At the same time, system (24) exist stable boundary equilibrium $E_1(0, 1.5152, 0.4545)$. See Figure 4.

(4) For $m = 2.2$, we have $m > m_2$. The system (16) has no interior equilibrium point, and boundary equilibrium $E_1(0, 1.5152, 0.4545)$ is globally asymptotically stable. See Figure 5.

VIII. Conclusion

In this paper, a commensal symbiosis system with the Allee effect and single feedback control is proposed and analyzed for the first time. The analysis of system (3) reveals that it has at most eight equilibrium points. This is quite different from the system containing two feedback control variables ([16], [18], [20]), which has a unique positive globally asymptotically stable equilibrium, and no bifurcation analysis was performed.

This paper aims to analyze the dynamic behavior of system (3) comprehensively. The complexity of the system‘s dynamic behavior rises with the number of equilibrium points; transcritical bifurcation and saddle-node bifurcation occur at different equilibrium points. This article establishes stability conditions for all possible equilibrium points and studies the global dynamics of interior equilibrium points using appropriate Lyapunov functions. The Sotomayor's bifurcation theorem is used to investigate the occurrence and specify the type of bifurcation.

By comparing different values of m, as shown in Figure 2-5, system (16) is very sensitive to the change in parameter value m. The parameter m, which reflects the strength of the Allee effect, plays a crucial role in changing the dynamic behavior of the system. As the value of m increases, the number of system equilibrium points first increases and then decreases. When m is large enough, it will cause the extinction of species x. This is similar to, but significantly different from, the commensal symbiosis system (2), which is affected by the Allee effect and two feedback controls. In addition, we also analyze the effects of the Allee effect and feedback control variables on the system. Whether it is the intensity of the Allee effect (m) or the influence degree (b and d) of the feedback control variables, it will have an important influence on the position of the equilibrium point.

It should be pointed out that the conditions for Theorem 4.1 are sufficient. During data simulation, it was found that

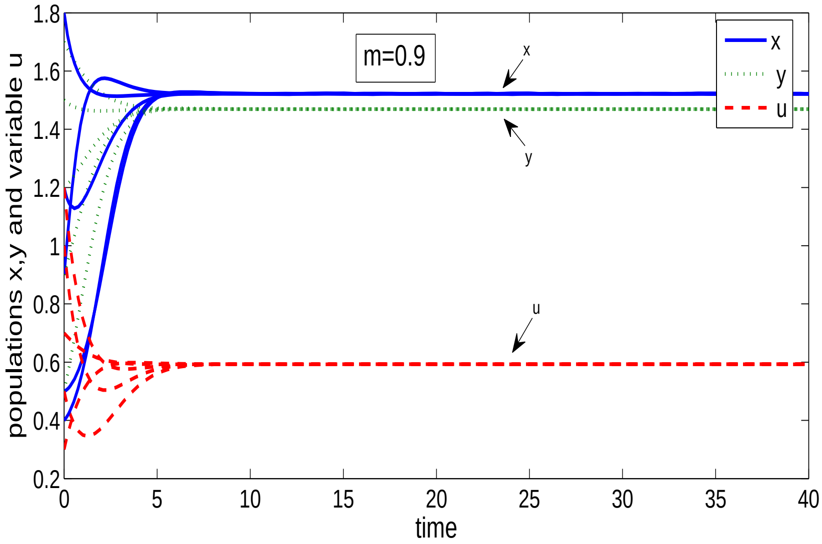

Fig. 2. Dynamic behaviors of system(16) when $m = 0.9$.

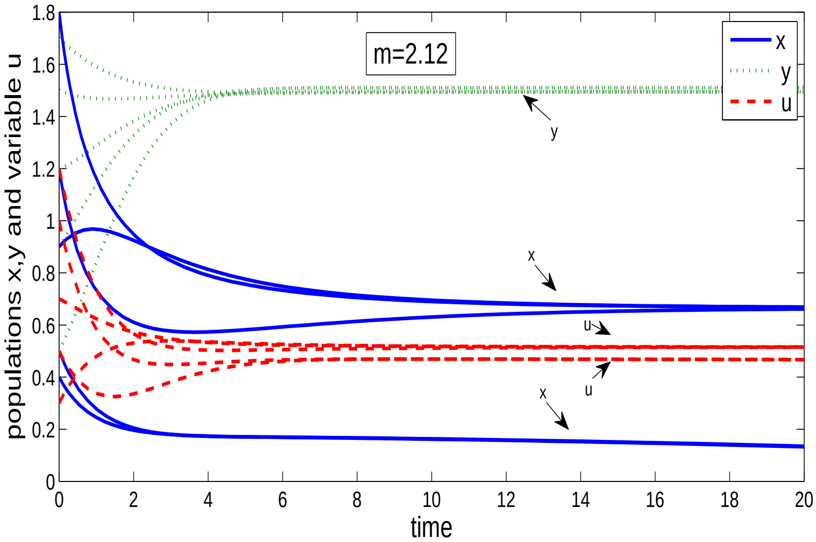

Fig. 3. Dynamic behaviors of system(16) when $m = 2.12$.

when $m > (1 - \frac{a^2dp}{4bcq})\alpha^2$, the equilibrium point may also be globally asymptotically stable, which indicates that there is still room for improvement in the conditions of Theorem 4.1.

References

[1] P. Beneden, "Animal parasites and messmates," *Animal parasites and messmates*, vol.19, Henry S. King, 1876.

[2] G. Sun, H. Sun, "The qualitative analysis of commensal symbiosis Model of two populations," *Mathematical Theory and Applications*, vol.23, no.3, pp. 65-68, 2003.

[3] N. Puspitasari, W. Kusumawinahyu, T. Trisilowati, "Dynamical analysis of the symbiotic model of commensalism in four populations with Michaelis-Menten type harvesting in the first commensal population," *Jurnal Teori dan Aplikasi Matematika*, vol.5, no.2, pp. 392-404, 2021.

[4] L. Xu, Y. Xue, X. Xie, et al. "Dynamic behaviors of an obligate commensal symbiosis model with Crowley-Martin functional responses," *Axioms*, vol.11, no.6, article ID 298, 2022.

[5] F. Chen, Q. Zhou, S. Lin, "Global stability of symbiotic medel of commensalism and parasitism with harvesting in commensal populations," *Wseas Transactions on Mathematics*, vol.21, no.1, pp. 424-432, 2022.

[6] Y. Xue, X. Xie, Q. Lin, "Almost periodic solutions of a commensalism system with Michaelis-Menten type harvesting on time scales," *Open Mathematics*, vol.17, no.1, pp. 1503-1514, 2019.

[7] Q. Zhou, S. Lin, F. Chen, et al. "Positive periodic solution of a discrete Lotka-Volterra commensal symbiosis model with Michaelis-Menten type harvesting," *Wseas Transactions on Mathematics*, vol.21, no.1, pp. 515-523, 2022.

[8] F. Chen, Y. Chen, Z. Li, et al. "Note on the persistence and stability property of a commensalism model with Michaelis-Menten harvesting and Holling type II commenaslistic benefit," *Applied Mathematics Letters*, vol.134, article ID 108381, 2022.

[9] Z. Wei, Y. Xia, T. Zhang, "Stability and bifurcation analysis of a commensal model with additive Allee effect and nonlinear growth rate," *International Journal of Bifurcation and Chaos*, vol.31, no.13, article ID 2150204, 2021.

[10] X. He, Z. Zhu, J. Chen, et al. "Dynamical analysis of a Lotka Volterra commensalism model with additive Allee effect," *Open Mathematics*, vol.20, no.1, pp. 646-665, 2022.

[11] L. Chen, T. Liu, F. Chen, "Stability and bifurcation in a two-patch

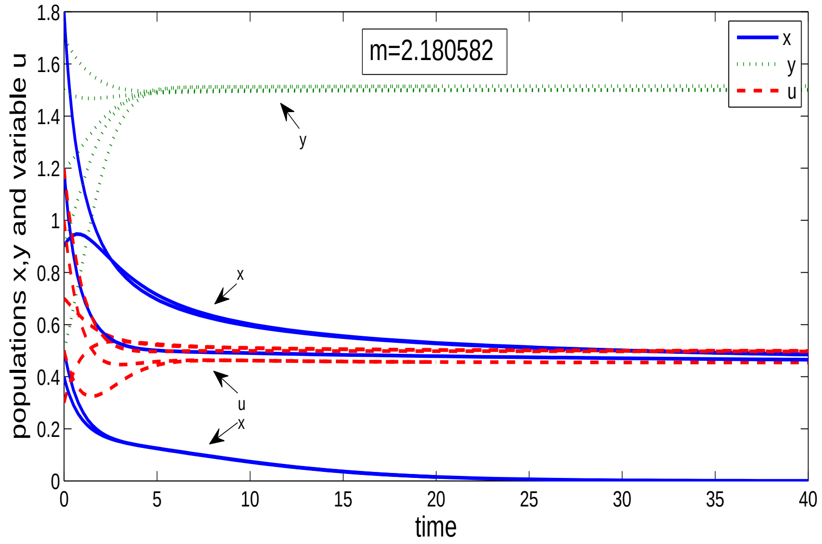

Fig. 4. Dynamic behaviors of system(16) when $m = 2.180582$.

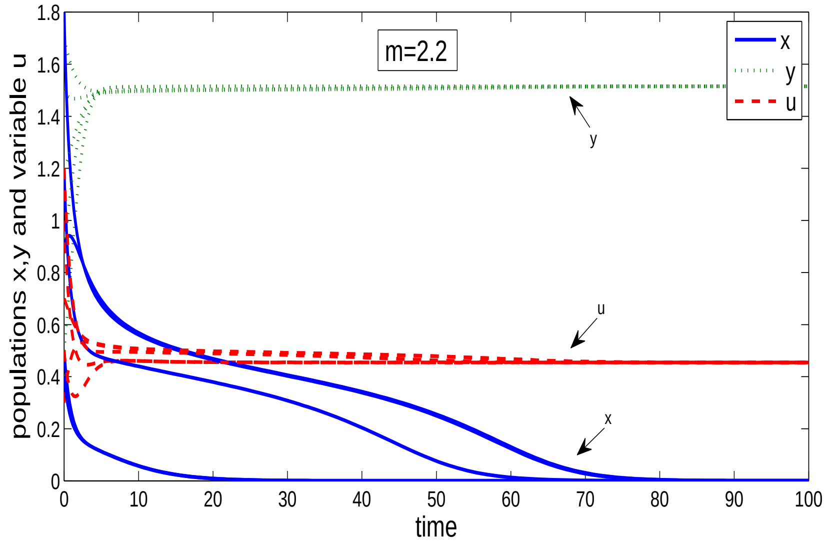

Fig. 5. Dynamic behaviors of system(16) when $m = 2.2$.

model with additive Allee effect," *AIMS Mathematics*, vol.7, no.1, pp. 536-551, 2022.

[12] I. Seval, "Stability and period-doubling bifurcation in a modified commensal symbiosis model with Allee effect," *Erzincan University Journal of Science and Technology*, vol.15, no.1, pp. 310-324, 2022.

[13] S. Jawad, "Study the dynamics of commensalism interaction with Michaels-Menten type prey harvesting," *Al-Nahrain Journal of Science*, vol.25, no.1, pp. 45-50, 2022.

[14] S. Chen, Y. Chong, F. Chen, "Periodic solution of a discrete commensal symbiosis model with Hassell-Varley type functional response," *Nonautonomous Dynamical Systems*, vol.9, no.1, pp. 170-181, 2022.

[15] Y. Chong, A. Kashyap, S. Chen, et al. "Dynamics analysis of a discrete-time commensalism model with additive Allee for the host species," *Axioms*, vol.12, no.11, article ID 1031, 2023.

[16] R. Han, F. Chen, "Global stability of a commensal symbiosis model with feedback controls," *Communications in Mathematical Biology and Neuroscience*, vol.2015, article ID 15, 2015.

[17] Y. Yang, K. Wang, "The stability of a commensal system that can't survive independently by one party with feedback controls," *Pure and Applied Mathematics*, vol.34, no.1, pp. 73-80, 2018.

[18] L. Xu, Q. Lin, C. Lei, "Dynamic behavior of commensal symbiosis system with both feedback control and Allee effect," *Journal of Shanghai Normal University (Natural Science Edition)*, vol.51, no.3, pp. 391-396, 2022.

[19] Y. Wang, T. Lv, L. Zhou, "Hopf bifurcation of commensal symbiosis systems with time delay and single feedback control," *Journal of North University of China (Natural Science Edition)*, vol.42, no.4, pp. 297-302, 2021.

[20] F. Chen, Y. Chong, S. Lin, "Global stability of a commensal symbiosis model with Holling II functional response and feedback controls," *Wseas Transactions on Systems and Control*, vol.17, no.1, pp. 279-286, 2022.

[21] N. Desneux, A. Decourtye, J. Delpuech, "The sublethal effects of pesticides on beneficial arthropods," *Annual Review of Entomology*, vol.52, pp. 81-106, 2007.

[22] F. Yao, H. Zheng, H. Li, et al. "Effect of chemotherapy on cell immunofunction in patients of lung cancer," *Chinese Journal of Clinical Oncology and Rehabilitation*, vol.8, no.2, pp. 213-218, 2001.

[23] L. Zhao, X. Xie, L. Yang, "Dynamic behaviors of a discrete Lotka-Volterra competition system with infinite delays and single feedback control," *Abstract and Applied Analysis*, vol.2014, article ID 867313, 2014.

[24] R. Han, F. Chen, "Stability of Lotka-Volterra cooperation system with single feedback control," *Annual of Applied Mathematics: English*,

vol.3, pp. 287-296, 2015.
[25] K. Yang, Z. Miao, F. Chen, "Influence of single feedback control variable on an autonomous Holling-II type cooperative system," *Journal of Mathematical Analysis and Applications*, vol.435, no.1, pp. 874-888, 2016.
[26] F. Chen, H. Wang, "Dynamic behaviors of a Lotka-Volterra competitive system with infinite delays and single feedback control," *abstract applied analysis*, vol.2016, article ID 43, 2016.
[27] Y. Xue, X. Xie, Q. Lin, et al. "Global attractivity and extinction of a discrete competitive system with infinite delays and single feedback control," *Discrete Dynamics in Nature and Society*, vol.2018, article ID 1893181, 2018.
[28] S. Yu, "Almost periodic solution for a modified Leslie-Gower system with single feedback control," *IAENG International Journal of Applied Mathematics,* vol.52, no.1, pp. 1-6, 2022.
[29] L. Perko, "Differential equations and dynamical systems," *Springer, New York, USA,* 3rd edition, 2001.
[30] Q. Zhu, S. Lin, R. Wu, F. Chen, "Dynamic behaviors of a commensalism model incorporating nonselective harvesting in a partial closure," *Wseas Transactions on Mathematics*, vol.22, pp. 798-806, 2023.
[31] X. Li, Q. Yue, F. Chen, "The dynamic behaviors of nonselective harvesting Lotka-Volterra predator-prey system with partial closure for populations and the fear effect of the prey species," *IAENG International Journal of Applied Mathematics*, vol. 53, no. 3, pp. 818-825, 2023.
[32] C. Huang, F. Chen, Q. Zhu, et al. "How the wind changes the Leslie-Gower predator-prey system?," *IAENG International Journal of Applied Mathematics,* vol. 53, no.3, pp. 907-915, 2023.
[33] Z. Zhu, F. Chen, L. Lai, et al. "Dynamic behaviors of a discrete May type cooperative system incorporating Michaelis-Menten type harvesting," *IAENG International Journal of Applied Mathematics*, vol. 50, no.3, pp. 458-467, 2020.
[34] Z. Zhu, R. Wu, F. Chen, et al. "Dynamic behaviors of a Lotka-Volterra commensal symbiosis model with non-selective Michaelis-Menten type harvesting," *IAENG International Journal of Applied Mathematics*, vol. 50, no.2, pp. 396-404, 2020.
[35] M. He, Z. Li, F. Chen, et al. "Dynamic behaviors of an N-species Lotka-Volterra model with nonlinear impulses," *IAENG International Journal of Applied Mathematics*, vol.50, no.1, pp. 22-30, 2020.
[36] K. Fang, J. Chen, Z. Zhu, et al. "Qualitative and bifurcation analysis of a single species logistic model with allee effect and feedback control," *IAENG International Journal of Applied Mathematics*, vol. 52, no. 2, pp. 320-326, 2022.

Lili Xu is currently an associate professor at the School of Mathematics and Physics, Ningde Normal University, Fujian Province. She received a Bachelor of Science from Fujian Agriculture and Forestry University in 2004 and a Master of Applied Mathematics from Xiamen University in 2010. Her research interests include biomathematics, option pricing, and ordinary differential equations.

Yalong Xue is now a lecturer at Ningde Normal University in Fujian Province. He graduated from Fuzhou University with a master's degree in 2016. His research interests include biomathematics and differential equations.

Qifa Lin is currently an associate professor at Ningde Normal University in Fujian Province. He graduated from Ningde Normal College in August 1991. He received a Bachelor of Science from Fujian Normal University in 1998 and a Master of Applied Mathematics from Xiamen University in 2007. His research interests include biomathematics and differential equations.

Fengde Chen works as a professor in the College of Mathematics and Statistics at Fuzhou University. He received a Bachelor's degree in Mathematics from Fujian Normal University in 1996 and a Master of Science degree in Mathematics from Fuzhou University in 1999. He obtained his Ph.D. degree in Mathematics in the area of differential equations from Peking University in 2002. His research interests include biomathematics and differential equations.