Strongly Vertex Perfectly Antimagic Total Graphs

S. Balasundar, R. Jeyabalan, R. Nishanthini and P. Swathi

***Abstract*—A totally antimagic total labeling (TAT) is a bijective mapping from $V(G) \cup E(G)$ to $\{1, 2, ..., |V(G)| + |E(G)|\}$ where all vertex and edge weights are distinct. Similarly to TAT, perfectly antimagic total labeling (PAT) uses distinctive weights for each vertex and edge. An instance of a PAT is "Strongly vertex perfectly antimagic total labeling" (SVPAT), where the weights of the vertices are larger than those of the edges.**

"Not every tree admits SVPAT." This is the result of an investigation into whether or not SVPAT labeling is present in specific graph families. The outcome proves that SVPAT cannot be achieved by joining any two graphs together; it must be a path graph. Ultimately, this manuscript illustrates the real-world significance of SVPAT labeling.

***Index Terms*—Antimagic, Vertex antimagic, Edge antimagic, Totally antimagic, Perfectly antimagic.**

I. Introduction

This paper employs only finite, simple, and undirected graphs. Consider a graph denoted as G, comprised of p vertices and q edges. Labeling is the process of assigning a set of positive integers to the elements of a graph. A total labeling for graph G is defined as $\mu : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$. The weight of a vertex, $wt_\mu(v)$, and an edge, $wt_\mu(uv)$, are calculated as follows: $wt_\mu(v) = \mu(v) + \sum_{w \in N(v)} \mu(vw)$ and, $wt_\mu(uv) = \mu(u) + \mu(v) + \mu(uv)$ respectively. Here, $N(v)$ represents the set of neighbors for vertex v. If all edge and vertex weights are distinct, the labeling is considered an edge-antimagic total (EAT) or vertex-antimagic total (VAT) labeling, abbreviated as EAT (VAT) [3]. An EAT (VAT) graph is a graph that can be labeled using the EAT (VAT) labeling method [1]. Hartsfield and Ringel [6] first proposed the idea of an antimagic graph. In edge labeling, as described by [6], vertex weights must be distinct and pairwise non-equal; this labeling is referred to as antimagic labeling. Further information about graph labeling can be found in [4], [5], [7], [8]. A labeling that exhibits both vertex-antimagic and edge-antimagic properties is known as a totally antimagic total (TAT) labeling. A graph is considered a TAT graph if it is capable of TAT labeling. The concept of TAT graphs was first introduced by Bača *et al.* [2].

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The study unveiled the complete absence of magic in wheels, stars, double stars, and cycles as total graphs. Total graphs are deemed perfect if they have distinct vertex and edge weights, as outlined by P. Swathi *et al.*[9]. The term "Perfectly antimagic total graphs" pertains to a graph possessing PAT labeling. This research specifically focuses on the strongly vertex perfectly antimagic total (SVPAT) graph, wherein all vertex weights are strictly greater than all edge weights. An SVPAT labeling is a PAT labeling with these conditions met. An SVPAT graph, in turn, possesses an SVPAT label. The sections that follow will present the results of the investigation into whether or not these graphs have SVPAT labeling.

Definition 1.1: The friendship graph F_n is a set of n triangles having a common center vertex.

Definition 1.2: The Prism D_n, $n \geq 3$ is a cubic graph representing a cartesian product of a path on two vertices with a cycle on n. Let $V(D_n) = \{u_{\bar{i}}, v_{\bar{i}}; 1 \leq \bar{i} \leq n\}$ be the vertex set and $E(D_n) = \{u_{\bar{i}}u_{\bar{i}+1}, v_{\bar{i}}v_{\bar{i}+1}, u_{\bar{i}}v_{\bar{i}}; 1 \leq \bar{i} \leq n\}$ where the indices are taken modulo n be the edge set of the prism D_n. The prism has $2n$ vertices and $3n$ edges.

Definition 1.3: A spider graph is a tree having at least one vertex of degree 3 and all others of degree 2 or less.

Definition 1.4: If G has order n, the graph is formed by taking one copy of G and n copies of H and joining the $\bar{i}^{th}$ vertex of G with an edge to every vertex in the $\bar{i}^{th}$ copy of H is known as the corona of G with a graph H, or $G \odot H$.

Definition 1.5: For $n \geq 3$ and $1 \leq k \leq \lfloor (n-1)/2 \rfloor$, the generalized Petersen graph $GP(n, k)$ is a linked cubic graph made up of an outer regular polygon $\{n\}$ (cycle graph C_n) and an inner star polygon $\{n, k\}$, with matching vertices in both polygons connected by edges.

II. Main Results

The following graphic illustrates the presence and absence of the SVPAT labeling in various graph families. This visual demonstrates that not all trees are compatible with SVPAT labeling.

Theorem 2.1: Every cycle graph C_n, $n \geq 3$ is SVPAT.

Proof. Define a total labeling $\mu : V(G) \cup E(G) \to \{1, 2, ..., 2n\}$ such that

$$\begin{aligned} \mu(v_{\bar{i}}) &= \bar{i},\ \forall \bar{i} = 1, 2, \ldots, n \\ \mu(v_{\bar{i}}v_{\bar{i}+1}) &= 2n + 1 - \bar{i},\ \forall \bar{i} = 1, 2, \ldots, n-1 \\ \mu(v_n v_1) &= n + 1. \end{aligned}$$

Given the total labeling μ above, the vertex weights are obtained as follows:

$$\begin{aligned} wt_\mu(v_1) &= 3n + 2. \\ wt_\mu(v_{\bar{i}}) &= 4n + 3 - \bar{i},\ \forall 2 \leq \bar{i} \leq n-1 \\ wt_\mu(v_n) &= 3n + 3. \end{aligned}$$

The edge weights are given by

$$\begin{aligned} wt_\mu(v_1v_n) &= 2n+2. \\ wt_\mu(v_{\bar{i}}v_{\bar{i}+1}) &= 2n+2+\bar{i}, \forall 1 \le \bar{i} \le n-1 \end{aligned}$$

from all the above discussion, all the vertex weights and all the edge weights are pairwise distinct.
The maximum edge weight of $v_{\bar{j}}v_n \in E(G)$,

$$\begin{aligned} wt_\mu^{max}(v_{\bar{j}}v_n) &= 2n+2+\bar{i} \\ &= 3n+1. \end{aligned}$$

The minimum vertex weight of $v \in V(G)$, $wt_\mu^{min}(v) = 3n+2$. Hence, $wt_\mu^{min}(v_{\bar{i}}) > wt_\mu^{max}(e)$. Since all the vertex weights are strictly greater than all the edge weights. Thus, C_n is SVPAT.

Theorem 2.2: Every friendship graph $F_n, n \ge 2$ is SVPAT.

Proof. Define a total labeling $\mu : V(G) \cup E(G) \to \{1, 2, ..., 5n+1\}$ in the following way.
For each $\bar{i} = 1, 2, \ldots, n$

$$\begin{aligned} \mu(v) &= 1 \\ \mu(v_{\bar{i}}) &= \bar{i}+1 \\ \mu(u_{\bar{i}}) &= n+1+\bar{i} \\ \mu(vv_{\bar{i}}) &= 2n+1+\bar{i} \\ \mu(vu_{\bar{i}}) &= 3n+1+\bar{i} \\ \mu(v_{\bar{i}}u_{\bar{i}}) &= 5n+2-\bar{i}. \end{aligned}$$

Vertex weights under the labeling μ are given by

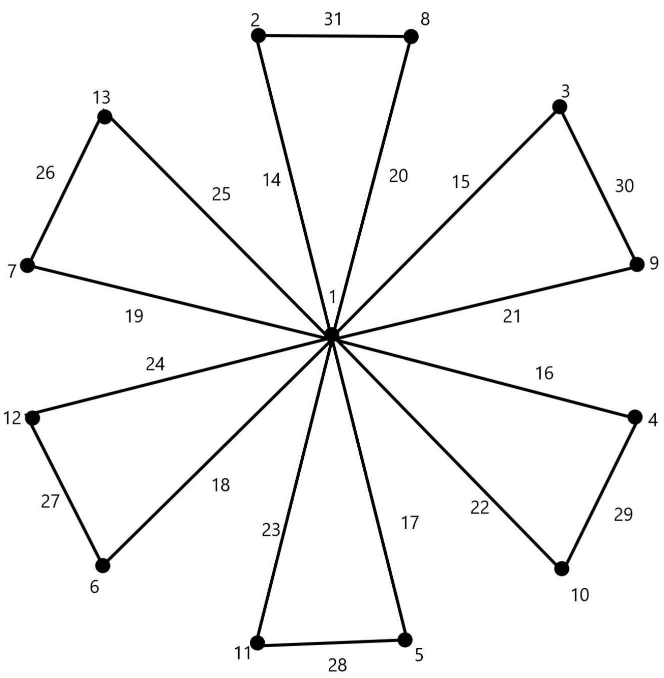

Fig. 1. Frienship graph F_6

$$\begin{aligned} wt_\mu(v_{\bar{i}}) &= \mu(v_{\bar{i}}) + \mu(v_{\bar{i}}v) + \mu(v_{\bar{i}}u_{\bar{i}}) \\ &= 7n+4+\bar{i}, \ \forall \ \bar{i} = 1, 2, ..., n \\ wt_\mu(u_{\bar{i}}) &= \mu(u_{\bar{i}}) + \mu(u_{\bar{i}}v) + \mu(v_{\bar{i}}u_{\bar{i}}) \\ &= 9n+4+\bar{i}, \ \forall \ \bar{i} = 1, 2, ..., n \end{aligned}$$

$$\begin{aligned} wt_\mu(v) &= \mu(v) + \sum \mu(vv_{\bar{i}}) + \sum \mu(vu_{\bar{i}}) \\ &= 1 + \frac{n}{2}(5n+3) + \frac{n}{2}(7n+3) \\ &= 6n^2+3n+1. \end{aligned}$$

It is simple to confirm that the vertex weights are pairwise distinct based on the vertex weights listed above.
For $1 \le \bar{i} \le n$, edge weights under the labeling μ are given by,

$$\begin{aligned} wt_\mu(vv_{\bar{i}}) &= 2n+3+2\bar{i}, \\ wt_\mu(vu_{\bar{i}}) &= 4n+3+2\bar{i}, \\ wt_\mu(v_{\bar{i}}u_{\bar{i}}) &= 6n+4+\bar{i}. \end{aligned}$$

The above edge weights show pairwise distinct edge weights. Since all the vertex weights and all the edge weights are pairwise distinct, F_n is PAT.

$$wt_\mu^{max}(e) = 7n+4.$$

$$wt_\mu^{min}(v) = 7n+5.$$

Since all the vertex weights are strictly greater than all the edge weights, F_n is SVPAT.

Theorem 2.3: For every odd positive integer n, $n \ge 3$ the prism D_n is SVPAT.

Proof. Define a total labeling $\mu : V(G) \cup E(G) \to \{1, 2, ..., p+q\}$ by

$$\begin{aligned} \mu(v_{\bar{i}}) &= \bar{i} \ \forall \ \bar{i} = 1, 2, ..., n \\ \mu(u_{\bar{i}}) &= n+\bar{i} \ \forall \ \bar{i} = 1, 2, ..., n \\ \mu(u_{\bar{i}}v_{\bar{i}}) &= 3n+1-\bar{i} \ \forall \ \bar{i} = 1, 2, ..., n \\ \mu(v_{\bar{i}}v_{\bar{i}+1}) &= 5n+1-\bar{i} \ \forall \ \bar{i} = 1, 2, ..., n-1 \\ \mu(u_{\bar{i}}u_{\bar{i}+1}) &= 4n+1-\bar{i} \ \forall \ \bar{i} = 1, 2, ..., n-1 \\ \mu(v_nv_1) &= 4n+1 \\ \mu(u_nu_1) &= 3n+1. \end{aligned}$$

The vertex weights under the labeling μ are

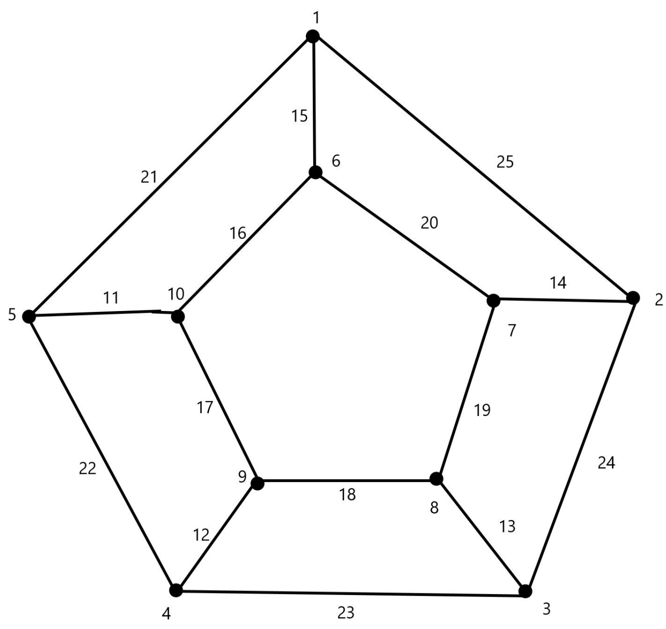

Fig. 2. Prism graph D_5

$$
\begin{aligned}
wt_\mu(v_1) &= 12n+2. \\
wt_\mu(v_{\bar{i}}) &= 13n+4-2\bar{i}. \\
wt_\mu(v_n) &= 11n+4. \\
wt_\mu(u_1) &= 11n+2 \\
wt_\mu(u_{\bar{i}}) &= 12n+4-2\bar{i}. \\
wt_\mu(u_n) &= 10n+4
\end{aligned}
$$

Based on the above vertex weights, it can be easily concluded that the vertex weights are pairwise distinct. Edge weights under the total labeling μ are given by

$$
\begin{aligned}
wt_\mu(v_1v_n) &= 5n+2, \\
wt_\mu(v_{\bar{i}}v_{\bar{i}+1}) &= 5n+2+\bar{i}. \\
wt_\mu(u_{\bar{i}}u_{\bar{i}+1}) &= 6n+2+\bar{i}, \forall 1 \le \bar{i} \le n-1 \\
wt_\mu(u_1u_n) &= 6n+2 \\
wt_\mu(u_{\bar{i}}v_{\bar{i}}) &= 4n+1+\bar{i}, \forall 1 \le \bar{i} \le n
\end{aligned}
$$

From the above equations $wt_\mu(u_{\bar{i}}v_{\bar{i}}) < wt_\mu(v_1v_n) < wt_\mu(v_{\bar{i}}v_{\bar{i}+1}) < wt_\mu(u_1u_n) < wt_\mu(u_{\bar{i}}u_{\bar{i}+1})$.
i.e.) All edge-weights are pairwise distinct.

$$wt_\mu^{max}(e) = 7n+1.$$

$$wt_\mu^{min}(v) = 10n+4.$$

Thus, all the vertex weights are strictly greater than all edge weights.

Corollary 2.4: The generalized peterson graph $GP(n,1)$ is SVPAT, for all odd $n \ge 3$.

Theorem 2.5: For every positive integer $n \ge 3$, the complete graph K_n is SVPAT.
Proof: From theorem 2.1, it is obvious that K_n is SVPAT for $n=3$. For $n \ge 4$, at every vertex in K_n, assign a label $1,2,...,n$. Under the vertex labeling established above, we now obtain the weight of all edges as $w(e_{\bar{i}}) \le w(e_{\bar{j}})$, for $1 \le \bar{i} < \bar{j} \le \frac{n(n-1)}{2}$. At this point, we assign s to $e_{\bar{i}}$ and t to $e_{\bar{j}}$, given that $n+1 \le s < t \le \frac{n(n-1)}{2}$. All complete graphs are SVPAT under this label.

Theorem 2.6: Let G be SVPAT, then there is no vertex of G having degree one.
Proof. Given that G is SVPAT, then all vertex weights are greater than all edge weights. Let us assume that there is a pendant vertex $v \in V(G)$. $wt_\mu(v) = \mu(v) + \mu(vu)$ where u is a neighbourhood of v and thus $wt_\mu(vu) = \mu(v)+\mu(vu)+\mu(u)$.
$i.e, wt_\mu(vu) > wt_\mu(v)$, which is contradiction to that G is SVPAT.

Corollary 2.7: Let G be any graph, then $G \odot nK_1$ is not SVPAT, $\forall n \ge 1$.

Lemma 2.8: Every tree is not SVPAT.
Proof. The proof presented in the theorem 2.6 makes this very obvious.

Corollary 2.9: For every path graph P_n, $\forall n \ge 1$, is not SVPAT.

Corollary 2.10: For every $n \ge 1$, the star graph S_n is not SVPAT.

Corollary 2.11: Every bistar graph $B_{(n,m)}$ as well as every regular bistar graph $B_{(n,n)}$ is not SVPAT, $\forall n,m \ge 1$.

Corollary 2.12: SVPAT does not apply to the spider graph.

Theorem 2.13: If G is a 2−regular SVPAT, then the sum of all the vertex labels is less than the sum of all edge labels.
Proof. Let G be a 2−regular graph. Define a SVPAT labeling $\mu : V(G) \cup E(G) \to \{1,2,...,p+q\}$, then all the vertex weights are strictly greater than all the edge weights.

$$\sum_{\bar{i}=1}^{p} wt_\mu(v_{\bar{i}}) > \sum_{\bar{j}=1}^{q} wt_\mu(e_{\bar{j}})$$

$$\sum_{\bar{i}=1}^{p} \mu(v_{\bar{i}}) + 2\sum_{\bar{j}=1}^{q} \mu(e_{\bar{j}}) > \sum_{\bar{j}=1}^{q} \mu(e_{\bar{j}}) + 2\sum_{\bar{i}=1}^{p} \mu(v_{\bar{i}})$$

$$i.e, \sum_{\bar{j}=1}^{q} \mu(e_{\bar{j}}) \quad > \quad \sum_{\bar{i}=1}^{p} \mu(v_{\bar{i}})$$

Therefore, the sum of all the vertex labels is less than the sum of all the edge labels.

III. Disjoint Union of Cartesian families of Graphs

Throughout this section, we show that the disjoint union of cycle graphs and prism graphs is both SVPAT. However, we show that the disjoint union of any graph with path is not SVPAT.

Theorem 3.1: Let $n \ge 3$ and $m \ge 1$ be positive integers, and a cycle C_n is SVPAT, then m copy of cycle C_n is SVPAT.
Proof. Assume C_n is SVPAT with the total labeling μ shown in theorem 2.1. For each vertex $v \in G$, we denote by symbol $v_{\bar{i}}$ the corresponding vertex in the $\bar{j}^{th}$ copy of C_n in mC_n, $\forall 1 \le \bar{i} \le n$ and $1 \le \bar{j} \le m$.

Let $u_{\bar{i}}v_{\bar{i}}$ be the edge corresponding to the edge uv in the $\bar{j}^{th}$ copy of C_n in mC_n. For $m \ge 1$, define the total labeling g of mC_n as follows.

$$
\begin{aligned}
g(v_{\bar{i}}) &= \mu(v_{\bar{i}}) + n(\bar{j}-1) \\
g(u_{\bar{i}}v_{\bar{i}}) &= \mu(uv) + n(2m-1-\bar{j})
\end{aligned}
$$

for each $\bar{i} = 1,2,...,n$ and $\bar{j} = 1,2,...,m$.
Vertex weights under the labeling g,

$$
\begin{aligned}
wt_g(v_{\bar{i}}) &= g(v_{\bar{i}}) \\
&\quad + \sum_{u \in N(v_{\bar{i}})} g(uv_{\bar{i}}), \forall \bar{i} = 1,2,...,n \\
&= \mu(v_{\bar{i}}) + n(\bar{j}-1) \\
&\quad + \sum_{u \in N(v_{\bar{i}})} \left[\mu(uv_{\bar{i}}) + n(2m-1-\bar{j})\right] \\
wt_g(v_{\bar{i}}) &= wt_\mu(v_{\bar{i}}) + n(\bar{j}-1) \\
&\quad +2n(2m-1-\bar{j})
\end{aligned}
$$

$\forall \bar{i} = 1,2,...,n$ and $\bar{j} = 1,2,...,m$.
The vertex weights under the labeling g are the following,
for $\bar{j}=1$, then $wt_g(v_{\bar{i}}) = wt_\mu(v_{\bar{i}}) + n(4m-4)$
for $\bar{j}=2$, then $wt_g(v_{\bar{i}}) = wt_\mu(v_{\bar{i}}) + n(4m-5)$
for $\bar{j}=3$, then $wt_g(v_{\bar{i}}) = wt_\mu(v_{\bar{i}}) + n(4m-6)$
$\vdots \quad \vdots \quad \vdots$
for $\bar{j}=m$, then $wt_g(v_{\bar{i}}) = wt_\mu(v_i) + n(3m-3)$.
By theorem 2.1, all the vertex weights of $v_{\bar{i}}$, $\bar{i} = 1,2,...,n$ under the labeling μ are distinct. The set of weights of

vertices $v_{\bar{i}}$, $\bar{i} = 1,2,...,n$ under the labeling g have the following ordering:

$wt_g(v_{\bar{i}})$(for $\bar{j} = 1$) $> wt_g(v_{\bar{i}})$(for $\bar{j} = 2$) $> wt_g(v_{\bar{i}})$(for $\bar{j} = m$), for all $\bar{i} = 1,2,...,n$.

Thus the set of all vertex weights in mC_n are distinct.

Edge weights under the labeling g,

$$\begin{aligned} wt_g(v_{\bar{i}}v_{\bar{i}+1}) &= g(v_{\bar{i}}) + g(v_{\bar{i}+1}) + g(v_{\bar{i}}v_{\bar{i}+1}) \\ &= wt_\mu(v_{\bar{i}}v_{\bar{i}+1}) + n[2m-3+\bar{j}] \end{aligned}$$

$\forall \bar{i} = 1,2,...,n-1$ and $\bar{j} = 1,2,...,m$

$$\begin{aligned} wt_g(v_1v_n) &= g(v_1) + g(v_n) + g(v_1v_n) \\ &= wt_\mu(v_1v_n) + n[2m-3+\bar{j}], \end{aligned}$$

for $\bar{j} = 1,2,...,m$.

By theorem 2.1, all the edge weights under the labeling μ are distinct. The set of weights of edges under the labeling g have the following ordering:

$wt_g(v_{\bar{i}}v_{\bar{i}+1})$(for $\bar{j} = 1$) $<$ $wt_g(v_{\bar{i}}v_{\bar{i}+1})$(for $\bar{j} = 2$) $<$ $wt_g(v_{\bar{i}}v_{\bar{i}+1})$(for $\bar{j} = m$), for all $\bar{i} = 1,2,...n$. Thus all the edge weights are distinct.

$$wt_g^{max}(e) = 3n+1+n[3m-3] = 3nm+1.$$

$$wt_g^{min}(v) = 3n+2+n[3m-3] = 3nm+2.$$

mC_n is SVPAT due to the fact that every vertex weight is strictly greater than every edge weight.

Theorem 3.2: If for every odd positive integer n, $n \geq 3$, the prism D_n is SVPAT, then mD_n, $m \geq 1$, is SVPAT.

Proof. Let D_n be SVPAT with the total labeling μ presented in theorem 2.3. for $m \geq 1$, define the total labeling g of mD_n as follows:

For $\bar{i} = 1,2,...,n$ & $\bar{j} = 1,2,...,m$

$$\begin{aligned} g(v_{\bar{i}}) &= \mu(v_{\bar{i}}) + n(\bar{j}-1), \\ g(u_{\bar{i}}) &= \mu(u_{\bar{i}}) + n(m+\bar{j}-2), \\ g(u_{\bar{i}}v_{\bar{i}}) &= \mu(u_{\bar{i}}v_{\bar{i}}) + n(3m-2-\bar{j}), \end{aligned}$$

for $\bar{i} = 1,2,...,n-1$ & $\bar{j} = 1,2,...,m$

$$\begin{aligned} g(v_{\bar{i}}v_{\bar{i}+1}) &= \mu(v_{\bar{i}}v_{\bar{i}+1}) + n(5m-4-\bar{j}), \\ g(u_{\bar{i}}u_{\bar{i}+1}) &= \mu(u_{\bar{i}}u_{\bar{i}+1}) + n(4m-3-\bar{j}), \end{aligned}$$

for $\bar{j} = 1,2,...,m$

$$\begin{aligned} g(v_1v_n) &= \mu(v_1v_n) + n(5m-4-\bar{j}), \\ g(u_1u_n) &= \mu(u_1u_n) + n(4m-3-\bar{j}), \end{aligned}$$

Vertex weights under the labeling g,

$$\begin{aligned} wt_g(v_{\bar{i}}) &= g(v_{\bar{i}}) + \sum_{v_k \in N(v_{\bar{i}})} g(v_k v_{\bar{i}}), \ \ \forall \bar{i} = 1,2,...,n \\ &= wt_\mu(v_{\bar{i}}) + n(13m-2\bar{j}-11), \\ wt_g(u_{\bar{i}}) &= g(u_{\bar{i}}) + \sum_{v_k \in N(u_{\bar{i}})} g(v_k u_{\bar{i}}) \ \ \forall \bar{i} = 1,2,...,n \\ &= wt_\mu(u_{\bar{i}}) + n(12m-2\bar{j}-10) \end{aligned}$$

The vertex weights are pairwise distinct based on the weights mentioned above. Edge weights under the labeling g,

$$\begin{aligned} wt_g(v_{\bar{i}}v_{\bar{i}+1}) &= g(v_{\bar{i}}) + g(v_{\bar{i}}v_{\bar{i}+1}) + g(v_{\bar{i}}v_{\bar{i}+1}) \\ &= n(5m-1+\bar{j}) + 2 + \bar{i}, \end{aligned}$$

for $\bar{i} = 1,2,...,n-1$ and $\bar{j} = 1,2,...,m$,

$$\begin{aligned} wt_g(v_1v_n) &= g(v_1) + g(v_n) + g(v_1v_n) \\ &= n(5m-1+\bar{j}) + 2, \end{aligned}$$

for $\bar{j} = 1,2,...,m$.

$$\begin{aligned} wt_g(u_{\bar{i}}u_{\bar{i}+1}) &= g(u_{\bar{i}}) + g(u_{\bar{i}+1}) + g(u_{\bar{i}}u_{\bar{i}+1}) \\ &= n(6m-1+\bar{j}) + 2 + \bar{i}, \end{aligned}$$

for all $\bar{i} = 1,2,...,n-1$ & $\bar{j} = 1,2,...,m$.

$$\begin{aligned} wt_g(u_1u_n) &= g(u_1) + g(u_n) + g(u_1u_n) \\ &= n(6m-1+\bar{j}) + 2, \end{aligned}$$

for all $\bar{j} = 1,2,...,m$.

$$\begin{aligned} wt_g(u_{\bar{i}}v_{\bar{i}}) &= g(u_{\bar{i}}) + g(v_{\bar{i}}) + g(u_{\bar{i}}v_{\bar{i}}) \\ &= n(4m-1+\bar{j}) + 1 + \bar{i}, \end{aligned}$$

for all $\bar{i} = 1,2,...,n$ & $\bar{j} = 1,2,...,m$.

It is evident from the above weights that the edge weights are pairwise distinct.

Maximum edge weight of $e \in E(G)$,

$$wt_g^{max}(e) = n(6m-1+\bar{j}) + 2 + \bar{i} = 7nm+1.$$

Minimum vertex weight of $v \in V(G)$,

$$wt_g^{min}(v) = wt_\mu(u) + n(12m-2\bar{j}-10) = 10mn+4.$$

Hence, mD_n is SVPAT.

Theorem 3.3: If G is any graph, then $G \cup nK_1$ is not SVPAT.

Proof. Let G be any $(p,q)-$graph. Assume that $G \cup nK_1$ is SVPAT, $\forall n \geq 1$ with the total labeling $\mu : V(G \cup nK_1) \cup E(G \cup nK_1) \rightarrow \{1,2,\ldots,p+q+n\}$ defined by

$$wt_\mu(u_{\bar{i}}) = \mu(u_{\bar{i}}) = p+q+\bar{i}, \ \ for \ \ 1 \leq \bar{i} \leq n$$

Let $u \in \bar{i}K_1$, for $1 \leq \bar{i} \leq n$. Without loss of generality, let $\mu(u) = p+q+1$.

If the label $p+q$ is labeled with any of the vertex $v \in V(G)$, then there exists an edge $vw \in E(G)$ such that

$$\begin{aligned} wt_\mu(vw) &= \mu(v) + \mu(vw) + \mu(w) \\ &= \mu(v) + p + q + \mu(w) \\ &> p+q+1, \end{aligned}$$

which contradicts the fact that G is SVPAT.

If $p+q$ is labeled with any of the edge $xy \in E(G)$, then

$$\begin{aligned} wt_\mu(xy) &= \mu(x) + \mu(xy) + \mu(y) \\ &= \mu(x) + p + q + \mu(y) \\ &> p+q+1, \end{aligned}$$

which is a contradiction.

Hence, $G \cup nK_1$, $\forall n \geq 1$, is not SVPAT.

Theorem 3.4: If G is any graph, then $G \cup nK_2$ is not SVPAT.

Proof. It follows from theorem 2.6 that if G has a vertex of degree one, it is not SVPAT.

Theorem 3.5: The disjoint union of any graph G with a path graph P_n is not SVPAT. i.e, $G \cup P_n$ is not SVPAT, $\forall n \geq 1$.

IV. Application

Many software industry projects are undertaken by individuals possessing diverse soft capabilities. It is these capabilities that serve as the vertices, while the workers themselves represent the edges. The label at the vertex indicates the number of individuals working as project heads, while the label at the edge signifies the number of people sharing work between two projects.

The weight assigned to the edges reflects the number of individuals possessing a specific soft capability who are employed by both projects that require it. Conversely, the vertex weight predicts the number of individuals with project lead responsibilities and their associated workers. Companies anticipate that projects will be completed successfully, with employees with soft skills contributing autonomously. The SVPAT concept facilitates determining the maximum and minimum number of workers employed by different projects, as well as their diverse capabilities. Ultimately, the SVPAT ideas will be transformed into software, with additional necessary characteristics for methods becoming available in the future.

V. Conclusion

Throught out this paper, we look into whether strongly vertex perfectly antimagic total (SVPAT) labeling exists or not in some graph families. ”Every tree doesn’t admit SVPAT” and ”the disjoint union of any graph and a path graph is not SVPAT” are both shown.

We also conclude by listing a few unsolved issues in the Strongly vertex perfectly antimagic total (SVPAT) graph.

Open Problem:1

Determine the fan graph $F_{1,n}$ is SVPAT, for all n.

Open Problem:2

Determine the Prism graph D_n is SVPAT, for every even positive integer n, $n \geq 4$.

Open Problem:3

Let $n \geq 2$ and $m \geq 1$ be positive integers, and a frienship graph F_n is SVPAT, then determine m copy of F_n is SVPAT.

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