Adjacent Vertex Strongly Distinguishing Total Coloring of Unicyclic Graphs

Li Zhou, Fei Wen, and Zepeng Li

***Abstract*—An *adjacent vertex strongly distinguishing total-coloring* of a graph G is a proper *total-coloring* such that no two adjacent vertices meet the same color set, where the color set of a vertex consists of all colors assigned on the vertex and its incident edges and neighbors. The minimum number of the required colors is called *adjacent vertex strongly distinguishing total chromatic number*, denoted by $\chi_{ast}(G)$. In this paper, we first prove that $\chi_{ast}(U) \leq \Delta(U) + 2$ for a unicyclic graph U with $\Delta(U) \geq 3$. Then we completely determine the adjacent vertex strongly distinguishing total chromatic number of the unicyclic graph U with $\Delta(U) = 3$, which further shows that the upper bound of $\chi_{ast}(U) \leq \Delta(U) + 2$ is sharp.**

***Index Terms*—adjacent vertex strongly distinguishing total-coloring, adjacent vertex strongly distinguishing total chromatic number, unicyclic graph.**

I. Introduction

THROUGHOUT this paper, let $G = (V, E)$ be a finite, simple and undirected graph with vertex set V and edge set E, and $|V| = \nu$. Moreover, suppose G contains no isolated edge. For a vertex $v \in V(G)$, if w is a neighbor of v, we denote by $v \sim w$, and all the neighbors of v in G are written as $N_G(v) = \{w|wv \in E(G)\}$. We use $d_G(v)$ (for short, $d(v)$) to denote the degree of vertex v in G. Clearly, $d(v) = |N_G(v)|$. Denote by $\Delta(G) = \max\{d(x)|x \in V(G)\}$ the maximum degree of G, and u_Δ the vertex u with degree Δ. For two sets A and B, we denote by $A \oplus B = (A \cup B) \setminus (A \cap B)$ the symmetric difference of A and B. A connected graph in which the number of edges equals to the number of vertices is called a *unicyclic graph* and denoted by U. The terminologies and notations used but undefined in this paper can be found in [1].

Let P_n, K_n and $C_n (n \geq 3)$ be the *path*, the *complete graph* and the *cycle* on n vertices, respectively. For a uncyclic graph U, let $\mathbf{C} = y_1 y_2 \cdots y_n y_1$ be the *basic cycle* of U. We denote by $L(n, k)$ the *lollipop graph*, which is composed of the basic cycle $\mathbf{C}$ by appending a path of length $k (\geq 1)$.

An *adjacent vertex distinguishing total-coloring* (AVDTC for short) of a graph G is a proper total-coloring of G such that no pair of adjacent vertices receive same color set, where the color set of each vertex consists of the colors assigned on the vertex and its incident edges. The *adjacent vertex distinguishing total chromatic number* of G, denoted by $\chi_{at}(G)$, is the minimum number k for which G admits a k-AVDTC. In 2005, Zhang et al. [2] first introduced the concept of AVDTC of graphs, and proposed the following conjecture:

Conjecture 1 ([11]). *Let G be a graph with $|V(G)| \geq 2$. Then $\chi_{at}(G) \leq \Delta + 3$.*

After then, scholars have carried out a lot of research on the conjecture [2–9]. Especially, Chen [10] confirmed that the conjecture holds for graphs G with $\Delta(G) = 3$. And Wang [11] completely characterized the *adjacent vertex distinguishing total-coloring* of planar graphs G with $\Delta \geq 14$. In 2008, Zhang et al.[12] put forward the concept of *adjacent vertex strongly distinguishing total-coloring* of graphs. Here, we cite the definition as follows.

Definition 1 ([12]). *Let G be a connected graphs of order at least 3, and f be a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$, where $C_f\langle u\rangle = \{f(u)\} \cup \{f(v)|uv \in E(G)\} \cup \{f(uv)|uv \in E(G)\}$. If f satisfies the following conditions:*

1) $\forall uv \in E(G), f(u) \neq f(v), f(u) \neq f(uv), f(v) \neq f(uv)$;
2) $\forall uv, uw \in E(G), v \neq w, f(uv) \neq f(uw)$;
3) $\forall uv \in E(G), C_f\langle u\rangle \neq C_f\langle v\rangle$.

Then f is called a k-adjacent vertex strongly distinguishing total-coloring of G ($k - AVSDTC$ for short). The minimum number of k is called the adjacent vertex strongly distinguishing total chromatic number of G and denoted by $\chi_{ast}(G)$. Clearly, $\chi_{ast}(G) = \min\{k|k - AVSDTC \text{ of } G\}$.

Meanwhile, they also obtained the adjacent vertex strongly distinguishing total chromatic numbers of some special graphs such as *cycle*, *path*, *complete graph* and *complete bipartite graph*. Based on these results, they proposed the following conjecture.

Conjecture 2 ([12]). *Let G be a graph with $|V(G)| \geq 3$. Then $\chi_{ast}(G) \leq \Delta + 3$.*

Aimed at Conjecture 2, this paper mainly considers the adjacent vertex strongly distinguishing total coloring of unicyclic graphs. More specifically, we first prove that $\chi_{ast}(U) \leq \Delta(U)+2$ for a unicyclic graph U with $\Delta(U) \geq 3$, and then completely determine the adjacent vertex strongly distinguishing total chromatic number of the unicyclic graph U with $\Delta(U) = 3$.

II. Preliminaries

Lemma 2 ([12], Lemma 2.1). *For any connected graph G with $|V(G)| \geq 3$, $\chi_{ast}(G) \geq \Delta + 1$. Moreover, if G has adjacent maximum degree vertices, then $\chi_{ast}(G) \geq \Delta + 2$.*

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Lemma 3 ([12], Theorem 2.5)**.** *Let C_n be a cycle of order n. Then*

$$\chi_{ast}(C_n) = \begin{cases} 4, & n \neq 4, 10 \text{ and } n \text{ is even;} \\ 5, & \text{otherwise.} \end{cases}$$

Lemma 4 ([12], Theorem 2.3)**.** *Let P_m be a path of order $m(\geq 3)$. Then*

$$\chi_{ast}(P_m) = \begin{cases} 4, & m \equiv 1(\mathrm{mod}2); \\ 5, & m \equiv 0(\mathrm{mod}2). \end{cases}$$

For a graph G, let us define two special paths of G: if there is a path $\widetilde{P}_{k+1} = uu_1u_2 \cdots\ u_k$ satisfied $d(u) = 3$, $d(u_k) = 1$, and $d(u_i) = 2$ for $i = 1, 2, \cdots k-1$, then $\widetilde{P}_{k+1}$ is called a *I*-path of G; if there is a path $\widetilde{P}_{k+1} = uu_1u_2 \cdots u_k$ satisfied $d(u) = d(u_k) = 3$, and $d(u_i) = 2$ for $i = 1, 2, \cdots k-1$, then $\widetilde{P}_{k+1}$ is called a *II*-path of G. For convenience, if $|A| = k \geq 1$ for a set A, then we call that A is a k-set. Based on above, we give a lemma in the following:

Lemma 5. *Let f be a $4 - AVSDTC$ of G, and $P = v_1v_2 \cdots v_k$ be a path contained in G, where $d(v_i) = 2$ for $i = 1, 2, \cdots, k$. Then the 3-set and the 4-set appear alternately on the vertices of P.*

Proof: According to Definition 1, $|C_f\langle v\rangle| \geq d(v) + 1$ for any $v \in V(G)$. Without loss of generality, we may suppose that $|C_f\langle v_1\rangle| = 3$, then $|C_f\langle v_2\rangle| = 4$ since otherwise, if $|C_f\langle v_2\rangle| = 3$, then $C_f\langle v_1\rangle = C_f\langle v_2\rangle$ due to $|C_f\langle v_1\rangle \cap C_f\langle v_2\rangle| \geq 3$, a contradiction. And we further get $|C_f\langle v_3\rangle| = 3$. In this way, one can conclude that $|C_f\langle v_i\rangle| = 3$ for $i \equiv 1 \pmod 2$, and $|C_f\langle v_i\rangle| = 4$ for $i \equiv 0 \pmod 2$. Similarly, if $|C_f\langle v_1\rangle| = 4$, one can also prove that $|C_f\langle v_i\rangle| = 4$ for $i \equiv 1 \pmod 2$, and $|C_f\langle v_i\rangle| = 3$ for $i \equiv 0 \pmod 2$. ■

Lemma 6. *For a graph G, let x be a vertex of G with $d(x) = d$, and y be a neighbor of x. Suppose f is a proper total-coloring of G. If $d(y) \leq \lfloor \frac{d-1}{2} \rfloor$ or $d(y) \geq 2d+1$, then $C_f\langle x\rangle \neq C_f\langle y\rangle$.*

Proof: For vertex $x \in V(G)$ with $d(x) = d$, we have $d+1 \leq |C_f\langle x\rangle| \leq 2d+1$. If $d(y) \leq \lfloor \frac{d-1}{2} \rfloor$, then $|C_f\langle y\rangle| \leq 2d(y)+1 \leq d < d+1 \leq |C_f\langle x\rangle|$; similarly, if $d(y) \geq 2d+1$, then $|C_f\langle y\rangle| \geq 2d+2 > 2d+1 \geq |C_f\langle x\rangle|$. Therefore, $C_f\langle x\rangle \neq C_f\langle y\rangle$. ■

Let x be a pendant vertex of G and y the just neighbor of x. If $d(y) \geq 3$, then by Lemma 6 $C_f\langle x\rangle \neq C_f\langle y\rangle$. Therefore, we can obtain the following corollary.

Corollary 7. *Let f be a total-coloring of G, and x a pendant vertex of G. If y is the neighbor of x with $d(y) \geq 3$, then $C_f\langle x\rangle \neq C_f\langle y\rangle$.*

III. Main results

Theorem 8. *Let U be a unicyclic graph with $\nu \geq 4$ vertices and $\Delta(U) \geq 3$. Then*

$$\chi_{ast}(U) \leq \Delta(U) + 2.$$

Proof: We will prove the theorem by induction on ν.

If $\nu = 4$, then $U \cong L(3,1)$, it is easy to see that $\chi_{ast}(U) = 5$.

Denote by $\mathbf{C}$ the basic cycle of U, and suppose that the theorem is true for the unicyclic graphs with fewer than ν vertices. We distinguish two cases in the following.

Case 1. There exists a pendant vertex of U whose neighbor is outside $\mathbf{C}$.

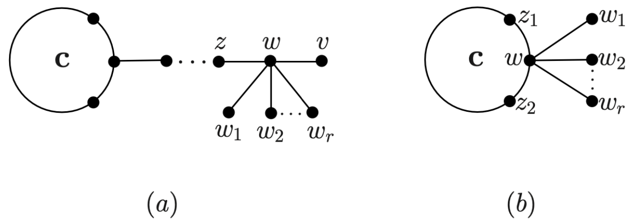

Fig. 1. The illustrations

Choosing a pendant vertex v of U such that $d_U(v, \mathbf{C}) = \max\{d_U(x, \mathbf{C}) | x \in V(U),\ d_U(x) = 1\}$, we suppose that w is the neighbor of v (see Fig. 1(a)). Let $N_U(w) = \{z, v, w_1, w_2, \cdots, w_r : 0 \leq r \leq \Delta - 2\}$, where $d_U(z) \geq 2$ and $d_U(w_i) = 1$ for $i = 1, 2, \cdots, r$. Suppose $U' = U - v$, then by assumption, $\chi_{ast}(U') \leq \Delta(U') + 2$ where $\Delta(U') \leq \Delta(U)$. Let f' be a $(\Delta(U') + 2) - AVSDTC$ of U'. Now we will extend f' to be a $(\Delta(U) + 2) - AVSDTC$ f for U. Set $f(v) = f'(zw)$, we distinct two subcases in the following.

Subcase 1.1. $r = 0$.

One can see that $d_U(w) = 2$. From Definition 1 we know that f should first be a proper total coloring of U. Thus, there are at most 2 forbidden colors for wv since $f(wv) \neq f'(w)$ and $f(wv) \neq f'(zw)$. On the other hand, if $|C_{f'}\langle z\rangle| - |C_{f'}\langle w\rangle| = 1$, then there is at most one color $f(wv) \in C_{f'}\langle z\rangle \setminus C_{f'}\langle w\rangle$ (In fact, there is just one forbidden color in $C_{f'}\langle z\rangle \setminus C_{f'}\langle w\rangle$) such that $C_f\langle z\rangle = C_f\langle w\rangle$. In addition, if $f(wv) = f'(z)$, it also leads to $C_f\langle w\rangle = C_f\langle v\rangle$. Hence, we have at least $(\Delta(U)+2) - 4 = \Delta(U) - 2 \geq 1$ available colors for wv.

Subcase 1.2. $r \geq 1$.

Clearly, $d_U(w) \geq 3$. For the edge wv, in order to ensure f is proper, there are at most Δ forbidden colors since $f(wv) \neq f'(w)$, $f(wv) \neq f'(zw)$ and $f(wv) \neq f'(ww_i)$ for $1 \leq i \leq r$, where $r \leq \Delta - 2$. Besides, if $|C_{f'}\langle z\rangle| - |C_{f'}\langle w\rangle| = 1$, then there is at most one forbidden color $f(wv) \in C_{f'}\langle z\rangle \setminus C_{f'}\langle w\rangle$ satisfying $C_f\langle z\rangle = C_f\langle w\rangle$. Note that $d_U(w) \geq 3$. From Corollary 7 we know that $C_f\langle w\rangle \neq C_f\langle v\rangle$ and $C_f\langle w\rangle \neq C_f\langle w_i\rangle$ for $1 \leq i \leq r$ (In what follows, if similar case occurs, one can prove it by Corollary 7, so we would not mention it again). Thus, there at least exists $(\Delta + 2) - (\Delta + 1) = 1$ available color for wv.

For other elements of U, we keep $f = f'$, and so, U admits a desired coloring f.

Case 2. The neighbor of each pendant vertex lies on $V(\mathbf{C})$.

Suppose that $w, z_1, z_2 \in V(\mathbf{C})$ and $wz_1 \in E(\mathbf{C}), wz_2 \in E(\mathbf{C})$ (see Fig. 1(b)). Let $N_U(w) = \{z_1, z_2, w_1, w_2, \cdots, w_r : 1 \leq r \leq \Delta - 2\}$, where $d_U(z_i) \geq 2$ for $i = 1, 2$, and $d_U(w_i) = 1$ for $i = 1, 2, \cdots, r$. Let $U' = U - w_r$. Then by assumption, $\chi_{ast}(U') \leq \Delta(U') + 2$ where $\Delta(U') \leq \Delta(U)$. Let f' be a $(\Delta(U') + 2) - AVSDTC$ of U'. Now we will extend f' to be desired coloring f of U.

Based on f', in order to ensure f is proper, there are at most Δ forbidden colors for ww_r since $f(ww_r) \neq f^{'}(w)$, $f(ww_r) \neq f^{'}(z_1w)$, $f(ww_r) \neq f^{'}(z_2w)$ and $f(ww_r) \neq f^{'}(ww_i)$ for $1 \leq i \leq r-1$, where $r \leq \Delta - 2$. Set $f(w_r) = f^{'}(z_1w)$ (if necessary, one can recolor it). Next, we will consider whether the color set of vertex w and its adjacent vertices is the same, and distinct two cases in the following.

(1) When $C_{f'}\langle z_1\rangle = C_{f'}\langle z_2\rangle$ and $|C_{f'}\langle z_1\rangle| - |C_{f'}\langle w\rangle| = 1$, there is at most one forbidden color such that $C_f\langle z_1\rangle = C_f\langle w\rangle = C_f\langle z_2\rangle$ (if $f(ww_r)$ is just the color of $C_{f'}\langle z_1\rangle \setminus C_{f'}\langle w\rangle$).

(2) When $C_{f'}\langle z_1\rangle \neq C_{f'}\langle z_2\rangle$, there are two subcases to consider.

i) $|C_{f'}\langle z_1\rangle| \neq |C_{f'}\langle z_2\rangle|$.

a) If $|C_{f'}\langle z_1\rangle| - |C_{f'}\langle w\rangle| = 1$, there is at most one forbidden color such that $C_f\langle z_1\rangle = C_f\langle w\rangle$.

b) If $|C_{f'}\langle z_2\rangle| - |C_{f'}\langle w\rangle| = 1$, there is at most one forbidden color such that $C_f\langle z_2\rangle = C_f\langle w\rangle$.

ii) $|C_{f'}\langle z_1\rangle| = |C_{f'}\langle z_2\rangle|$.

a) If $|C_{f'}\langle z_1\rangle| - |C_{f'}\langle w\rangle| = 1$ and $C_{f'}\langle w\rangle \nsubseteq C_{f'}\langle z_2\rangle$, there is at most one forbidden color such that $C_f\langle z_1\rangle = C_f\langle w\rangle$.

b) If $|C_{f'}\langle z_2\rangle| - |C_{f'}\langle w\rangle| = 1$ and $C_{f'}\langle w\rangle \nsubseteq C_{f'}\langle z_1\rangle$, there is at most one forbidden color such that $C_f\langle z_2\rangle = C_f\langle w\rangle$.

item [c)]If $|C_{f'}\langle z_1\rangle| - |C_{f'}\langle w\rangle| = 1$, $C_{f'}\langle w\rangle \subsetneqq C_{f'}\langle z_1\rangle$ and $C_{f'}\langle w\rangle \subsetneqq C_{f'}\langle z_2\rangle$, then we will recolor the vertex w_r. It is easy to see that $C_{f'}\langle w\rangle = C_{f'}\langle z_1\rangle \cap C_{f'}\langle z_2\rangle$ and $|C_{f'}\langle z_1\rangle \oplus C_{f'}\langle z_2\rangle| = 2$. Therefore, for f there exist two colors in $C_{f'}\langle z_1\rangle \oplus C_{f'}\langle z_2\rangle$ to assign w_r and ww_r, such that $C_f\langle z_1\rangle \neq C_f\langle w\rangle$ and $C_f\langle z_2\rangle \neq C_f\langle w\rangle$.

Hence, there at most exists one forbidden color in cases (1) and (2), except for the two cases, it just ensures that $f(ww_r)$ is a proper coloring in U.

Consequently, there at least exists $(\Delta+2) - (\Delta+1) \geq 1$ available color for ww_r to extend f' to f, such that $C_f\langle z_1\rangle \neq C_f\langle w\rangle$ and $C_f\langle w\rangle \neq C_f\langle z_2\rangle$. So we get a desired coloring f of U, and thus $\chi_{ast}(U) \leq \Delta(U) + 2$.

The proof is completed. ■

Theorem 9. *Let U be a unicyclic graph with $\Delta(U) = 3$ on $\nu(\geq 4)$ vertices, and $\mathbf{C} = y_1y_2\cdots y_ny_1$ the basic cycle of U. If U satisfies the following conditions:*

(1) *there exists two maximum degree adjacent vertices u_Δ and v_Δ in U;*

(2) *no two maximum degree vertices u_Δ and v_Δ are adjacent in U, but*

(a) *$n = 4, 10$ or n is odd;*

(b) *$n \geq 6$ $(n \neq 10)$ is even and U has a $I-$ even path or a $II-$ odd path;*

then $\chi_{ast}(U) = 5$; otherwise $\chi_{ast}(U) = 4$.

Proof: Let U be a unicyclic graph with $\Delta(U) = 3$ on $\nu \geq 4$ vertices. For convenience, we denote by $\mathbf{C} = y_1y_2\cdots y_ny_1$ the basic cycle of U. Now, we distinguish two cases in the following.

Case 1. There exists two maximum degree vertices $u_\Delta, v_\Delta \in V(U)$ such that $u_\Delta \sim v_\Delta$.

In this case, $\nu \geq 5$ since $\nu = 4$ is impossible. From Lemma 2 we know that $\chi_{ast}(U) \geq 5$. Then we prove the conclusion by induction on ν.

If $\nu = 5$, then we have $n = 3$, it is obvious that $\chi_{ast}(U) = 5$. Now, we assume that the conclusion is true for the unicyclic graphs with fewer than ν vertices.

Choosing a pendant vertex v of U such that $d_U(v, \mathbf{C}) = \max\{d_U(x, \mathbf{C}) | x \in V(U)$ and $d_U(x) = 1\}$, suppose that w is the neighbor of v. Let $N_U(w) = \{v, z\}$ if $d_U(w) = 2$, or $N_U(w) = \{v, z, u\}$ if $d_U(w) = 3$, where $2 \leq d_U(z) \leq 3$ and $d_U(u) = 1$. Suppose $U^{'} = U - v$, then by assumption, $U^{'}$ has a $5-AVSDTC\ f^{'}$.

Since $\forall x \in V(U')$, $|C_{f'}\langle x\rangle| \geq 3$, we have $3 \leq |C_{f'}\langle w\rangle| \leq 5$ and $3 \leq |C_{f'}\langle z\rangle| \leq 5$.

In what following, if $d_{U'}(w) = a$, $d_{U'}(z) = b$, $|C_{f'}\langle w\rangle| = c$ and $|C_{f'}\langle z\rangle| = d$, then we can abbreviate this ordered sequence as "*abcd* " for short. Now, we distinct two cases for $d_{U'}(w)$ below.

(1) When $d_{U'}(w) = 1$, we have $|C_{f'}\langle w\rangle| = 3$, and further get $4 \leq |C_{f'}\langle z\rangle| \leq 5$. Thus, $d_{U'}(w)$, $d_{U'}(z)$, $|C_{f'}\langle w\rangle|$ and $|C_{f'}\langle z\rangle|$ have 4 combinations: 1234, 1235, 1334 and 1335.

(2) When $d_{U'}(w) = 2$, we have $3 \leq |C_{f'}\langle w\rangle| \leq 5$, however, $|C_{f'}\langle w\rangle| = 3$ implies that $C_{f'}\langle w\rangle = C_{f'}\langle u\rangle$ because of $|C_{f'}\langle w\rangle \cap C_{f'}\langle u\rangle| \geq 3$, and thus, we have $4 \leq |C_{f'}\langle w\rangle| \leq 5$. Here $d_{U'}(w)$, $d_{U'}(z)$, $|C_{f'}\langle w\rangle|$ and $|C_{f'}\langle z\rangle|$ have 8 combinations: 2243, 2244, 2245, 2253, 2254, 2344, 2354 and 2345, except for 2255, 2355, 2343 and 2353 since if 2255 and 2355 occur, then $|C_{f'}\langle w\rangle| = |C_{f'}\langle z\rangle| = 5$, which leads to $C_{f'}\langle w\rangle = C_{f'}\langle z\rangle$ in $U^{'}$, a contradiction, and if 2343 and 2353 occur, then $d_{U'}(z) = 3$, one can further get $|C_{f'}\langle z\rangle| \geq 4$, clearly, they are also impossible.

If 1234 and 1334 occur, set

$$f(x) = \begin{cases} f^{'}(wz), & x = v; \\ t, & x = wv, \text{ and } t \in \{1,2,3,4,5\} \setminus C_{f'}\langle z\rangle; \\ f^{'}(x), & \text{otherwise}. \end{cases}$$

Then it follows from $|C_{f'}\langle w\rangle| = 3$, $|C_{f'}\langle z\rangle| = 4$ and $f(wv) \in \{1,2,3,4,5\} \setminus C_{f'}\langle z\rangle$ that $|C_f\langle w\rangle| = 4$ and $C_f\langle w\rangle \neq C_{f'}\langle z\rangle$. Since $C_{f'}\langle z\rangle = C_f\langle z\rangle$ we have $C_f\langle w\rangle \neq C_f\langle z\rangle$. In addition, one can get $C_f\langle w\rangle \neq C_f\langle v\rangle$ due to $|C_f\langle v\rangle| = 3$.

If 2243, 2245, 2253, 2254, 2345 and 2354 occur, set

$$f(x) = \begin{cases} f^{'}(wz), & x = v; \\ t, & x = wv, \text{ and } t \in C_{f'}\langle w\rangle \setminus \\ & \{f^{'}(w), f^{'}(wz), f^{'}(wu)\}; \\ f^{'}(x), & \text{otherwise}. \end{cases}$$

Then one can get $C_f\langle w\rangle = C_{f'}\langle w\rangle$ since $f(wv) \in C_{f'}\langle w\rangle \setminus \{f^{'}(w), f^{'}(wz), f^{'}(wu)\}$, and thus $C_f\langle w\rangle \neq C_f\langle z\rangle$. Furthermore, by Corollary 7 we have that $C_f\langle w\rangle \neq C_f\langle v\rangle$ and $C_f\langle w\rangle \neq C_f\langle u\rangle$.

If 1235, 1335, 2244 and 2344 occur, set

$$f(x)=\begin{cases} f^{'}(wz), & x=v; \\ t, & x=wv \text{ and } t\in\{1,2,3,4,5\}\setminus C_{f^{'}}\langle w\rangle; \\ f^{'}(x), & \text{otherwise.} \end{cases}$$

Then we have $||C_f\langle w\rangle|-|C_f\langle z\rangle||=1$ due to $||C_{f'}\langle w\rangle|-|C_{f'}\langle z\rangle||=0$, or 2 and $f(wv)\in\{1,2,3,4,5\}\setminus C_{f'}\langle w\rangle$, and so, $C_f\langle w\rangle\neq C_f\langle z\rangle$. Meanwhile, by Corollary 7 it follows that $C_f\langle w\rangle\neq C_f\langle v\rangle$ and $C_f\langle w\rangle\neq C_f\langle u\rangle$.

Therefore, U admits a $5-AVSDTC$.

Case 2. No two maximum degree vertices u_Δ and v_Δ are adjacent in U.

Subcase 2.1. U has only one maximum degree vertex, namely $U\cong L(n,k)$.

According to Lemma 2 we know that $\chi_{ast}(U)\geq 4$. Let $\widetilde{p}_{k+1}=uu_1u_2\cdots u_k$ be a *I*-path, where $d(u_k)=1$. Without loss of generality, suppose $u=y_1$, clearly $d(y_1)=3$. Assume by contradiction that, U has a $4-AVSDTC$ $f:V(U)\cup E(U)\rightarrow\{1,2,3,4\}$.

Subcase 2.1.1. When $n=4$, $U\cong L(4,\nu-4)$.

Without loss of generality, let $f(y_1)=1$, $f(y_2)=2$, $f(y_1y_2)=3$. According to $d(y_1)=3$ we have $|C_f\langle y_1\rangle|=4$, which leads to $|C_f\langle y_2\rangle|=|C_f\langle y_4\rangle|=3$, then we only get $f(y_2y_3)=1$, $f(y_3)=3$. By Lemma 5 we have $|C_f\langle y_3\rangle|=4$, which implies that $f(y_3y_4)=4$ or $f(y_4)=4$.

- If $f(y_3y_4)=4$, then $f(y_4)=2$, and we further have $|C_f\langle y_4\rangle|=4$, hence, it results in $C_f\langle y_4\rangle=C_f\langle y_1\rangle$, a contradiction;
- If $f(y_4)=4$, then $f(y_3y_4)=2$, so we get $|C_f\langle y_4\rangle|=4$, it also a contradiction.

Subcase 2.1.2. When $n=10$, $U\cong L(10,\nu-10)$.

Since $d(y_1)=3$ and $\chi_{ast}(U)=4$, we have $|C_f\langle y_i\rangle|=4, i\equiv 1 \pmod 2$; $|C_f\langle y_i\rangle|=3, i\equiv 0 \pmod 2$ by Lemma 5. Without loss of generality, let $f(y_1)=1$, $f(y_2)=2$, $f(y_1y_2)=3$. According to $|C_f\langle y_2\rangle|=3$ we have $C_f\langle y_2\rangle=\{1,2,3,\}$, then $f(y_2y_3)=1$, $f(y_3)=3$. Now, we distinct three steps to deduce a contradiction.

Step 1: since U has a $4-AVSDTC$ f and $|C_f\langle y_3\rangle|=4$ we have $f(y_3y_4)=4$ or $f(y_4)=4$, but $4\notin C_f\langle y_2\rangle$, thus $C_f\langle y_2\rangle\neq C_f\langle y_4\rangle$.

Step 2: we suppose, by a contradiction that $C_f\langle y_6\rangle=C_f\langle y_2\rangle=\{1,2,3\}$. From $|C_f\langle y_5\rangle|=4$ we have $f(y_4y_5)=4$ or $f(y_4)=4$, which leads to $f(y_3y_4)=f(y_4y_5)$, or $f(y_3y_4)=f(y_4)$, or $f(y_4y_5)=f(y_4)$, a contradiction. Let $f(y_4)=4$. From $|C_f\langle y_4\rangle|=3$ we know $f(y_3y_4)=2$, $f(y_4y_5)=3$, $f(y_5)=2$, $C_f\langle y_4\rangle=\{2,3,4\}$. $C_f\langle y_6\rangle=\{1,2,3\}$ implies that $f(y_5y_6)=1$, $f(y_6)=3$, $f(y_6y_7)=2$, $f(y_7)=1$. Since $|C_f\langle y_8\rangle|=3$, $f(y_8y_9)=1$. And because $|C_f\langle y_{10}\rangle|=3$, $f(y_9y_{10})=1$, a contradiction. So $C_f\langle y_2\rangle\neq C_f\langle y_6\rangle$.

Step 3: From $|C_f\langle y_3\rangle|=4$ we have either $f(y_3y_4)=4$ or $f(y_4)=4$.

- If $f(y_3y_4)=4$, by $|C_f\langle y_4\rangle|=3$ we have $C_f\langle y_4\rangle=\{1,3,4\}$ or $\{2,3,4\}$ and $f(y_4y_5)=3$, $f(y_5)=4$. No matter what $f(y_4)=1$ or $f(y_4)=2$, it will lead to $C_f\langle y_4\rangle\neq C_f\langle y_6\rangle$.
- If $f(y_4)=4$, from $|C_f\langle y_4\rangle|=3$ we know $f(y_3y_4)=2$, $f(y_4y_5)=3$, $f(y_5)=2$, $C_f\langle y_4\rangle=\{2,3,4\}$. And because $|C_f\langle y_5\rangle|=4$ we have $f(y_5y_6)=1$ or $f(y_6)=1$. So $C_f\langle y_4\rangle\neq C_f\langle y_6\rangle$.

In the same way, we can obtain that the five color set $C\ _f\langle y_2\rangle$, $C_f\langle y_4\rangle$, $C_f\langle y_6\rangle$, $C_f\langle y_8\rangle$ and $C_f\langle y_{10}\rangle$ are different from each other. But $C\ _4^3=4<5$, a contradiction.

Subcase 2.1.3. $n\geq\ 3$ is odd.

Note that $d(y_1)=3$. By Lemma 5 one can see that $|C_f\langle y_i\rangle|=3, i\equiv 0 \pmod 2$ and $|C_f\langle y_i\rangle|=4, i\equiv 1 \pmod 2$. However, $|C_f\langle y_n\rangle|=|C_f\langle y_1\rangle|=4$, a contradiction.

Subcase 2.1.4. $n\geq 6$ $(n\neq 10)$ is even and U has a *I*-even path $\widetilde{p}_{k+1}$.

Since $d(y_1)=3$, $|C_f\langle y_1\rangle|=4$. By Lemma 5, we have $|C_f\langle u_i\rangle|=3, i\equiv 1 \pmod 2$; $|C_f\langle u_i\rangle|=4, i\equiv 0 \pmod 2$. However, $k-1$ is odd, it therefore leads to a contradiction since $|C_f\langle u_{k-1}\rangle|=|C_f\langle u_k\rangle|=3$.

From above all, we have $\chi_{ast}(U)\geq 5$.

We next prove $\chi_{ast}(U)=5$ by considering the length of *I*-path $\widetilde{P}_{k+1}$. For $k=1$, $\widetilde{P}_{k+1}=y_1u_1$, $d(y_1)=3$. By Lemma 3, $\chi_{ast}(C_n)=5$ if $n=4,10$, or n is odd. Suppose $g:V(\mathbf{C})\cup E(\mathbf{C})\rightarrow\{1,2,3,4,5\}$. For each vertex of $\mathbf{C}$, $3\leq|C_g\langle y_i\rangle|\leq 5$ since $d(y_i)=2$, $i=1,2$ and n. Then we color the y_1u_1 and u_1.

In the following, if $|C_g\langle y_1\rangle|=a$, $|C_g\langle y_2\rangle|=b$ and $|C_g\langle y_n\rangle|=c$, then we can abbreviate this ordered sequence as "*abc*" for short. We will distinct three cases for $|C_g\langle y_1\rangle|$ below.

(1) When $|C_g\langle y_1\rangle|=3$, by Lemma 5, $|C_g\langle y_i\rangle|\geq 4$ for $i=2,n$. Thus, $|C_g\langle y_1\rangle|$, $|C_g\langle y_2\rangle|$ and $|C_g\langle y_n\rangle|$ have 4 combinations: 344, 345, 354 and 355.

(2) When $|C_g\langle y_1\rangle|=4$, $|C_g\langle y_1\rangle|$, $|C_g\langle y_2\rangle|$ and $|C_g\langle y_n\rangle|$ have 9 combinations: 433, 434, 435, 443, 444, 445, 453, 454 and 455.

(3) When $|C_g\langle y_1\rangle|=5$, $|C_g\langle y_1\rangle|$, $|C_g\langle y_2\rangle|$ and $|C_g\langle y_n\rangle|$ have 4 combinations: 533, 534, 543 and 544 since if 535 and 545 occur, then $C_g\langle y_1\rangle=C_g\langle y_n\rangle$, and if 553, 554 and 555 occur, then $C_g\langle y_1\rangle=C_g\langle y_2\rangle$, all those are impossible.

If 344 occurs, then $|\{1,2,3,4,5\}\setminus\ C_g\langle y_1\rangle|=2$. Thus, there exist two colors to assign y_1u_1 and u_1, which leads to $|C_f\langle y_1\rangle|=5$, and so $C_f\langle y_1\rangle\neq C_f\langle y_2\rangle$ and $C_f\langle y_1\rangle\neq C_f\langle y_n\rangle$. Moreover, it follows from Corollary 7 that $C_f\langle y_1\rangle\neq C_f\langle u_1\rangle$.

If 345 occurs, set

$$f(x)=\begin{cases} g(y_1y_2), & x=u_1; \\ t, & x=y_1u_1, \text{ and } t\in C_g\langle y_n\rangle\setminus\ C_g\langle y_2\rangle; \\ g(x), & \text{otherwise.} \end{cases}$$

From $f(y_1u_1)\in C_g\langle y_n\rangle\setminus\ C_g\langle y_2\rangle$ we have $|C_f\langle y_1\rangle|=4$ and $C_f\langle y_1\rangle\neq C_f\langle y_2\rangle$. In addition, we can get $C_f\langle y_1\rangle\neq C_f\langle y_n\rangle$ as $|C_f\langle y_n\rangle|=5$. Moreover, by Corollary 7 we have $C_f\langle y_1\rangle\neq C_f\langle u_1\rangle$. Similarly, the conclusion is also true whence 354 occurs.

If 355, 433, 434, 443 and 444 occur, set

$$f(x)=\begin{cases} g(y_1y_2), & x=u_1; \\ t, & x=y_1u_1 \text{ and } t\in\{1,2,3,4,5\}\setminus C_g\langle y_1\rangle; \\ g(x), & \text{otherwise.} \end{cases}$$

Then we get $||C_f\langle y_1\rangle|-|C_f\langle y_i\rangle||=\ 1$ or 2 since $||C_g\langle y_1\rangle|-|C_g\langle y_i\rangle||=0,1,2$ and $f(y_1u_1)\in\{1,2,3,4,5\}\setminus C_g\langle y_1\rangle$ for $i=2,n$. Thus, $C_f\langle y_1\rangle\neq C_f\langle y_2\rangle$ and $C_f\langle y_1\rangle\neq$

$C_f\langle y_n\rangle$. Identically, by Corollary 7 we have $C_f\langle y_1\rangle \neq C_f\langle u_1\rangle$.

If 435, 445, 453, 454, 455, 533, 534, 543 and 544 occur, set

$$f(x)=\begin{cases} g(y_1y_2), & x=u_1;\\ t, & x=y_1u_1,\ \text{and } t\in\ C_g\langle y_1\rangle\setminus\\ & \{g(y_1),g(y_1y_2),g(y_1y_n)\};\\ g(x), & \text{otherwise.}\end{cases}$$

Then $C_f\langle y_1\rangle = C_g\langle y_1\rangle$ due to $f(y_1u_1) \in C_g\langle y_1\rangle \setminus \{g(y_1), g(y_1y_2), g(y_1y_n)\}$, and thus, $C_f\langle y_1\rangle \neq C_f\langle y_2\rangle$ and $C_f\langle y_1\rangle \neq C_f\langle y_n\rangle$. Moreover, it follows from Corollary 7 that $C_f\langle y_1\rangle \neq C_f\langle u_1\rangle$. Thus, $\chi_{ast}(U)=5$.

For $k=2$, $\widetilde{P}_{k+1}=y_1u_1u_2$. Base on $k=1$ above, $d(y_1)=3$ and $d(u_1)=1$, so we get $4\leq |C_g\langle y_1\rangle|\leq 5$, $|C_g\langle u_1\rangle|=3$. We then color u_1u_2 and u_2. Now, $|C_g\langle y_1\rangle|$ and $|C_g\langle u_1\rangle|$ have 2 combinations: 43 (i.e., $|C_g\langle y_1\rangle|=4$ and $|C_g\langle u_1\rangle|=3$) and 53.

If 43 occurs, then

$$f(x)=\begin{cases} g(y_1u_1), & x=u_2;\\ t, & x=u_1u_2,\ \text{and } t\in\{1,2,3,4,5\}\setminus\\ & \ C_g\langle y_1\rangle;\\ g(x), & \text{otherwise.}\end{cases}$$

Thus, $C_f\langle y_1\rangle \neq C_f\langle u_1\rangle$ and $|C_f\langle u_1\rangle|=4$ due to $f(u_1u_2)\in\{1,2,3,4,5\}\setminus\ C_g\langle y_1\rangle$. Note that $|C_f\langle u_2\rangle|=3$, so we have $C_f\langle u_1\rangle\neq C_f\langle u_2\rangle$.

If 53 occurs, set

$$f(x)=\begin{cases} g(y_1u_1), & x=u_2;\\ t, & x=u_1u_2,\ \text{and } t\in\{1,2,3,4,5\}\setminus\\ & \ C_g\langle u_1\rangle;\\ g(x), & \text{otherwise.}\end{cases}$$

Then, it follows from $f(u_1u_2)\in\{1,2,3,4,5\}\setminus\ C_g\langle u_1\rangle$ that $|C_f\langle u_1\rangle|=4$. Besides, $|C_f\langle y_1\rangle|=5$ and $|C_f\langle u_2\rangle|=3$. Thus, $C_f\langle y_1\rangle\neq C_f\langle u_1\rangle$ and $C_f\langle u_1\rangle\neq C_f\langle u_2\rangle$. Hence $\chi_{ast}(U)=5$.

For $k\geq 3$, we prove the conclusion by induction on $\nu\geq 6$. If $\nu=6$, then we have $n=3$, it is obvious that $\chi_{ast}(U)=5$. Assume that the conclusion is true for the unicyclic graphs with fewer than ν vertices. Choosing a pendant vertex v of U such that $d_U(v,\mathbf{C})=\max\{d_U(x,\mathbf{C})|x\in V(U),\ d_U(x)=1\}$ and $v\sim w\sim z\sim s$. Suppose $U^{'}=U-\{v,w\}$, then by assumption, $U^{'}$ has a $5-AVSDTC\ f^{'}$. Let $f^{'}: V(U)\cup E(U)\rightarrow\{1,2,3,4,5\}$. Then we set

$$f(x)=\begin{cases} f^{'}(s), & x=zw;\\ f^{'}(sz), & x=w;\\ t, & x=wv \text{ and } t\in\{1,2,3,4,5\}\setminus C_{f^{'}}\langle z\rangle;\\ f^{'}(zw), & x=v;\\ f^{'}(x), & \text{otherwise.}\end{cases}$$

Because $f(zw)=f^{'}(s)$ and $f(w)=f^{'}(sz)$ we have $C_{f^{'}}\langle z\rangle=C_f\langle z\rangle$ and $|C_f\langle z\rangle|=3$, and further get $C_f\langle s\rangle\neq C_f\langle z\rangle$. In addition, from $f(wv)\in\{1,2,3,4,5\}\setminus C_{f^{'}}\langle z\rangle$ and $f(v)=f^{'}(zw)$ we know that $|C_f\langle w\rangle|=4$ and $C_f\langle z\rangle\neq C_f\langle w\rangle$. Note that $|C_f\langle v\rangle|=3$. So we have $C_f\langle w\rangle\neq C_f\langle v\rangle$. Thus, $\chi_{ast}(U)=5$.

Subcase 2.2. U has at least two non-adjacent vertices with maximum degree. From Lemma 2 we know that $\chi_{ast}(U)\geq 4$. For convenience, we denote by $\mathbf{C}=y_1y_2\cdots y_ny_1$ the basic cycle of U, $\widetilde{p}_{k+1}=uu_1u_2\cdots u_k$ is a *I*-path or *II*-path, where $d(u)=3$. Assume by contradiction that, U has a $4-AVSDTC\ f: V(U)\cup E(U)\rightarrow\{1,2,3,4\}$.

Subcase 2.2.1. $n=4$.

Without loss of generality, we assume that $d(y_1)=d(y_3)=3$, $f(y_1)=1$, $f(y_2)=2$, $f(y_1y_2)=3$. According to $d(y_1)=d(y_3)=3$ we have $|C_f\langle y_1\rangle|=|C_f\langle y_3\rangle|=4$. By Lemma 5, $|C_f\langle y_2\rangle|=|C_f\langle y_4\rangle|=3$. Thus $f(y_2y_3)=1$, $f(y_3)=3$. If $C_f\langle y_4\rangle=C_f\langle y_2\rangle=\{1,2,3\}$, then $f(y_4)=2$, $f(y_1y_4)=3$, further we know $f(y_1y_4)=f(y_1y_2)=3$, a contradiction. So $C_f\langle y_2\rangle\neq C_f\langle y_4\rangle$, it means that $f(y_3y_4)=4$, or $f(y_4)=4$, or $f(y_1y_4)=4$.

- If $f(y_3y_4)=4$, then $f(y_1y_4)=2$, we further get $|C_f\langle y_4\rangle|=4$, thus, it is a contradiction because $C_f\langle y_4\rangle=C_f\langle y_3\rangle$.
- If $f(y_4)=4$, then $f(y_3y_4)=2$, which leads to $|C_f\langle y_4\rangle|=4$, a contradiction.
- If $f(y_1y_4)=4$, then $f(y_4)=2$, so $|C_f\langle y_4\rangle|=4$, it also a contradiction.

Subcase 2.2.2. $n=10$.

We assume that U has only two maximum degree vertices. Without loss of generality, let $d(y_1)=d(y_3)=3$, $f(y_1)=1$, $f(y_2)=2$, $f(y_1y_2)=3$. By Lemma 5, we have $|C_f\langle y_i\rangle|=3, i\equiv 0 \pmod 2$; $|C_f\langle y_i\rangle|=4, i\equiv 1 \pmod 2$. From $|C_f\langle y_2\rangle|=3$ we know $C_f\langle y_2\rangle=\{1,2,3\}$, then $f(y_2y_3)=1$, $f(y_3)=3$. We distinct two steps to deduce a contradiction.

Step 1: suppose $C_f\langle y_4\rangle=C_f\langle y_2\rangle=\{1,2,3\}$, then $f(y_3y_4)=2$, $f(y_4)=1$, $f(y_4y_5)=3$, $f(y_5)=2$. We have $f(y_5y_6)=4$ or $f(y_6)=4$ since $|C_f\langle y_5\rangle|=4$. Similar to the proof as Subcase 2.1.2 of Theorem 9 we get $C_f\langle y_4\rangle\neq C_f\langle y_6\rangle$. Hence, it suggests that $C_f\langle y_6\rangle=\{1,2,4\}$ or $\{2,3,4\}$.

- If $C_f\langle y_6\rangle=\{1,2,4\}$, then $f(y_6y_7)=2$. According to Subcase 2.1.2 of Theorem 9, we can obtain that $C_f\langle y_4\rangle$, $C_f\langle y_6\rangle$, $C_f\langle y_8\rangle$ and $C_f\langle y_{10}\rangle$ are different from each other. And from $f(y_1)=1$ we know $C_f\langle y_{10}\rangle=\{1,3,4\}$, $C_f\langle y_8\rangle=\{2,3,4\}$. Thus, $f(y_1y_{10})=4$, $f(y_{10})=3$, $f(y_9y_{10})=1$, $f(y_9)=4$, and we further have $f(y_7y_8)=4$. According to $f(y_6y_7)=2$ and $C_f\langle y_8\rangle=\{2,3,4\}$ we get $f(y_8)=2$, $f(y_7)=3$, it results $C_f\langle y_6\rangle=\{1,2,3,4\}$, a contradiction. Hence, $C_f\langle y_2\rangle\neq C_f\langle y_4\rangle$.
- If $C_f\langle y_6\rangle=\{2,3,4\}$, we can also get $C_f\langle y_2\rangle\neq C_f\langle y_4\rangle$.

Step 2: suppose $C_f\langle y_6\rangle=C_f\langle y_2\rangle=\{1,2,3\}$, according to $|C_f\langle y_3\rangle|=|C_f\langle y_5\rangle|=4$, it is easy to see that $f(y_4)=4$. From $|C_f\langle y_4\rangle|=3$ we have $f(y_3y_4)=2$, $f(y_4y_5)=3$, $f(y_5)=2$, so $C_f\langle y_4\rangle=\{2,3,4\}$. And because $C_f\langle y_6\rangle=\{1,2,3\}$ we get $f(y_5y_6)=1$, $f(y_6)=3$, $f(y_6y_7)=2$, $f(y_7)=1$. From $|C_f\langle y_8\rangle|=3$ we know $f(y_8y_9)=1$. However, since $|C_f\langle y_{10}\rangle|=3$ and $f(y_1)=1$, $f(y_9y_{10})=1$, it also a contradiction because $f(y_8y_9)=f(y_9y_{10})$. So $C_f\langle y_2\rangle\neq C_f\langle y_6\rangle$.

In the same way, we can obtain $C_f\langle y_2\rangle$, $C_f\langle y_4\rangle$, $C_f\langle y_6\rangle$, $C_f\langle y_8\rangle$ and $C_f\langle y_{10}\rangle$ are different from each other. But $C_4^3=4<5$, there is at least one pair of vertices with the same color set, a contradiction. Analogously, we can prove that $\chi_{ast}(U) \geq 5$ if there are three, four and five maximum vertices on the basic cycle $\mathbf{C}$.

Subcase 2.2.3. When either n is odd, or $n \geq 6$ $(n \neq 10)$ is even and U has a $I-$ even path, the proof is similar to Subcases 2.1.3 and 2.1.4, respectively.

Subcase 2.2.4. $n \geq 6$ $(n \neq 10)$ is even and U has a $II-$ odd path.

Let one of such paths be $\widetilde{P}_{k+1} = uu_1u_2\cdots u_k$, where $d(u) = d(u_k) = 3$, $d(u_i) = 2, i = 1, 2, \cdots, k-1$, $k \equiv 1 \pmod 2$. Without loss of generality, we assume that $u = y_1$. Since $d(y_1) = d(u_k) = 3$, $|C_f\langle y_1\rangle| = |C_f\langle u_k\rangle| = 4$. By Lemma 5, then we obtain $|C_f\langle u_i\rangle| = 3$, $i \equiv 1 \pmod 2$, and $|C_f\langle u_i\rangle| = 4$, $i \equiv 0 \pmod 2$. Hence, $|C_f\langle u_{k-1}\rangle| = |C_f\langle u_k\rangle| = 4$, a contradiction.

From all above, thus $\chi_{ast}(U) \geq 5$. Next we prove $\chi_{ast}(U) = 5$ by induction on $\nu \geq 6$.

If $\nu = 6$, then we have $n = 4$, it is obvious that $\chi_{ast}(U) = 5$. Now, we suppose that the conclusion is true for the unicyclic graphs with fewer than ν vertices.

Choosing a pendant vertex v of U such that$d_U(v, \mathbf{C}) = max\{d_U(x, \mathbf{C})|x \in V(U), d_U(x) = 1\}$, we suppose that w is the neighbor of v. Let $N_U(w) = \{v, z\}$ if $d_U(w) = 2$, where $2 \leq d_U(z) \leq 3$; or $N_U(w) = \{v, z, u\}$ if $d_U(w) = 3$, where $d_U(z) = 2$ and $d_U(u) = 1$. Suppose $U^{'} = U - v$, then by assumption, $U^{'}$ has $5 - AVSDTC$ f.

Since $\forall x \in V(U')$, $|C_{f'}\langle x\rangle| \geq 3$, we have $3 \leq |C_{f'}\langle w\rangle| \leq 5$ and $3 \leq |C_{f'}\langle z\rangle| \leq 5$.

(1) When $d_{U'}(w) = 1$, we have $|C_{f'}\langle w\rangle| = 3$, and further get $4 \leq |C_{f'}\langle z\rangle| \leq 5$. Now, $d_{U'}(w)$, $d_{U'}(z)$, $|C_{f'}\langle w\rangle|$ and $|C_{f'}\langle z\rangle|$ have 4 combinations: 1234, 1235, 1334 and 1335.

(2) When $d_{U'}(w) = 2$, we have $4 \leq |C_{f'}\langle w\rangle| \leq 5$, and further get $d_{U'}(z) = 2$ since no two maximum degree vertices are adjacent. Thus, $d_{U'}(w)$, $d_{U'}(z)$, $|C_{f'}\langle w\rangle|$ and $|C_{f'}\langle z\rangle|$ have 5 combinations: 2243, 2244, 2245, 2253 and 2254 but 2255. If 2255 occur, then $|C_{f'}\langle w\rangle| = |C_{f'}\langle z\rangle| = 5$, which results in $C_{f'}\langle w\rangle = C_{f'}\langle z\rangle$ in $U^{'}$, a contradiction.

If 1234 and 1334 occur, set

$$f(x) = \begin{cases} f^{'}(wz), & x = v; \\ t, & x = wv \text{ and } t \in \{1,2,3,4,5\} \setminus C_{f'}\langle z\rangle; \\ f^{'}(x), & \text{otherwise.} \end{cases}$$

From $|C_{f'}\langle w\rangle| = 3$, $|C_{f'}\langle z\rangle| = 4$ and $f(wv) \in \{1,2,3,4,5\} \setminus C_{f'}\langle z\rangle$ we know that $C_f\langle w\rangle \neq C_{f'}\langle z\rangle$ and $|C_f\langle w\rangle| = 4$. In addition, by $C_{f'}\langle z\rangle = C_f\langle z\rangle$ we have $C_f\langle w\rangle \neq C_f\langle z\rangle$, meanwhile, $C_f\langle w\rangle \neq C_f\langle v\rangle$ due to $|C_f\langle v\rangle| = 3$.

If 2243, 2245, 2253 and 2254 occur, set

$$f(x) = \begin{cases} f^{'}(wz), & x = v; \\ t, & x = wv, \text{ and } t \in C_{f'}\langle w\rangle \setminus \\ & \{f^{'}(w), f^{'}(wz), f^{'}(wu)\}; \\ f^{'}(x), & \text{otherwise.} \end{cases}$$

Then we get $C_f\langle w\rangle = C_{f'}\langle w\rangle$ because $f(wv) \in C_{f'}\langle w\rangle \setminus \{f^{'}(w), f^{'}(wz), f^{'}(wu)\}$, and so $C_f\langle w\rangle \neq C_f\langle z\rangle$. Besides these, we also have $C_f\langle w\rangle \neq C_f\langle v\rangle$ and $C_f\langle w\rangle \neq C_f\langle u\rangle$ by Corollary 7.

If 1235, 1335 and 2244 occur, set

$$f(x) = \begin{cases} f^{'}(wz), & x = v; \\ t, & x = wv \text{ and } t \in \{1,2,3,4,5\} \setminus C_{f'}\langle w\rangle; \\ f^{'}(x), & \text{otherwise.} \end{cases}$$

Hence, we have $||C_f\langle w\rangle| - |C_f\langle z\rangle|| = 1$ since $||C_{f'}\langle w\rangle| - |C_{f'}\langle z\rangle|| = 0, 2$ and $f(wv) \in \{1,2,3,4,5\} \setminus C_{f'}\langle w\rangle$, and so, $C_f\langle w\rangle \neq C_f\langle z\rangle$. Also by Corollary 7, one can get $C_f\langle w\rangle \neq C_f\langle v\rangle$ and $C_f\langle w\rangle \neq C_f\langle u\rangle$. Thus, U admits a $5 - AVSDTC$.

Subcase 2.3. $n \geq 6$ $(n \neq 10)$ is even and U doesn't contain both *I*-even paths and *II*-odd paths.

From Lemma 2 we know that $\chi_{ast}(U) \geq 4$. We next prove $\chi_{ast}(U) = 4$ by considering $d_U(v, \mathbf{C}) = \max\{d_U(x, \mathbf{C})|x \in V(U), d_U(x) = 1\}$.

When $d_U(v, \mathbf{C}) = 1$, we have $d(y_i) = 3$ for some i, we denote by y_iu_i the pendant edges of y_i. By Lemma 3 $\chi_{ast}(\mathbf{C}) = 4$ if $n \geq 6$ $(n \neq 10)$ is even. For convenience, we may suppose $g : V(\mathbf{C}) \cup E(\mathbf{C}) \to \{1,2,3,4\}$. Base on the coloring of $\mathbf{C}$, we then color all pendant edges of U. Let

$$f(x) = \begin{cases} g(y_iy_{i+1}), & x = u_i; \\ t, & x = y_iu_i, \text{ and } t \in C_g\langle y_i\rangle \setminus \\ & \{g(y_i), g(y_iy_{i+1}), g(y_{i-1}y_i)\}; \\ g(x), & \text{otherwise} \end{cases}$$

where the subscripts are taken modulo n.

Now, we have $C_f\langle y_i\rangle = C_g\langle y_i\rangle$ due to $f(u_i) = g(y_iy_{i+1})$ and $f(y_iu_i) \in C_g\langle y_i\rangle \setminus \{g(y_i), g(y_iy_{i+1}), g(y_{i-1}y_i)\}$, and thus, $C_f\langle y_i\rangle \neq C_f\langle y_{i+1}\rangle$. Furthermore, by Corollary 7 it deduces that $C_f\langle y_i\rangle \neq C_f\langle u_i\rangle$. So we get $\chi_{ast}(U) = 4$.

Clearly, $d_U(v, \mathbf{C}) = 2$ is impossible since G has no adjacent vertices with maximum degree, which will lead to a *I*-even paths in G, a contradiction.

When $d_U(v, \mathbf{C}) \geq 3$, we prove the conclusion by induction on $\nu \geq 7$. For $\nu = 7$, we have $n = 6$, it is obvious that $\chi_{ast}(U) = 4$. Assume that the conclusion is true for the unicyclic graphs with fewer than ν vertices. We select a pendant vertex v of U such that $d_U(v, \mathbf{C}) = \max\{d_U(x, \mathbf{C})|x \in V(U), d_U(x) = 1\}$ and $v \sim w \sim z \sim s$, and then distinct two cases in the following.

(1) If $d_U(w) = 2$,we denote by $N_U(w) = \{v, z\}$ where $d_U(z) = 2$. Suppose $U^{'} = U - \{v, w\}$, then by assumption, $U^{'}$ has a $4 - AVSDTC$ $f^{'}$. Let $f^{'} : V(U) \cup E(U) \to \{1,2,3,4\}$. Then we set

$$f(x) = \begin{cases} f^{'}(s), & x = zw; \\ f^{'}(sz), & x = w; \\ t, & x = wv \text{ and } t \in \{1,2,3,4\} \setminus C_{f'}\langle z\rangle; \\ f^{'}(zw), & x = v; \\ f^{'}(x), & \text{otherwise.} \end{cases}$$

Since $f(zw) = f^{'}(s)$ and $f(w) = f^{'}(sz)$ we have $C_{f'}\langle z\rangle = C_f\langle z\rangle$ and $|C_f\langle z\rangle| = 3$, and further get $C_f\langle s\rangle \neq C_f\langle z\rangle$. In addition, it follows from $f(wv) \in \{1,2,3,4\} \setminus C_{f'}\langle z\rangle$ and $f(v) = f^{'}(zw)$ that $|C_f\langle w\rangle| = 4$ and $C_f\langle z\rangle \neq C_f\langle w\rangle$. We notice that $|C_f\langle v\rangle| = 3$, and so $C_f\langle w\rangle \neq C_f\langle v\rangle$.

(2) If $d_U(w) = 3$, we denote by $N_U(w) = \{v, z, u\}$ where $d_U(z) = 2$ and $d_U(u) = 1$. Suppose $U^{'} = U - v$, then by assumption, $U^{'}$ has a $4 - AVSDTC$ $f^{'}$.

Since $d_U(w) = 3$, $d_{U'}(w) = 2$. Then we have $|C_{f'}\langle w\rangle| = 4$. Thus, $d_{U'}(w)$, $d_{U'}(z)$, $|C_{f'}\langle w\rangle|$ and $|C_{f'}\langle z\rangle|$ have just one combination 2243 because 2244 results in $C_{f'}\langle z\rangle = C_{f'}\langle w\rangle$ in U'. Hence, for the case of 2243, we set

$$f(x) = \begin{cases} f'(wz), & x = v; \\ t, & x = wv, \text{ and } t \in C_{f'}\langle w\rangle \setminus \\ & \{f'(w), f'(wz), f'(wu)\}; \\ f'(x), & \text{otherwise.} \end{cases}$$

Then $C_f\langle w\rangle = C_{f'}\langle w\rangle$ due to $f(wv) \in C_{f'}\langle w\rangle \setminus \{f'(w), f'(wz), f'(wu)\}$, and thus, $C_f\langle w\rangle \neq C_f\langle z\rangle$. Furthermore, by Corollary 7 it deduces that $C_f\langle w\rangle \neq C_f\langle v\rangle$ and $C_f\langle w\rangle \neq C_f\langle u\rangle$.

Together with above, U has a $4 - AVSDTC$, and thus the proof follows.

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