Bond Pricing under CIR Process with Threshold Setting

Haoyan Zhang, Lei Tang, Fu Wang, and Yunli Du,

***Abstract*—This paper presents an examination of the conditional characteristic function, first hitting time problem and bond pricing under the CIR process with a threshold setting, also known as CIRT. To investigate the conditional characteristic function of CIRT, we solve an equation associated with the infinitesimal generator and its domain. Using a similar approach, we solve the bond pricing problem by introducing the sharpe ratio. All the results we were able to get are in closed form. Sensitive analysis is provided under various coefficients using graphical representations.**

***Index Terms*—CIR process, threshold, characteristic function, first hitting time, bond pricing.**

I. Introduction

BONDS are long-term borrowing instruments that are issued as securities by businesses, governments, and other organizations to raise money. Although the history of bonds dates back thousands of years, contemporary industry is credited with the creation and growth of the bond market. The British government began using bonds in the early 19th century to generate funds, and by the early 20th century, the US bond market had grown to be the biggest in the world. One of the most significant marketplaces in the financial sector is the bond market. Bond pricing techniques have been studied by numerous academics, and bond theory is comparatively developed and comprehensive.

The term structure of interest rates, as a critical derivative of interest rates, has attracted a lot of scholars' attention. Vasicek [20] used the Ornstein-Uhlenbeck process to model the mean-reverting dynamic of interest rates. Cox et al. [5] proposed an intertemporal general equilibrium asset pricing model to study the term structure of interest rates. Ho and Lee [11] derived an arbitrage-free interest rate movements model (AR model) to price interest rate contingent claims relative to the observed complete term structure of interest rates. Health et al. [9] presented a unifying theory for valuing contingent claims under a stochastic term structure of interest rates. Heston [10] selected stochastic volatility process to characterize this dynamic. Among others, we refer to Calvo-Garrido et al. [1], Chan et al. [2], Choi et al. [3], Duffee [7], Fan et al. [8], Malinska [13], Munk [14], [15], Tian and Zhang [17], Zhang and Tian [23], etc. More recently, Wei et al. [21] concerned with parameter estimation problem for squared radial Ornstein-Uhlenbeck process driven by α-stable noises; Wei and Xu [22] studied h least squares estimation for Ornstein-Uhlenbeck process, which contains sub-fractional Brownian processes; Ni et al. [16] explored a class of p-Laplacian fractional hybrid-Sturm-Liouville-Langevin integro-differential equations, and the corresponding functional boundary value conditions involves Caputo-Hadamard fractional derivative; Udoye et al. [19] investigated Ornstein-Uhlenbeck operator serving for sensitivity analysis involving Levy processes, and further Udoye et al. [18] derived expressions for the greeks from parameters and computed the sensitivities of the parameters of zero-coupon bond under the variance gamma process, respectively. However, they almost focused on the theoretical results like stochastic integro-differential equation and parameters estimation. But the term structure of interest rates and derivative pricing are also topics worth exploring.

The relationship between a zero-coupon bond's yield-to-maturity and bond maturity is known as the term structure of interest rates. Certain models have been devised for the valuation of zero-coupon bonds, assuming that the probability distribution of an interest rate, or some other variable, at some future moment, follows a log-normal distribution. This assumption, however, fails to take into account the reality that interest rates fluctuate over time, making these models inappropriate for evaluating interest in derivatives like callable bonds and American-style swap options. We provide the CIRT to characterize the term structure of interest rates in order to get around this restriction. The conditional characteristic function and bond pricing problem are then resolved using this approach. The federal funds rate trajectory in the US from 2005 to 2020 is depicted in Figure 1. It is clear that the interest rate starts to go downward after 2008 and stays there for a long time till 2015. This illustrates many mean-reverting drifts at various points in time. Our suggestion is to employ the CIRT to simulate the dynamic behavior of interest rates in order to tackle the issue of non-persistent long-term mean levels and mean reversion rates over time.

In fact, threshold models have been examined in the field of bond pricing theory; Dong and Wang [6] and Lemke and Archontakis [12] are two examples of these studies. Nevertheless, they either allowed the short-term interest rate to follow an autoregressive process with the intercept flipping endogenously, or they merely took into account the regime change that was dictated endogenously by the underlying financial asset. On the other hand, both the drift and diffusion coefficients are threshold terms in our model. This enables a more thorough depiction of the dynamic behavior of interest

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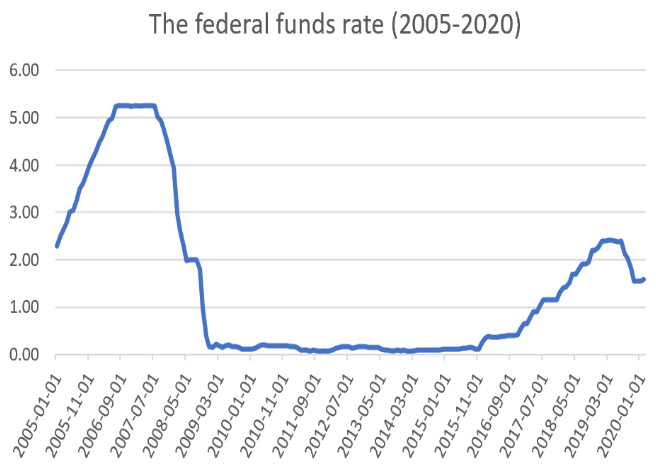


Fig. 1. The federal funds rate from 2005 to 2020.

rates, accounting for variations in volatility as well as mean reversion. This method offers a framework for pricing interest rate derivatives that is more precise and adaptable. To be more exact, the CIRT is described as

$$dX_t = \kappa(\theta(X_t) - X_t)dt + \sigma(X_t)\sqrt{X_t}dW_t, \qquad (1)$$

where W_t is a standard Brownian motion defined on the probability space, $\kappa > 0$ is the speed of mean reversion, $\theta(\cdot)$ is the threshold long-term mean, $\sigma(\cdot)$ is the threshold volatility of interest rate. The functions $\theta(\cdot)$ and $\sigma(\cdot)$ threshold by level a and b are denoted as follows:

$$\theta(X_t) = \begin{cases} \theta_1, & X_t \geq a, \\ \theta_2, & X_t < a, \end{cases} \qquad (2)$$

and

$$\sigma(X_t) = \begin{cases} \sigma_1, & X_t \geq b, \\ \sigma_2, & X_t < b. \end{cases} \qquad (3)$$

On the one hand, our model simplifies to the CIR process suggested by Cox et al. [5], if there is no threshold phenomena in equation (1), expanding the generality of our model. However, the threshold levels are not always the same, and we can select various coefficients in equations (2) and (3) to better fit our model to the real market. Thus, this study has three goals in mind: (1) Solve the conditional characteristic function using martingale methods; (2) Calculate the price of valuable zero-coupon bonds under this dynamic; and (3) Give numerical results to elucidate the dynamic scheme. The underlying interest rate model is described as a CIRT.

The rest of our paper is organized as follows. Section 2 derives the conditional characteristic function of CIRT. The explicit solution to bond price when the underlying asset satisfies our model is argued in Section 3. Numerical results about the bond price with different coefficients and bond's term are provided in section 4. Section 5 concludes.

II. Conditional characteristic function and first hitting time problem of CIRT

This section intends to calculate the conditional characteristic function and first hitting time problem of CIRT model X_t, which is defined by

$$\psi(X_{t+\tau}, u, \tau; X_t) = E[\exp(iuX_{t+\tau})|X_t], \qquad \tau = T - t,$$

and

$$E_0[e^{-\alpha\tau_x}],$$

where $\tau_x = \inf\{t|X_t = x\}$ and E_0 denotes the expectation starting from 0, respectively.

Theorem2.1 If X_t satisfies the SDE (1), then if $a > b$, the conditional characteristic function of X_t is If X_t satisfies the SDE (1), then if $a > b$, the conditional characteristic function of X_t is

$$\psi(X_{t+\tau}, u, \tau; X_t) = \begin{cases} \exp\left\{-\frac{2\kappa\theta_1}{\sigma_1^2}\ln\left(1-\frac{iu}{c_1}\right) + \frac{iue^{-\kappa\tau}}{1-iu/c_1}X_t\right\}, & X_t \geq a, \\ \exp\left\{-\frac{2\kappa\theta_2}{\sigma_1^2}\ln\left(1-\frac{iu}{c_1}\right) + \frac{iue^{-\kappa\tau}}{1-iu/c_1}X_t\right\}, & b < X_t < a \\ \exp\left\{-\frac{2\kappa\theta_2}{\sigma_2^2}\ln\left(1-\frac{iu}{c_2}\right) + \frac{iue^{-\kappa\tau}}{1-iu/c_2}X_t\right\}, & X_t \leq b. \end{cases}$$

If $a \leq b$, the conditional characteristic function of X_t is

$$\psi(X_{t+\tau}, u, \tau; X_t) = \begin{cases} \exp\left\{-\frac{2\kappa\theta_1}{\sigma_1^2}\ln\left(1-\frac{iu}{c_1}\right) + \frac{iue^{-\kappa\tau}}{1-iu/c_1}X_t\right\}, & X_t \geq b, \\ \exp\left\{-\frac{2\kappa\theta_1}{\sigma_2^2}\ln\left(1-\frac{iu}{c_2}\right) + \frac{iue^{-\kappa\tau}}{1-iu/c_2}X_t\right\}, & a < X_t < b \\ \exp\left\{-\frac{2\kappa\theta_2}{\sigma_2^2}\ln\left(1-\frac{iu}{c_2}\right) + \frac{iue^{-\kappa\tau}}{1-iu/c_2}X_t\right\}, & X_t \leq a. \end{cases}$$

The coefficient c_i, for $i = 1, 2$, is denoted by

$$c_i = -\frac{2\kappa}{\sigma_i^2(1-e^{-\kappa\tau})}.$$

Proof Recalling the definition of the conditional characteristic function of X_t at the beginning of this section and applying Ito's formula, we have,

$$d\psi = \left[-\frac{\partial\psi}{\partial\tau} + \frac{\partial\psi}{\partial X_t}\kappa(\theta(X_t) - X_t) + \frac{1}{2}\frac{\partial^2\psi}{\partial X_t^2}\sigma^2(X_t)X_t\right]dt + \frac{\partial\psi}{\partial X_t}\sigma(X_t)\sqrt{X_t}dW_t.$$

Because for any $s < \tau$,

$$\begin{aligned} E[\psi(X_{t+\tau}, u, \tau; X_{t+s})|X_t] &= E[(E[\exp(iuX_{t+\tau})|X_{t+s}])|X_t] \\ &= E[\exp(iuX_{t+\tau})|X_t] \\ &= \psi(X_{t+\tau}, u, \tau; X_t), \end{aligned}$$

proving that conditional expectation $\psi(X_{t+\tau}, u, \tau; X_t)$ is a martingale, it means that drift coefficient in dψ equals to 0, i.e.,

$$-\frac{\partial\psi}{\partial\tau} + \frac{\partial\psi}{\partial X_t}\kappa(\theta(X_t) - X_t) + \frac{1}{2}\frac{\partial^2\psi}{\partial X_t^2}\sigma^2(X_t)X_t = 0, \quad (2.1)$$

subject to the boundary condition $\psi(X_T, u, 0|X_T) = \exp(iuX_T)$ when $\tau = 0$. To solve the last equation, we apply the method of undeterminated coefficients. Suppose that the display of the solution to (2.1) takes the form of

$$\psi(X_{t+\tau}, u, \tau; X_t) = \exp\{C(\tau) + D(\tau)X_t)\}, \qquad (2.2)$$

with the boundary condition $C(0) = 0$ and $D(0) = iu$.

Based on (2.2), the first and second order partial derivatives are shown as below

$$\begin{aligned} \frac{\partial\psi}{\partial\tau} &= \psi[C'(\tau) + D'(\tau)X_t], \\ \frac{\partial\psi}{\partial X} &= \psi D(\tau), \\ \frac{\partial^2\psi}{\partial X^2} &= \psi D^2(\tau). \end{aligned}$$

Substituting these equations in (2.1) implies

$$\begin{aligned}&-C^{'}(\tau)+\kappa\theta(X_t)D(\tau)\\&+\left[\frac{1}{2}\sigma^2(X_t)D^2(\tau)-\kappa D(\tau)-D^{'}(\tau)\right]X_t=0.\end{aligned}$$

Note that the above equation holds for any X_t, thus we have

$$\begin{cases}\frac{1}{2}\sigma^2(X_t)D^2(\tau)-\kappa D(\tau)-D^{'}(\tau)=0,\\ -C^{'}(\tau)+\kappa\theta(X_t)D(\tau)=0.\end{cases} \tag{2.3}$$

The first equality in (2.3) is the Riccati equation meeting the boundary conditions. After separating variables in the above Riccati equation, we get

$$\frac{dD(\tau)}{\frac{1}{2}\sigma^2(X_t)D^2(\tau)-\kappa D(\tau)}=d\tau. \tag{2.4}$$

Setting $d_1=\frac{2\kappa}{\sigma^2(X_t)}$, (2.4) becomes

$$\left(\frac{1}{D(\tau)}-\frac{1}{D(\tau)-d_1}\right)dD(\tau)=-\kappa d\tau. \tag{2.5}$$

After integrating (2.5), we see

$$\ln\left[\frac{D(\tau)}{D(\tau)-d_1}\right]=-\kappa\tau+a_1.$$

It is straightforward by letting $a_2=e^{a_1}$ to have

$$D(\tau)=\frac{-a_2e^{-\kappa\tau}d_1}{1-a_2e^{-\kappa\tau}}. \tag{2.6}$$

Recalling the boundary condition $D(0)=iu$, then $a_2=\frac{iu}{iu-d_1}$. Substituting a_2 and d_1 into (2.6) results in

$$D(\tau)=\frac{iue^{-\kappa\tau}}{1-iu/c_i}, \tag{2.7}$$

where $c_i=\frac{2\kappa}{\sigma_i^2(1-e^{\kappa\tau})}$. Obviously, by the boundary condition and the expression of $D(\tau)$, we compute the result for $C(\tau)$ and finish this proof.

Remark Directly obtaining the density function could be challenging at times. As an alternative, obtaining the characteristic function appears more manageable, and using the connection provided by, one may compute the transition density and the conditional moment.

$$E[X_t^n]=(-i)^n\psi^{(n)}(X_t,0),$$

and

$$f(X_t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp(-iuX_t)\psi(X_t,u)\mathrm{d}u,$$

respectively.

So far, we have the explicit formulations for the conditional characteristic function of X_t. Next, we study the the Laplace transform of first hitting time under X_t.

Theorem2.2 Setting $\delta_1=\frac{\kappa}{\theta_1\sigma_1^2}$, $\delta_2=\frac{\kappa}{\theta_2\sigma_2^2}$, $\delta_{12}=\frac{\kappa}{\theta_1\sigma_2^2}$, $\delta_{21}=\frac{\kappa}{\theta_2\sigma_1^2}$, $\nu_1=\frac{2\kappa}{\theta_1\sigma_1^2}-1$, $\nu_2=\frac{2\kappa}{\theta_2\sigma_2^2}-1$, $\nu_{12}=\frac{2\kappa}{\theta_1\sigma_2^2}-1$ and $\nu_{21}=\frac{2\kappa}{\theta_2\sigma_1^2}$, then the Laplace transform of first hitting time τ_x of CIRT starting at 0 is denoted by (i) if $a>b$,

$$E_0[e^{-\alpha\tau_x}]=$$
$$\begin{cases}\frac{(\frac{\sqrt{2\kappa}}{\sigma_1}\sqrt{x})^{\nu_1+1}e^{-x\kappa/\sigma_1^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa x}{\sigma_1^2})}=\frac{1}{\Phi(\alpha/\kappa,\nu_1+1;2\kappa x/\sigma_1^2)}, & X_t\geq a,\\ \frac{(\frac{\sqrt{2\kappa}}{\sigma_1}\sqrt{x})^{\nu_{21}+1}e^{-x\kappa/\sigma_1^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{21}+1),\frac{\nu_{21}}{2}}(\frac{2\kappa x}{\sigma_1^2})}=\frac{1}{\Phi(\alpha/\kappa,\nu_{21}+1;2\kappa x/\sigma_1^2)},\\ \quad b<X_t<a,\\ \frac{(\frac{\sqrt{2\kappa}}{\sigma_2}\sqrt{x})^{\nu_2+1}e^{-x\kappa/\sigma_2^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa x}{\sigma_2^2})}=\frac{1}{\Phi(\alpha/\kappa,\nu_2+1;2\kappa x/\sigma_2^2)}, & X_t\leq b;\end{cases}$$

(ii) if $a\leq b$,

$$E_0[e^{-\alpha\tau_x}]=$$
$$\begin{cases}\frac{(\frac{\sqrt{2\kappa}}{\sigma_1}\sqrt{x})^{\nu_1+1}e^{-x\kappa/\sigma_1^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa x}{\sigma_1^2})}=\frac{1}{\Phi(\alpha/\kappa,\nu_1+1;2\kappa x/\sigma_1^2)}, & X_t\geq b,\\ \frac{(\frac{\sqrt{2\kappa}}{\sigma_2}\sqrt{x})^{\nu_{12}+1}e^{-x\kappa/\sigma_2^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{12}+1),\frac{\nu_{12}}{2}}(\frac{2\kappa x}{\sigma_2^2})}=\frac{1}{\Phi(\alpha/\kappa,\nu_{12}+1;2\kappa x/\sigma_2^2)},\\ \quad a<X_t<b,\\ \frac{(\frac{\sqrt{2\kappa}}{\sigma_2}\sqrt{x})^{\nu_2+1}e^{-x\kappa/\sigma_2^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa x}{\sigma_2^2})}=\frac{1}{\Phi(\alpha/\kappa,\nu_2+1;2\kappa x/\sigma_2^2)}, & X_t\leq a,\end{cases}$$

where the Whittaker functions $M_{k,\mu}(z)=z^{\mu+1/2}e^{-z/2}\Phi(\mu-k+\frac{1}{2},2\mu+1;z)$.

Proof From Chou and Lin [4], we know that

$$E_0[e^{-\alpha\tau_x}]=\frac{2^{2\alpha/\kappa-\nu-1}x^{\nu/2}(\sqrt{2\kappa}/\sigma)^{2\alpha/\kappa}\Gamma(\alpha/\kappa)}{\Gamma(\nu+1)\int_0^\infty I_\nu(\eta\sqrt{x})e^{-\frac{\sigma^2\eta^2}{8\kappa}}\eta^{\frac{2\alpha}{\kappa}-\nu-1}d\eta}. \tag{2.8}$$

Assuming that $\mu=\frac{1}{2}(\frac{2\alpha}{\kappa}-\nu-1)$, $2\hat{a}^{1/2}=\sqrt{x}$, $p=\frac{\sigma^2}{8\kappa}$ and introducing a variable of integration in (2.8) by setting $t=\eta^2$, we derive that

$$\begin{aligned}&\int_0^\infty I_\nu(\eta\sqrt{x})e^{-\frac{\sigma^2\eta^2}{8\kappa}}\eta^{\frac{2\alpha}{\kappa}-\nu-1}d\eta\\=&\frac{1}{2}\int_0^\infty e^{-pt}t^{\mu-\frac{1}{2}}I_\nu(2\hat{a}^{1/2}t^{1/2})dt\\=&\frac{\Gamma(\frac{\alpha}{\kappa})e^{x\kappa/\sigma^2}(\frac{8\kappa}{\sigma^2})^{\frac{1}{2}(\frac{2\alpha}{\kappa}-\nu-1)}}{\sqrt{x}\Gamma(\nu+1)}M_{-\frac{1}{2}(\frac{2\alpha}{\kappa}-\nu-1),\frac{\nu}{2}}(2\kappa x/\sigma^2),\end{aligned}$$

which reduces (2.8) to the form of

$$\begin{aligned}E_0[e^{-\alpha\tau_x}]=&\frac{(\frac{\sqrt{2\kappa}}{\sigma}\sqrt{x})^{\nu+1}e^{-x\kappa/\sigma^2}}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu+1),\frac{\nu}{2}}(\frac{2\kappa x}{\sigma^2})}\\=&\frac{1}{\Phi(\alpha/\kappa,\nu+1;2\kappa x/\sigma^2)}.\end{aligned}$$

To end this proof, we replace ν by $\nu_{1,2},\nu_{12},\nu_{21}$ and σ by $\sigma_{1,2},\sigma_{12},\sigma_{21}$, respectively.

Corollary2.1 The Laplace transform of the first hitting time $\tau_{y\to x}=\inf\{t|X_t=x\}$ of CIRT starting at $y(0<y<$

$x)$ is denoted by (i) if $a > b$,

$$E_y[e^{-\alpha\tau_{y\to x}}] = \begin{cases} \left(\frac{x}{y}\right)^{\frac{\nu_1+1}{2}} e^{\kappa(y-x)/\sigma_1^2} \frac{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa y}{\sigma_1^2})}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa x}{\sigma_1^2})} \\ = \frac{\Phi(\alpha/\kappa,\nu_1+1;2\kappa y/\sigma_1^2)}{\Phi(\alpha/\kappa,\nu_1+1;2\kappa x/\sigma_1^2)}, \quad X_t \geq a, \\ \left(\frac{x}{y}\right)^{\frac{\nu_{21}+1}{2}} e^{\kappa(y-x)/\sigma_1^2} \frac{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{21}+1),\frac{\nu_{21}}{2}}(\frac{2\kappa y}{\sigma_1^2})}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{21}+1),\frac{\nu_{21}}{2}}(\frac{2\kappa x}{\sigma_1^2})} \\ = \frac{\Phi(\alpha/\kappa,\nu_{21}+1;2\kappa y/\sigma_1^2)}{\Phi(\alpha/\kappa,\nu_{21}+1;2\kappa x/\sigma_1^2)}, \quad b < X_t < a, \\ \left(\frac{x}{y}\right)^{\frac{\nu_2+1}{2}} e^{\kappa(y-x)/\sigma_2^2} \frac{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa y}{\sigma_2^2})}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa x}{\sigma_2^2})} \\ = \frac{\Phi(\alpha/\kappa,\nu_2+1;2\kappa y/\sigma_2^2)}{\Phi(\alpha/\kappa,\nu_2+1;2\kappa x/\sigma_2^2)}, \quad X_t \leq b; \end{cases}$$

(ii) if $a \leq b$,

$$E_y[e^{-\alpha\tau_{y\to x}}] = \begin{cases} \left(\frac{x}{y}\right)^{\frac{\nu_1+1}{2}} e^{\kappa(y-x)/\sigma_1^2} \frac{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa y}{\sigma_1^2})}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa x}{\sigma_1^2})} \\ = \frac{\Phi(\alpha/\kappa,\nu_1+1;2\kappa y/\sigma_1^2)}{\Phi(\alpha/\kappa,\nu_1+1;2\kappa x/\sigma_1^2)}, \quad X_t \geq b, \\ \left(\frac{x}{y}\right)^{\frac{\nu_{12}+1}{2}} e^{\kappa(y-x)/\sigma_2^2} \frac{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{12}+1),\frac{\nu_{12}}{2}}(\frac{2\kappa y}{\sigma_2^2})}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{12}+1),\frac{\nu_{12}}{2}}(\frac{2\kappa x}{\sigma_2^2})} \\ = \frac{\Phi(\alpha/\kappa,\nu_{12}+1;2\kappa y/\sigma_2^2)}{\Phi(\alpha/\kappa,\nu_{12}+1;2\kappa x/\sigma_2^2)}, \quad a < X_t < b, \\ \left(\frac{x}{y}\right)^{\frac{\nu_2+1}{2}} e^{\kappa(y-x)/\sigma_2^2} \frac{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa y}{\sigma_2^2})}{M_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa x}{\sigma_2^2})} \\ = \frac{\Phi(\alpha/\kappa,\nu_2+1;2\kappa y/\sigma_2^2)}{\Phi(\alpha/\kappa,\nu_2+1;2\kappa x/\sigma_2^2)}, \quad X_t \leq a; \end{cases}$$

Proof For $0 < y < x$, we have $\tau_{0\to x} = \tau_{0\to y} + \tau_{y\to x}$ where $\tau_{0\to y}$ and $\tau_{y\to x}$ are independent because of the strong Markov property. Hence,

$$E_y[e^{-\alpha\tau_{y\to x}}] = \frac{E_0[e^{-\alpha\tau_{0\to x}}]}{E_0[e^{-\alpha\tau_{0\to y}}]},$$

which finishes the proof.

Analogously, we obtain the Laplace transform of $\tau_{y\to x}$ in the case $0 < x < y$. Note that here we have to use the modified Bessel functions of the third kind K_ν instead of the modified Bessel functions of the first kind I_ν such that the required uniform integrability assumption is satisfied.

The Laplace transform of the first hitting time $\tau_{y\to x} = \inf\{t | X_t = x\}$ of CIRT starting at $y (0 < x < y)$ is denoted by (i) if $a > b$,

$$E_y[e^{-\alpha\tau_{y\to x}}] = \begin{cases} \left(\frac{x}{y}\right)^{\frac{\nu_1+1}{2}} e^{\kappa(y-x)/\sigma_1^2} \frac{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa y}{\sigma_1^2})}{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa x}{\sigma_1^2})} \\ = \frac{\Psi(\alpha/\kappa,\nu_1+1;2\kappa y/\sigma_1^2)}{\Psi(\alpha/\kappa,\nu_1+1;2\kappa x/\sigma_1^2)}, \quad X_t \geq a, \\ \left(\frac{x}{y}\right)^{\frac{\nu_{21}+1}{2}} e^{\kappa(y-x)/\sigma_1^2} \frac{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{21}+1),\frac{\nu_{21}}{2}}(\frac{2\kappa y}{\sigma_1^2})}{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{21}+1),\frac{\nu_{21}}{2}}(\frac{2\kappa x}{\sigma_1^2})} \\ = \frac{\Psi(\alpha/\kappa,\nu_{21}+1;2\kappa y/\sigma_1^2)}{\Psi(\alpha/\kappa,\nu_{21}+1;2\kappa x/\sigma_1^2)}, \quad b < X_t < a, \\ \left(\frac{x}{y}\right)^{\frac{\nu_2+1}{2}} e^{\kappa(y-x)/\sigma_2^2} \frac{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa y}{\sigma_2^2})}{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa x}{\sigma_2^2})} \\ = \frac{\Psi(\alpha/\kappa,\nu_2+1;2\kappa y/\sigma_2^2)}{\Psi(\alpha/\kappa,\nu_2+1;2\kappa x/\sigma_2^2)}, \quad X_t \leq b; \end{cases}$$

(ii) if $a \leq b$,

$$E_y[e^{-\alpha\tau_{y\to x}}] = \begin{cases} \left(\frac{x}{y}\right)^{\frac{\nu_1+1}{2}} e^{\kappa(y-x)/\sigma_1^2} \frac{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa y}{\sigma_1^2})}{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_1+1),\frac{\nu_1}{2}}(\frac{2\kappa x}{\sigma_1^2})} \\ = \frac{\Psi(\alpha/\kappa,\nu_1+1;2\kappa y/\sigma_1^2)}{\Psi(\alpha/\kappa,\nu_1+1;2\kappa x/\sigma_1^2)}, \quad X_t \geq b, \\ \left(\frac{x}{y}\right)^{\frac{\nu_{12}+1}{2}} e^{\kappa(y-x)/\sigma_2^2} \frac{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{12}+1),\frac{\nu_{12}}{2}}(\frac{2\kappa y}{\sigma_2^2})}{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_{12}+1),\frac{\nu_{12}}{2}}(\frac{2\kappa x}{\sigma_2^2})} \\ = \frac{\Psi(\alpha/\kappa,\nu_{12}+1;2\kappa y/\sigma_2^2)}{\Psi(\alpha/\kappa,\nu_{12}+1;2\kappa x/\sigma_2^2)}, \quad a < X_t < b, \\ \left(\frac{x}{y}\right)^{\frac{\nu_2+1}{2}} e^{\kappa(y-x)/\sigma_2^2} \frac{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa y}{\sigma_2^2})}{W_{\frac{1}{2}(-\frac{2\alpha}{\kappa}+\nu_2+1),\frac{\nu_2}{2}}(\frac{2\kappa x}{\sigma_2^2})} \\ = \frac{\Psi(\alpha/\kappa,\nu_2+1;2\kappa y/\sigma_2^2)}{\Psi(\alpha/\kappa,\nu_2+1;2\kappa x/\sigma_2^2)}, \quad X_t \leq a; \end{cases}$$

where the Whittaker functions $W_{k,\mu}(z) = z^{\mu+1/2}e^{-z/2}\Psi(\mu - k + \frac{1}{2}, 2\mu + 1; z)$.

Proof Using the same assumptions as the proof of Theorem 2.2, we have

$$\begin{aligned} &\int_0^\infty K_\nu(\eta\sqrt{x})e^{-\frac{\sigma^2\eta^2}{8\kappa}}\eta^{\frac{2\alpha}{\kappa}-\nu-1}d\eta \\ =&\frac{1}{2}\int_0^\infty e^{-pt}t^{\mu-\frac{1}{2}}K_\nu(2\hat{a}^{1/2}t^{1/2})dt \\ =&\frac{\Gamma(\frac{\alpha}{\kappa})\Gamma(\frac{\alpha}{\kappa}-\nu)e^{x\kappa/\sigma^2}(\frac{8\kappa}{\sigma^2})^{\frac{1}{2}(\frac{2\alpha}{\kappa}-\nu-1)}}{2\sqrt{x}}\times \\ &W_{-\frac{1}{2}(\frac{2\alpha}{\kappa}-\nu-1),\frac{\nu}{2}}(2\kappa x/\sigma^2), \end{aligned}$$

Hence

$$\begin{aligned} E_0[e^{-\alpha\tau_x}] &= \frac{2^{2\alpha/\kappa-\nu-1}x^{\nu/2}(\sqrt{2\kappa}/\sigma)^{2\alpha/\kappa}\Gamma(\alpha/\kappa)}{\Gamma(\nu+1)\int_0^\infty K_\nu(\eta\sqrt{x})e^{-\frac{\sigma^2\eta^2}{8\kappa}}\eta^{\frac{2\alpha}{\kappa}-\nu-1}d\eta} \\ &= \frac{x^{(\nu+1)/2}(\sqrt{2\kappa}/\sigma)^{\nu+1}e^{-x\kappa/\sigma^2}}{\Gamma(\nu+1)\Gamma(\frac{\alpha}{\kappa}-\nu)W_{-\frac{1}{2}(\frac{2\alpha}{\kappa}-\nu-1),\frac{\nu}{2}}(2\kappa x/\sigma^2)}. \end{aligned}$$

With

$$E_y[e^{-\alpha\tau_{y\to x}}] = \frac{E_0[e^{-\alpha\tau_{0\to x}}]}{E_0[e^{-\alpha\tau_{0\to y}}]},$$

and replacing ν by $\nu_{1,2}, \nu_{12}, \nu_{21}$ and σ by $\sigma_{1,2}, \sigma_{12}, \sigma_{21}$, respectively, the result follows immediately.

III. Bond pricing under CIRT

The goal of this section is to calculate the bond price in the case where (1) represents the underlying asset, r_t. The term structure of interest rates was studied by Cox et al. [5] using an intertemporal general equilibrium asset pricing model (Feller's branching process), as is widely known. This diffusion was termed after the Cox-Ingersoll-Ross (CIR) process in the literature today because of their exceptional contributions. In contrast, to understand the fluctuating drift and diffusion components in our research, we model the underlying asset using the CIRT, which may indicate some shift effects.

More precisely, suppose that the finance market considered in our paper is arbitrary-free and the default-free zero-coupon bond price at time t is denoted by $f(r_t, \tau)$ or $f(t, T, r_t)$, where T is the maturity and $\tau = T - t$ is the bond's term. Let $f(t, T, r_t)$ be the bond price based on r_t with the maturity

T at time t. By Ito's formula, we have,

$$\begin{aligned} df(t,T,r_t) =& \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial r}dr + \frac{1}{2}\sigma^2 r \frac{\partial^2 f}{\partial r^2}dt \\ =& \mu_f(t,r_t) f dt + \sigma_f(t,r_t) f dW_t, \end{aligned}$$

where

$$\mu_f(t,r_t)f := \frac{\partial f}{\partial t} + \kappa(\theta(r)-r)\frac{\partial f}{\partial r} + \frac{1}{2}\sigma^2(r) r \frac{\partial^2 f}{\partial r^2}, \quad (3.1)$$

$$\sigma_f(t,r_t)f := \sigma(r)\sqrt{r}\frac{\partial f}{\partial r}. \quad (3.2)$$

What follows next is our main result about the bond price under CIRT.

Theorem3.1 Suppose that the underlying zero coupon interest rate satisfies (1) and λ is the sharpe index in the modern market which is arbitrary-free. If $a > b$, then the bond price $f(t,T,r_t)$ of the zero coupon interest rate with the maturity time T is

$$f(t,T,r_t) = \begin{cases} \exp\left\{\frac{2\kappa\theta_1}{\sigma_1^2}\ln\left(\frac{2\gamma_1 e^{(\kappa+\gamma_1+\lambda)/2}}{2\gamma_1+(\kappa+\gamma_1+\lambda)(e^{\gamma_1\tau}-1)}\right) \right. \\ \left. +\frac{-2(e^{\gamma_1\tau}-1)}{2\gamma_1+(\kappa+\gamma_1+\lambda)(e^{\gamma_1\tau}-1)}r\right\}, \quad r \geq a, \\ \exp\left\{\frac{2\kappa\theta_2}{\sigma_1^2}\ln\left(\frac{2\gamma_1 e^{(\kappa+\gamma_1+\lambda)/2}}{2\gamma_1+(\kappa+\gamma_1+\lambda)(e^{\gamma_1\tau}-1)}\right) \right. \\ \left. +\frac{-2(e^{\gamma_1\tau}-1)}{2\gamma_1+(\kappa+\gamma_1+\lambda)(e^{\gamma_1\tau}-1)}r\right\}, \quad b < r < a \\ \exp\left\{\frac{2\kappa\theta_2}{\sigma_2^2}\ln\left(\frac{2\gamma_2 e^{(\kappa+\gamma_2+\lambda)/2}}{2\gamma_2+(\kappa+\gamma_2+\lambda)(e^{\gamma_2\tau}-1)}\right) \right. \\ \left. +\frac{-2(e^{\gamma_2\tau}-1)}{2\gamma_2+(\kappa+\gamma_2+\lambda)(e^{\gamma_2\tau}-1)}r\right\}, \quad r \leq b. \end{cases} \quad (3.3)$$

If $a \leq b$, then the bond price $f(t,T,r_t)$ of the zero coupon interest rate with the maturity time T is

$$f(t,T,r_t) = \begin{cases} \exp\left\{\frac{2\kappa\theta_1}{\sigma_1^2}\ln\left(\frac{2\gamma_1 e^{(\kappa+\gamma_1+\lambda)/2}}{2\gamma_1+(\kappa+\gamma_1+\lambda)(e^{\gamma_1\tau}-1)}\right) \right. \\ \left. +\frac{-2(e^{\gamma_1\tau}-1)}{2\gamma_1+(\kappa+\gamma_1+\lambda)(e^{\gamma_1\tau}-1)}r\right\}, \quad r \geq b, \\ \exp\left\{\frac{2\kappa\theta_1}{\sigma_2^2}\ln\left(\frac{2\gamma_2 e^{(\kappa+\gamma_2+\lambda)/2}}{2\gamma_2+(\kappa+\gamma_2+\lambda)(e^{\gamma_2\tau}-1)}\right) \right. \\ \left. +\frac{-2(e^{\gamma_2\tau}-1)}{2\gamma_2+(\kappa+\gamma_2+\lambda)(e^{\gamma_2\tau}-1)}r\right\}, \quad a < r < b \\ \exp\left\{\frac{2\kappa\theta_2}{\sigma_2^2}\ln\left(\frac{2\gamma_2 e^{(\kappa+\gamma_2+\lambda)/2}}{2\gamma_2+(\kappa+\gamma_2+\lambda)(e^{\gamma_2\tau}-1)}\right) \right. \\ \left. +\frac{-2(e^{\gamma_2\tau}-1)}{2\gamma_2+(\kappa+\gamma_2+\lambda)(e^{\gamma_2\tau}-1)}r\right\}, \quad r \leq a. \end{cases}$$

The coefficient γ_i, for $i = 1, 2$, is denoted by

$$\gamma_i = \sqrt{(\kappa+\lambda)^2 + 2\sigma_i^2}.$$

Proof In the bond pricing theory, if a bond market is arbitrary-free, the sharpe ratio of trading bonds with different terms should be equal. Thanks to Vasicek [20] choosing the market price of risk i.e., sharpe ratio

$$\lambda(r,t) = \frac{-\lambda\sqrt{r_t}}{\sigma}, \quad (3.4)$$

in our paper, we introduce the same ratio in (3.4) by

$$\frac{\mu_f(t,r_t) - r_t}{\sigma_f(t,r_t)} = \lambda(t,r_t) = \frac{-\lambda\sqrt{r_t}}{\sigma(r)}. \quad (3.5)$$

With the expressions of (3.1), (3.2) and (3.5), we establish

$$\frac{1}{2}\frac{\partial^2 f}{\partial r^2}\sigma^2(r)r + \frac{\partial f}{\partial r}[\kappa(\theta(r)-r) - \lambda r] - \frac{\partial f}{\partial \tau} - rf = 0. \quad (3.6)$$

Similar idea to section 2, we naturally suppose that the solution to (3.6) takes the form of

$$f(t,T,r_t) = \exp\{A(\tau) + B(\tau)r\},$$

with the boundary condition $A(0) = B(0) = 0$. Taking the partial derivatives for f results in

$$\begin{aligned} \frac{\partial f}{\partial \tau} &= f[A'(\tau) + B'(\tau)r], \\ \frac{\partial f}{\partial r} &= fB(\tau), \\ \frac{\partial^2 f}{\partial r^2} &= fB^2(\tau). \end{aligned}$$

Substitute them into (3.6), then we obtain

$$\begin{aligned} & r\left[\frac{1}{2}\sigma(r)^2 B(\tau)^2 - (\lambda+\kappa)B(\tau) - B(\tau)' - 1\right] \\ & + \kappa\theta(r)B(\tau) - A'(\tau) = 0, \end{aligned}$$

which holds for arbitrary r. It suggests that

$$\begin{cases} \frac{1}{2}\sigma(r)^2 B(\tau)^2 - (\lambda+\kappa)B(\tau) - B(\tau)' - 1 = 0, \\ \kappa\theta(r)B(\tau) - A'(\tau) = 0. \end{cases}$$

Parallel to the calculation in section 3. It is directly by setting $\gamma_i = \sqrt{(\kappa+\lambda)^2 + 2\sigma_i^2}$ to solve the equations and we complete this proof.

Obviously, if $\theta_1 = \theta_2$ and $\sigma_1 = \sigma_2$ in Theorem 3.1, interest rate model becomes the CIR process and the bond price will not be piecewise and our results could cover equation (23) in Cox et al. [5].

IV. Numerical Results

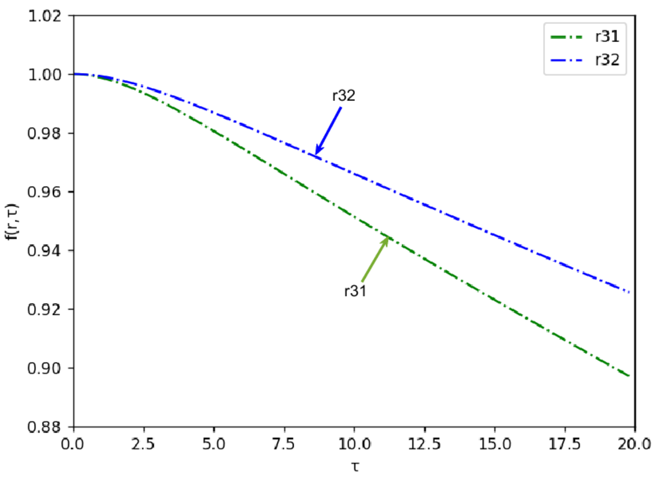
In this section, we will present the numerical analysis by displaying the bond price under the CIRT. Without loss of generality, we only consider the assumption $a > b$ in the result (3.3). More precisely, we set $a = 0.04$, $b = 0.02$, $\kappa = 0.1$, $\lambda = 0.2$, $\theta_1 = 0.04$, $\sigma_1 = 0.2$, $\theta_2 = 0.02$, $\sigma_2 = 0.1$ as the common parameters, if not selected to be the impact factor.

A. The impact of τ and r on bond pricing

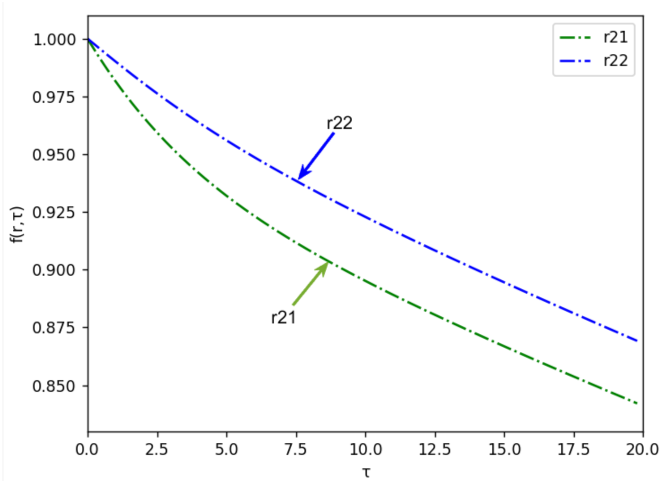
The bond price display concerning the impact on τ and r is shown in this subsection. Figure 2 makes it clear that when τ rises, bond price falls. The fact that emerges in the bond pricing theory and this phenomenon are consistent. The present value of future cash flows determines a bond's value on the date of issuance. The discounted present value will therefore decrease with an increase in the maturity date. Furthermore, figure 2 makes it clear that when r rises, the bond price falls. This occurrence is consistent with the idea that the discount price will be low when you obtain a "big" interest rate. The interest rate in each image 2 is $r_{\cdot 1} > r_{\cdot 2}$; as a result, the green line is lower.

B. The impact of θ on bond pricing

This subsection shows the display of the bond price about the impact on θ. It is obviously to see from figure 3 that bond price decreases as θ increases. The analysis of θ is similar to r, because θ is the long-term mean of r.

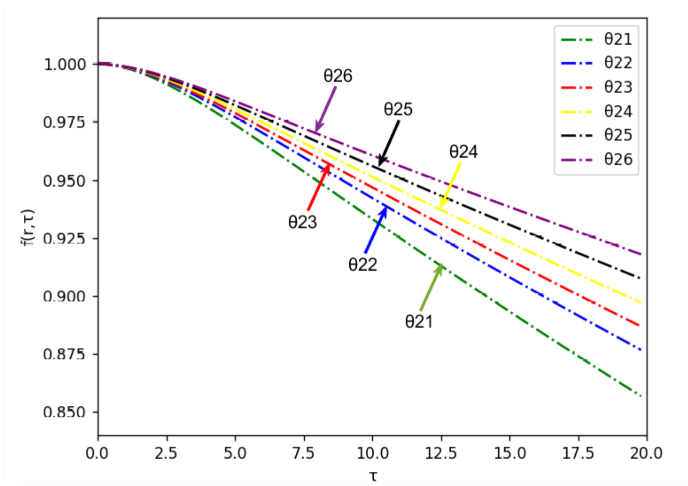


(a) Bond price when $r_{31} = 0.02$ and $r_{32} = 0.01$.

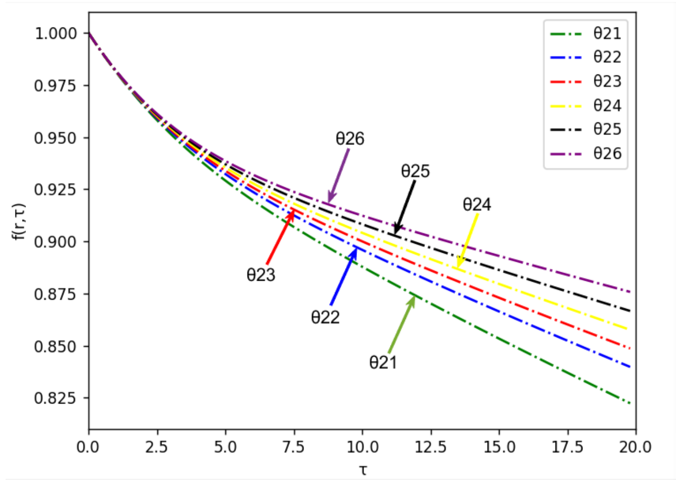


(b) Bond price when $r_{21} = 0.04$ and $r_{22} = 0.03$.

Fig. 2. The impact of τ and r on bond pricing. The left and right graphs correspond to the bond price under CIRT $r_{31} = 0.02$ and $r_{32} = 0.01$ compared to $r_{21} = 0.04$ and $r_{22} = 0.03$.



(a) Bond price when $r_3 = 0.02$.



(b) Bond price when $r_2 = 0.04$.

Fig. 3. The impact of θ on bond pricing. The left and right graphs correspond to the bond price under CIRT when $r_3 = 0.02$ and $r_2 = 0.04$, respectively. In each graph, the green, blue, red, yellow, black, purple lines represent $\theta_{21} = 0.04$, $\theta_{22} = 0.03$, $\theta_{23} = 0.025$, $\theta_{24} = 0.02$, $\theta_{25} = 0.015$, $\theta_{26} = 0.01$ respectively.

C. The impact of σ on bond pricing

The bond price display about the impact on σ is shown in this subsection. Figure 4 makes it clear that bond price rises in proportion to σ. This phenomena aligns with the idea that higher risk can result in higher returns. Bond prices may rise as a result of investors' increased demand during periods of heightened market risk. Higher volatility may also make bonds more risky and unpredictable. Risk-takers are more inclined to pursue higher gains at this rate.

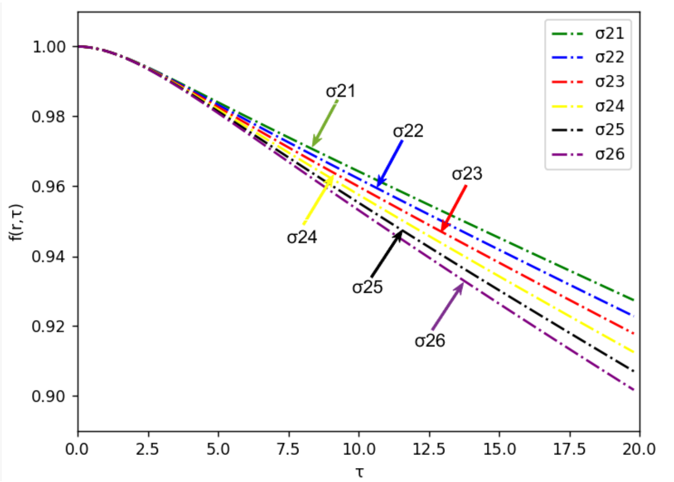
D. The impact of κ on bond pricing

In this subsection, the bond price's impact on κ is displayed. The speed at which the mean reverts increases with κ, suggesting that the interest rate will approach the long-term mean θ more quickly. Different instances need to be considered, and in our numerical results, we let $\theta = 0.04$. Figure 5(a) illustrates how a large κ will cause the interest rate to rise to θ more quickly because $r_3 < \theta$. Accordingly, bond price at identical τ may fall when κ increases ($r \uparrow \theta$ more quickly), as seen in figure 5(a). In contrast, large κ will cause the interest rate to fall down to θ more quickly because $r_1 > \theta$. This is evident when we concentrate on the numerical results in figure 5(c). Thus the bond price at same τ increases as κ increases ($r \downarrow \theta$ more quickly). In the between, there exists cross phenomenon in figure 5(b). The reason may be that when r_2 is set to be around or equal to long-term mean θ, the bond price seems not to be sensitive to the change of κ.

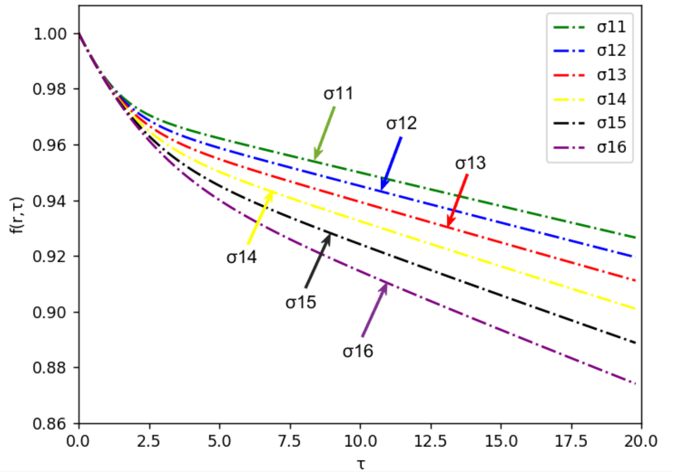
E. The impact of λ on bond pricing

The impact of the bond price on λ is displayed in this subsection. Figure 6 makes it clear that bond price grows as λ increases. The sharpe ratio, or market price of risk, is denoted by λ. The additional return that investors are ready to pay for taking on a risk is known as the "market price of risk". An elevated λ signifies a higher projected yield per unit risk, therefore translating into a higher bond price.

In conclusion, you stand to gain more money if you take on greater risk. The influence of τ, r, θ, σ, and λ all satisfy the bond pricing intuitions, according to the sensitivity analysis presented above. Of them, σ and λ indicate comparable analysis, while r and θ have similar analysis. However, it is difficult to provide a clear explanation for θ because the



(a) Bond price when $r_3 = 0.02$.



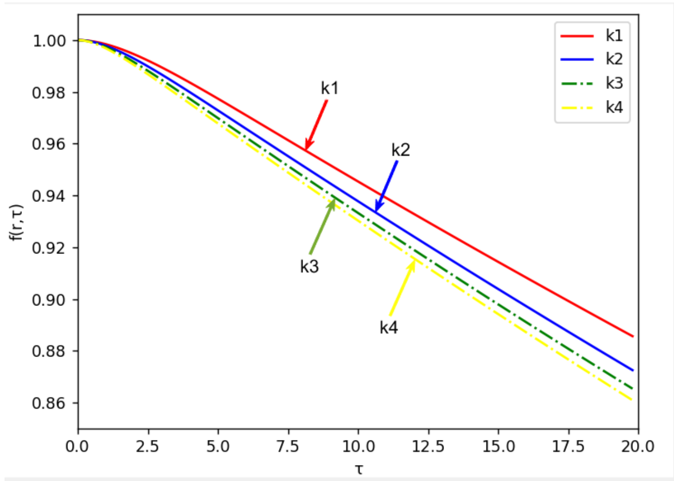
(b) Bond price when $r_2 = 0.04$.

Fig. 4. The impact of σ on bond pricing. The left and right graphs correspond to the bond price under CIRT when $r_3 = 0.02$ and $r_2 = 0.04$, respectively. In the left graph, the green, blue, red, yellow, black, purple lines represent $\sigma_{21} = 0.4$, $\sigma_{22} = 0.35$, $\sigma_{23} = 0.3$, $\sigma_{24} = 0.25$, $\sigma_{25} = 0.2$, $\sigma_{26} = 0.15$, respectively. In the right graph, the green, blue, red, yellow, black, purple lines represent $\sigma_{11} = 0.8$, $\sigma_{12} = 0.7$, $\sigma_{13} = 0.6$, $\sigma_{14} = 0.5$, $\sigma_{15} = 0.4$, $\sigma_{16} = 0.3$, respectively.

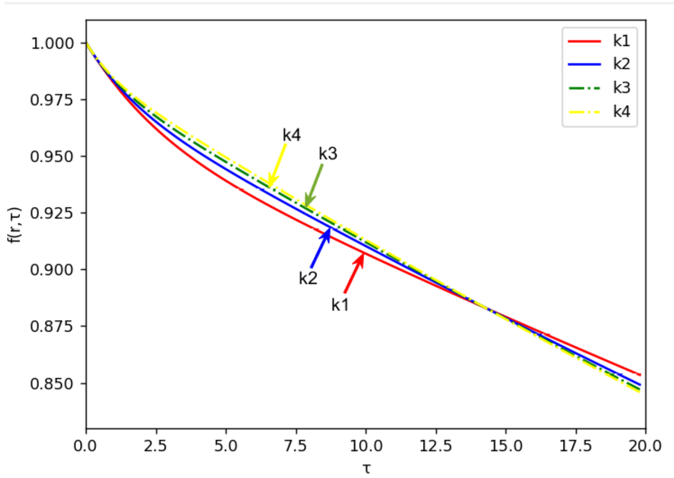
long-term anticipation θ could cause interest rates to rise in either an upward or downward direction. Furthermore, a change in the interest rate will result in a completely different pricing.

V. Conclusion

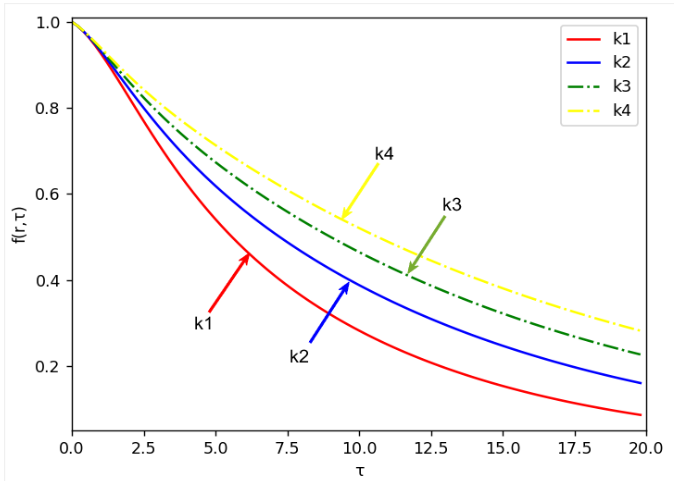
We have examined the theoretical characteristics and financial uses of CIRT in this paper. We take into consideration the threshold stochastic process, which includes but is not limited to CIR processes, in light of the shortcomings of conventional Vasicek or CIR models. These processes handle the coefficients of the stochastic drift and diffusion term as distinct values for particular dynamic study. Based on this, we create the pricing differential equation for the bond price under our model, compute the conditional characteristic function, and offer the closed-form solution of the bond price together with the boundary conditions. Our estimates and pricing descriptions are gratefully supported by numerical results.



(a) Bond price when $r_3 = 0.02$.



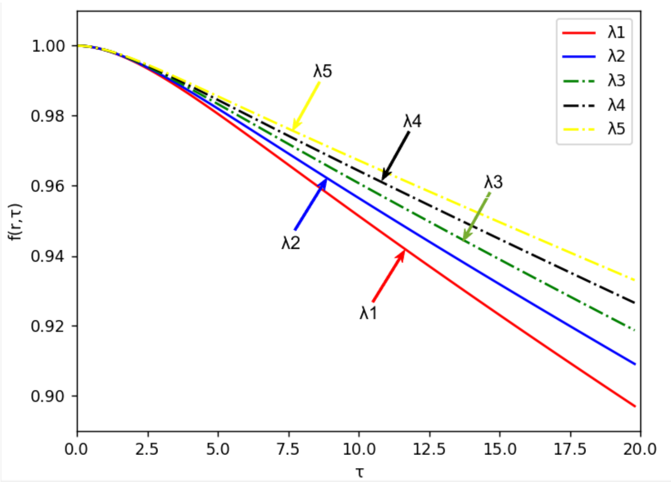
(b) Bond price when $r_2 = 0.04$.



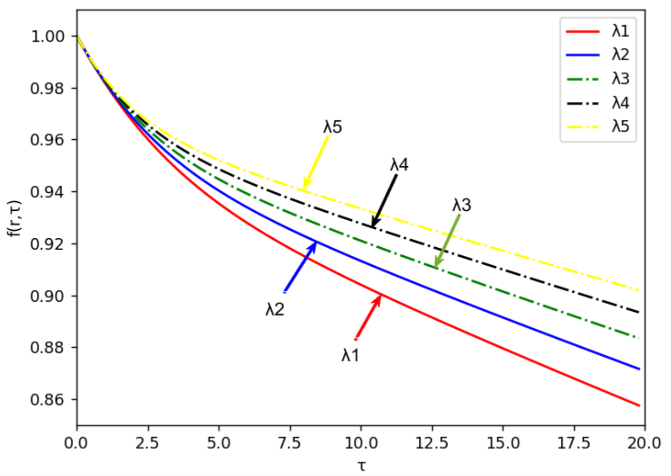
(c) Bond price when $r_1 = 0.1$.

Fig. 5. The impact of κ on bond pricing. The first(a), second(b) and third(c) graphs correspond to the bond price under CIRT when $r_3 = 0.02$, $r_2 = 0.04$ and $r_3 = 0.1$, respectively. In each of the graph, the red, blue, green, yellow lines represent $\kappa_1 = 0.2$, $\kappa_2 = 0.4$, $\kappa_3 = 0.6$, $\kappa_4 = 0.8$ respectively.

In future study, it is significant for us (1) to investigate the bond price under other threshold diffusions, e.g. regime-switching and jump effect, and (2) to consider other derivative pricing problems such as option or swaption.



(a) Bond price when $r_3 = 0.02$.



(b) Bond price when $r_2 = 0.04$.

Fig. 6. The impact of λ on bond pricing. The left and right graphs correspond to the bond price under CIRT when $r_3 = 0.02$ and $r_2 = 0.04$, respectively. In each graph, the red, blue, green, black, yellow lines represent $\lambda_1 = 0.2$, $\lambda_2 = 0.25$, $\lambda_3 = 0.3$, $\lambda = 0.35$, $\lambda_5 = 0.4$ respectively.

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