Some Studies on Clique-free Sets of a Graph Using Clique Degree Conditions

Anusha Laxmana, Sayinath Udupa Nagara Vinayaka, Vinay Madhusudanan and Prathviraj Nagaraja

***Abstract*—Cliques are maximal complete subgraphs of a graph. A vertex v is said to vc-cover a clique C if v is in the clique C. A set S of vertices of a graph G is called a vc-covering set of G if every clique of G is vc-covered by some vertex in S. The cardinality of the smallest vc-covering set of G is called the vc-covering number, denoted as $\alpha_{vc}(G)$. In this paper, we define new parameters such as strong (weak) vc-covering number and strong (weak) clique-free number, and we establish a relationship between them. We present an algorithm to find these numbers and obtain some bounds for the newly defined parameters. In addition, we define a partial order on the vertex set of a graph using clique degree conditions and study some of its properties.**

***Index Terms*—vc-degree, clique-free set, vc-covering set, vc-poset.**

I. Introduction

In this paper, by a graph $G = (V, E)$ we mean a simple and undirected graph of order $|V| = p$ and size $|E| = q$, where V and E, respectively denote the vertex set and the edge set of G. The terminologies and notations used here are as in [4], [7]. A clique is a maximal complete subgraph of a graph. If a vertex v is in the clique C, we say that v is incident on C or v vc-covers C. The vc-degree of a vertex u denoted by $d_{vc}(u)$ is the number of cliques vc-covered by u. Note that $d_{vc}(u) \leq d(u)$, for any $u \in V(G)$ and if G is an acyclic graph, then $d_{vc}(u) = d(u)$, for all $u \in V(G)$. We denote minimum and maximum vc-degree of vertices of G by $\delta_{vc} = \delta_{vc}(G)$ and $\Delta_{vc} = \Delta_{vc}(G)$, respectively. A graph G is called vc-regular if every vertex has the same vc-degree. S. G. Bhat [2] initiated the study of vc-covering sets in graphs. A vc-covering set of a graph G is a set S of vertices in G such that every clique of G is vc-covered by some vertex in S and the vc-covering number of G denoted by $\alpha_{vc}(G)$ is the minimum cardinality of a vc-covering set of G. A set S of vertices in G is said to be a clique-free set [[2] pp. 56-96] if the subgraph induced by S does not induce any clique of G. The maximum cardinality of a clique-free set of G is the clique-free number of G and it is denoted by $\beta_{vc}(G)$. These two parameters satisfy the following relationship: $\alpha_{vc}(G) + \beta_{vc}(G) = p$. A vertex v is said to be a polycliqual vertex if it is incident on more than one clique. A clique is a pendant clique if only one polycliqual vertex is incident on it. A graph G is called a clique star if every clique of G is a pendant clique. For a survey on cliques refer to [6].

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II. Strong vc-covering sets and strong clique-free sets of a graph

Motivated by the study of strong (weak) vertex coverings and independent sets of a graph by S. S. Kamath and R. S. Bhat [5], we define the following.

Definition 1. *Let $G = (V, E)$ be a graph. A vertex $u \in V$ strongly (weakly) vc-covers a clique C of G if u vc-covers C and $d_{vc}(u) \geq d_{vc}(w)$ $(d_{vc}(u) \leq d_{vc}(w))$ for every vertex w vc-covering C.*

Definition 2. *A set $S \subseteq V$ is said to be a strong (weak) vc-covering set of G if elements of S strongly (weakly) vc-cover all the cliques of G. The strong (weak) vc-covering number $\alpha_{svc}(G)$ $(\alpha_{wvc}(G))$ is the minimum cardinality of a strong (weak) vc-covering set of G.*

Definition 3. *A set $W \subseteq V$ is said to be a strong (weak) clique-free set of G if W is a clique-free set of G and for any vertex $u \in W$ and for any clique C incident on u, there exists a vertex $v \in V - W$ such that v weakly (strongly) vc-covers C. The maximum cardinality of a strong (weak) clique-free set of G is the strong (weak) clique-free number denoted by $\beta_{svc}(G)$ $(\beta_{wvc}(G))$.*

Remark 1.
(i) For a null p-vertex graph $\overline{K_p}$, we assume that $\alpha_{svc}(\overline{K_p}) = \alpha_{wvc}(\overline{K_p}) = 0$ and $\beta_{svc}(\overline{K_p}) = \beta_{wvc}(\overline{K_p}) = p$.
(ii) Let $G = (V, E)$ be a non-trivial and non-null graph and $u_{\Delta_{vc}}$ and $(w_{\delta_{vc}})$ be vertices of G of the maximum and minimum vc-degrees, respectively. vc-degree. Then $V - \{u_{\delta_{vc}}\}$ is a strong vc-covering set of G and $V - \{u_{\Delta_{vc}}\}$ is a weak vc-covering set of G. Further, $\{u_{\Delta_{vc}}\}$ is a strong clique-free set of G and $\{u_{\delta_{vc}}\}$ is a weak clique-free set of G.
(iii) If G is a vc-regular graph, then $\alpha_{vc}(G) = \alpha_{svc}(G) = \alpha_{wvc}(G)$ and $\beta_{vc}(G) = \beta_{svc}(G) = \beta_{wvc}(G)$.

Example 1. *Consider the graph G_1 given in the Figure 1. Note that $d_{vc}(v_1) = d_{vc}(v_2) = d_{vc}(v_3) = 1$, $d_{vc}(v_4) = d_{vc}(v_5) = d_{vc}(v_6) = 2$ and $d_{vc}(v_7) = d_{vc}(v_8) = d_{vc}(v_9) = 3$. Then $\{v_4, v_5, v_6, v_7\}$ is a vc-covering set of G_1 of minimum cardinality, $\{v_4, v_5, v_6, v_7, v_8\}$ is a strong vc-covering set of G_1 of minimum cardinality and $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ is a weak vc-covering set of G_1 of minimum cardinality Hence, $\alpha_{vc}(G_1) = 4$, $\alpha_{svc}(G_1) = 5$ and $\alpha_{wvc}(G_1) = 7$. Also, we observe that $\{v_1, v_2, v_3, v_8, v_9\}$ is a clique-free set of maximum cardinality, $\{v_1, v_2, v_3, v_9\}$ is a weak clique-free set of maximum cardinality and $\{v_8, v_9\}$*

is a strong clique-free set of maximum cardinality. Therefore, $\beta_{vc}(G_1) = 5$, $\beta_{svc}(G_1) = 2$ *and* $\beta_{wvc}(G_1) = 4$.

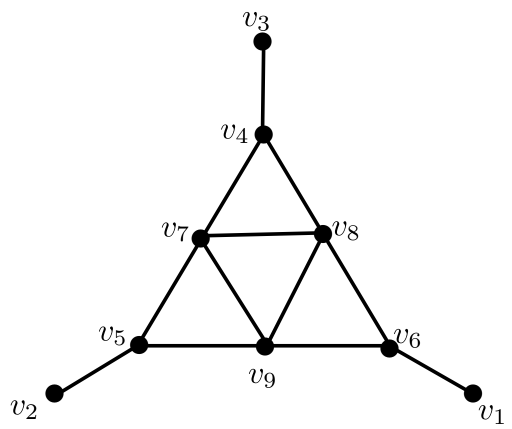

Fig. 1. Graph G_1 with strong vc-covering number=5, weak vc-covering number=7, strong clique-free number=2 and weak clique-free number=4

A. Gallai-type results

Lemma 1. *Let* $G = (V, E)$ *be a graph. For any set* $S \subseteq V$,

(i) S *is a strong vc-covering set of* G *if and only if* $V - S$ *is a weak clique-free set of* G.
(ii) S *is a weak vc-covering set of* G *if and only if* $V - S$ *is a strong clique-free set of* G.

Proof: Let S be a strong vc-covering set of G. Since every clique is vc-covered by some vertex in S, at least one vertex of every clique is in S. Hence $\langle V - S \rangle$ cannot induce any clique of G. Thus, $W = V - S$ is a clique-free set of G. Let $u \in W$ and C be a clique incident on u. Then, there exists a vertex $v \in S$ such that v strongly vc-covers C. Hence, W is a weak clique-free set of G. On the other hand, let W be a weak clique-free set of G. Suppose that $V - W$ is not a vc-covering set. Then there exists a clique C in G which is not vc-covered by any vertex in $V - W$. This implies all the vertices of C are in W. Then $\langle W \rangle$ contains the clique C, which is a contradiction. Thus, $S = V - W$ is a vc-covering set of G. Suppose there exists a clique C in G which is not strongly vc-covered by any vertex in S. Then there is a vertex say u in the clique C such that $d_{vc}(u) > d_{vc}(w)$, for every vertex $w \in V(C) \cap S$. This implies, $u \in W$ and there is no vertex in S which strongly covers C, which is a contradiction. Thus, S is a strong vc-covering set of G. Hence (i) follows. With similar arguments, we can prove (ii). ■

The following theorem depicts the relationship between the newly defined parameters.

Theorem 2. *For any graph* $G = (V, E)$ *of order* p,

(i) $\alpha_{svc}(G) + \beta_{wvc}(G) = p$
(ii) $\alpha_{wvc}(G) + \beta_{svc}(G) = p$.

Proof: Let S be a strong vc-covering set of G such that $|S| = \alpha_{svc}(G)$. Then by Lemma 1, $V - S$ is a weak clique-free set of G. Hence, $\beta_{wvc}(G) \geq |V - S| = p - |S|$. Therefore, $\alpha_{svc}(G) + \beta_{wvc}(G) \geq p$. Again, if W is a weak clique-free set of G such that $|W| = \beta_{wvc}(G)$. Then $V - W$ is a strong vc-covering set by Lemma 1. This implies that, $\alpha_{svc}(G) \leq |V - W| = p - |W|$. Hence, $\alpha_{svc}(G) + \beta_{wvc}(G) \leq p$. Then, from the above inequalities (i) follows. With similar arguments, we can prove (ii). ■

Proposition 3. *Let* $G = (V, E)$ *be a graph of order* p.

(i) If there exists a strong vc-covering set of G *which is also a strong clique-free set of* G*, then* $\alpha_{svc}(G) + \alpha_{wvc}(G) \leq p$.
(ii) If there exists a weak vc-covering set of G *which is also a weak clique-free set of* G*, then* $\beta_{svc}(G) + \beta_{wvc}(G) \geq p$.

Proof: Let S be a strong vc-covering set of G which is also a strong clique-free set of G. Then, by Lemma 1, $V - S$ is a weak vc-covering set of G. Hence, $\alpha_{wvc}(G) \leq |V - S| = p - |S|$. Also, since strong vc-covering set of G, we have $\alpha_{svc}(G) \leq |S|$. By adding the two results we get, $\alpha_{svc}(G) + \alpha_{wvc}(G) \leq p$. By, similar argument, we get $\beta_{svc}(G) + \beta_{wvc}(G) \geq p$. ■

Remark 2. *From the Remark 1 and Theorem 2, the following holds: if* G *is a non-trivial and non-null graph, then* $1 \leq \alpha_{svc}(G) \leq p-1$, $1 \leq \alpha_{wvc}(G) \leq p-1$, $1 \leq \beta_{svc}(G) \leq p-1$ *and* $1 \leq \beta_{wvc}(G) \leq p - 1$.

Theorem 4. *For any graph* G,

(i) $\alpha_{svc}(G) \leq \alpha_{wvc}(G)$
(ii) $\beta_{svc}(G) \leq \beta_{wvc}(G)$.

Proof: Let $S = \{v_1, v_2, ..., v_k\}$ be a strong vc-covering set of G such that $d_{vc}(v_1) \leq d_{vc}(v_2) \leq \cdots \leq d_{vc}(v_k)$. Let $\mathcal{C}$ be the collection of all cliques in G. If there is $w \in N[v_1]$ such that $d_{vc}(w) = 1$, then we name w by u_1. Otherwise, we choose $u_1 \in N[v_1]$ such that $d_{vc}(u_1)$ is minimum and u_1 weakly vc-covers maximum number of elements of $\mathcal{C}$. If v_1 itself satisfies the above condition, then we take $u_1 = v_1$. Let $D_1 = \{u_1\}$. We remove the cliques which are weakly vc-covered by u_1 from the set $\mathcal{C}$. Now for i, $2 \leq i \leq k$, choose $u_i \in N[v_i] - D_{i-1}$ such that either $d_{vc}(u_i) = 1$ or $d_{vc}(u_i)$ is minimum and u_i weakly vc-covers maximum number of elements of $\mathcal{C}$. If v_i itself satisfies the above condition, then we take $u_i = v_i$. Let $D_i = D_{i-1} \cup \{u_i\}$. We remove the cliques which are weakly vc-covered by u_i from the set $\mathcal{C}$. By continuing this process, we get $D_k = \{u_1, u_2, \ldots, u_k\}$ such that $D_k \subseteq W$, for some weak vc-covering set W of minimum cardinality. Hence, $\alpha_{svc}(G) = |S| = |D_k| \leq |W| = \alpha_{wvc}$. Thus, (i) holds. With similar arguments, we can prove (ii). ■

B. Algorithm to find the strong vc-covering number and weak clique-free number of a graph

In this section we present algorithms to find the strong (weak) clique-free number and vc-covering number of a graph. We make use of vcdeg() function which gives the number of cliques incident on the given vertex at that moment and sort() function which sorts the vertices of G in the descending order of the vc-degree.

Input: The vertex set V and the set of all cliques $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of G
Output: $\alpha_{svc}(G)$ and $\beta_{wvc}(G)$
Algorithm:
$\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$
$v[\,] = [v_1, v_2, \ldots, v_p]$
$V = sort(V)$

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d_vc v[ ] = [vcdeg(v_1), vcdeg(v_2), ..., vcdeg(v_p)]
S = {v[1]}
for (i = 1; i ≤ k; i = i + 1)
  if v[1] ∈ C_i then
    𝒞 = 𝒞 − {C_i}
  end if
end for
for (i = 2; i ≤ p; i = i + 1)
  if vcdeg(v[i]) ≥ 1 then
    x[ ] = [v[i]]
    for (j = 3; j ≤ p; j = j + 1)
      if (d_vc v[i] = d_vc v[j]) then
        x = x ∪ [v[j]]
      end if
    end for
    if x ≠ [v[i]] then
      x = sort(x)
      if (vcdeg(x[1]) > vcdeg(v[i])) then
        temp = v[i]
        v[i] = x[1]
        x[1] = temp
      end if
    end if
    S = S ∪ {v[i]}
  end if
  for (l = 1; l ≤ k; l = l + 1)
    if v[i] ∈ C_l then
      𝒞 = 𝒞 − {C_l}
    end if
  end for
end for
α_svc(G) = |S|
β_wvc(G) = |V − S|
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Note: In a similar manner, we can construct an algorithm to find $\alpha_{wvc}(G)$ and $\beta_{svc}(G)$.

Remark 3. *The time complexity of the algorithm 2.1. in worst case scenario is $O(n(2n+1))$.*

C. Construction of a graph with arbitrarily large difference between $\alpha_{vc}(G)$ and $\alpha_{svc}(G)$

Consider the graph G_1 as given in the Figure 1. We have $\alpha_{vc}(G_1) - \alpha_{svc}(G_1) = 1$. Let ${G_1}^{(1)} = {G_2}^{(1)} = {G_3}^{(1)} = G_1$ and we rename v_j by ${v_{ij}}^{(1)}$, for all $i, j, 1 \le i \le 3, 1 \le j \le 9$. Let G_2 be the graph obtained by joining the vertices v_i of G_1 and ${v_{i1}}^{(1)}$ of ${G_i}^{(1)}$ for all $i, 1 \le i \le 3$, as shown in the Figure 2. Note that $\{v_7, v_8, {v_{11}}^{(1)}, {v_{21}}^{(1)}, {v_{31}}^{(1)}, {v_{17}}^{(1)}, {v_{18}}^{(1)}, {v_{15}}^{(1)}, {v_{16}}^{(1)}, {v_{27}}^{(1)}, {v_{28}}^{(1)}, {v_{25}}^{(1)}, {v_{26}}^{(1)}, {v_{37}}^{(1)}, {v_{38}}^{(1)}, {v_{35}}^{(1)}, {v_{36}}^{(1)}\}$ is a strong vc-covering set of G_2 of minimum cardinality and $\{v_7, v_6, {v_{11}}^{(1)}, {v_{21}}^{(1)}, {v_{31}}^{(1)}, {v_{17}}^{(1)}, {v_{15}}^{(1)}, {v_{16}}^{(1)}, {v_{27}}^{(1)}, {v_{25}}^{(1)}, {v_{26}}^{(1)}, {v_{37}}^{(1)}, {v_{35}}^{(1)}, {v_{36}}^{(1)}\}$ is a vc-covering set of G_2 of minimum cardinality. Hence, $sn_0(G_2) - n_0(G_2) = 3$. Similarly, for $n \ge 1$, let ${G_i}^{(n)} = G_1$, for all $i, 1 \le i \le 3 \times 2^{n-1}$ and rename v_j by ${v_{ij}}^{(n)}$, for all $i, j, 1 \le i \le 3 \times 2^{n-1}, 1 \le j \le 9$. Consider the graph G_{n+1} obtained by joining the vertex ${v_{i1}}^{(n)}$ of ${G_i}^{(n)}$ and a pendant vertex of G_n, for all $i, 1 \le i \le 3 \times 2^{n-1}$. Then $\{v_7, v_8\} \cup \left(\bigcup_{j=1}^{n} \bigcup_{i=1}^{j} \{{v_{1k}}^{(i)}, {v_{2k}}^{(i)}, \ldots, {v_{(3\times 2^{i-1})k}}^{(i)}\}_{k=1,7,8} \right) \cup \bigcup_{i=1}^{3\times 2^{n-1}} \{{v_{ik}}^{(n)}\}_{k=5,6}$ is a strong vc-covering set of minimum cardinality of G_{n+1} and $\{v_7, v_6\} \cup \left(\bigcup_{j=1}^{n} \bigcup_{i=1}^{j} \{{v_{1k}}^{(i)}, {v_{2k}}^{(i)}, \ldots, {v_{(3\times 2^{i-1})k}}^{(i)}\}_{k=1,6,7} \right) \cup \bigcup_{i=1}^{3\times 2^{n-1}} \{{v_{i6}}^{(n)}\}$ is a vc-covering set of minimum cardinality of G_{n+1}. Therefore, $\alpha_{svc}(G) - \alpha_{vc}(G) = 3 \times 2^{n-1}$. Hence, it is possible to find a graph G with arbitrarily large difference between $\alpha_{vc}(G)$ and $\alpha_{svc}(G)$.

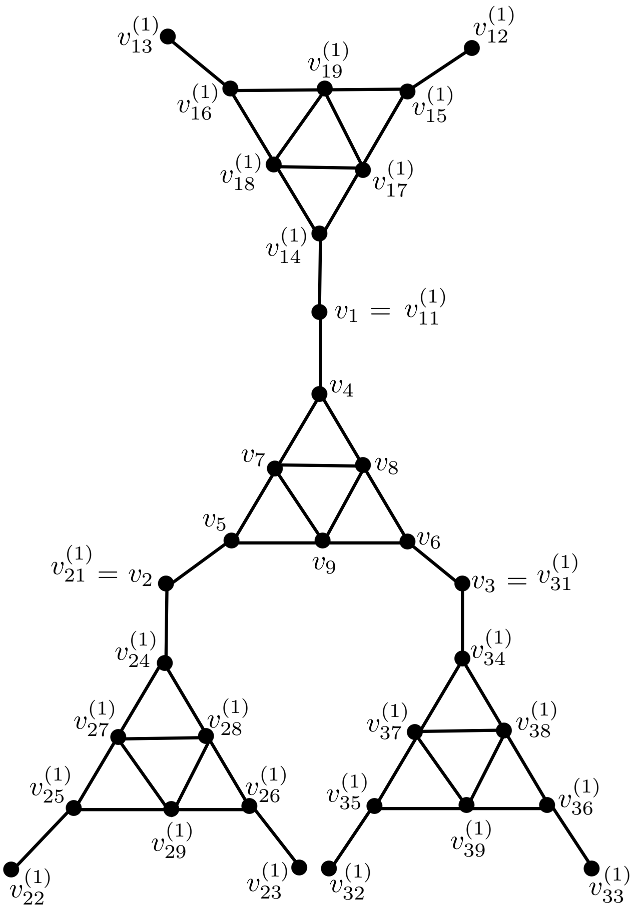

Fig. 2. Graph G_2 obtained by joining the vertices v_i of G_1 and ${v_{i1}}^{(1)}$ of ${G_i}^{(1)}$ for all $i, 1 \le i \le 3$.

Remark 4. *Consider a clique star graph G with n cliques. Then, there exists a unique vertex u (say) in G with $d_{vc}(u) = n$ and $d_{vc}(w) = 1$, for any vertex w other than u in G. We observe that the difference between $\alpha_{vc}(G) = \alpha_{svc}(G) = 1$ and $\alpha_{wvc}(G) = n$ is $n - 1$, which can be made arbitrarily large.*

D. Bounds on strong vc-covering number and weak vc-covering number of a graph

Proposition 5. *Let c be the number of cliques in a graph G and c_p be the number of pendant cliques in G. Then $c_p \le \alpha_{wvc} \le c$ and equality holds if and only if G is a clique star.*

Proof: Clearly, $\alpha_{wvc} \le c$. Suppose there is a pendant clique K of G. Then, there exists a vertex $u \in K$ such that $d_{vc}(u) > 1$ and $d_{vc}(v) = 1$, for all $v \in K$ such that $v \ne u$. Then, every minimal weak vc-covering set of G contains exactly one vertex of K with vc-degree 1. Hence,

$c_p \leq \alpha_{wvc}$. Note that G is a clique star if and only if every clique of G is a pendant clique if and only if $c = \alpha_{wvc} = c_p$. ■

Definition 4. *A polycliqual vertex u of a graph G is a support polycliqual vertex if it is incident on a pendant clique of G. Otherwise, it is called a non-support polycliqual vertex.*

Definition 5. *A polycliqual vertex u is said to be a weak non-support polycliqual vertex if it is a non-support polycliqual vertex and $d_{vc}(u) \leq d_{vc}(w)$ for every polycliqual vertex w adjacent to u.*

Proposition 6. *Let G be a connected graph which is not complete. Let p_c be the number of polycliqual vertices in G. Then, $\alpha_{svc}(G) \leq p_c$ and equality holds if and only if G is a weak non-support polycliqual vertex free graph.*

Proof: Suppose there is a vertex u in G which is not polycliqual. Let C be a clique incident on u. Since G is connected graph which is not complete, there exists a polycliqual vertex say v incident on C. Note that $d_{vc}(u) < d_{vc}(v)$. Choose a polycliqual vertex w incident on C with the maximum vc-degree. Then C is strongly covered by w. This implies that u does not belong to any minimal strong vc-covering set of G. Hence, $\alpha_{svc}(G) \leq p_c$. Now, assume that $\alpha_{svc}(G) = p_c$. Suppose there exists a weak non-support polycliqual vertex say u in G. Then $d_{vc}(u) \leq d_{vc}(v)$, for all polycliqual vertex v adjacent to u. Then u does not uniquely strongly vc-cover any clique of G. Hence, for any minimal strong vc-covering set S of G, there exists a polycliqual vertex in the closed neighbourhood of u which does not belong to S. This implies that $\alpha_{svc}(G) \leq p_c - 1$, which is a contradiction. Conversely, let G be a weak non-support polycliqual vertex free graph. Then every polycliqual vertex of G uniquely strongly vc-covers a clique of G. Hence, the set of all polycliqual vertices of G is the minimum strong vc-covering set of G. Therefore, $\alpha_{svc}(G) = p_c$. ■

III. Partial ordering relation on the vertex set of a graph using vc-degree

A partial order on a set P is a binary relation on P which is reflexive, antisymmetric and transitive. A set on which a partial order is defined is called a partially ordered set or briefly a poset. The length of a poset $(P, \leq)$ denoted by $l(P)$ is defined as $l(P) = \max\{|C| - 1 : \text{C is a maximal subchain of } P\}$. A lattice is a poset in which every pair of elements has a greatest lower bound and a least upper bound. For a survey refer to [3].

Definition 6. *We define a partial order $\leq$ on the vertex set V of a graph G as follows: for any two vertices u and w, $u \leq w$ if either $u = w$ or there exists a path between u and w say $u = v_1, v_2, \ldots, v_n = w$ such that $d_{vc}(v_1) < d_{vc}(v_2) < \cdots < d_{vc}(v_n)$. We call the poset $(V, \leq)$ as the vc-poset of G.*

Example 2. *Consider the graph G given in the Figure 3. Then $d_{vc}(v_1) = 4$, $d_{vc}(v_2) = d_{vc}(v_6) = d_{vc}(v_{10}) = d_{vc}(v_{11}) = 3$, $d_{vc}(v_3) = 2$, $d_{vc}(v_4) = d_{vc}(v_5) = d_{vc}(v_7) = d_{vc}(v_8) = d_{vc}(v_9) = d_{vc}(v_{12}) = d_{vc}(v_{13}) = 1$. Then $v_4 < v_3 < v_2 < v_1$, $v_5 < v_2 < v_1$, $v_{12} < v_{10} < v_1$, $v_7 < v_6 < v_1$, $v_8 < v_6 < v_1$, $v_9 < v_6 < v_1$ and $v_{13} < v_{11}$. The Hasse diagram of the vc-poset $(V, \leq)$ is given in the Figure 4.*

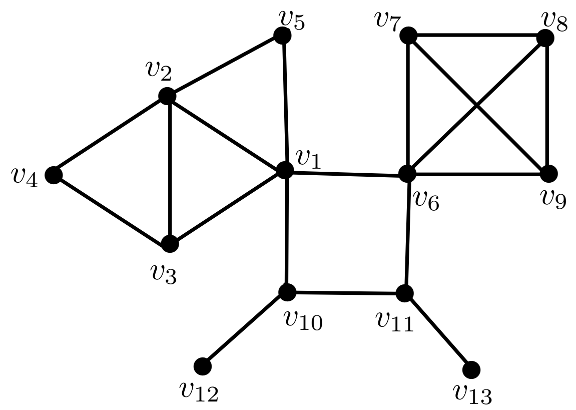

Fig. 3. Graph G with $\Delta_{vc}(G) = 4$ and $\delta_{vc}(G) = 1$

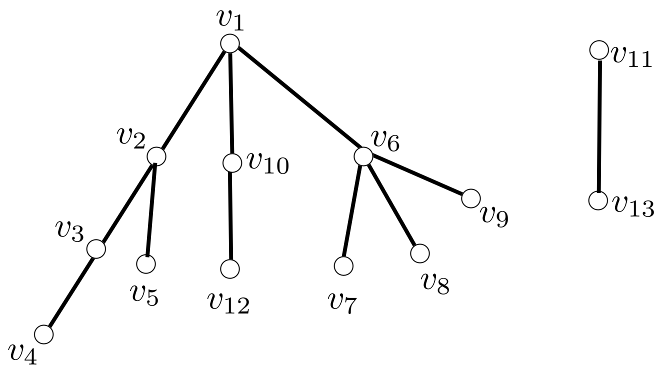

Fig. 4. vc-poset $(V, \leq)$ of the graph G

Proposition 7. *For a connected graph $G = (V, E)$, the Hasse diagram of the vc-poset $(V, \leq)$ is connected if and only if for any $u, w \in V$, there exists a path $u = v_1, v_2, \ldots, v_n = w$ in G such that $d_{vc}(v_i) \neq d_{vc}(v_{i+1})$, for all i, $1 \leq i \leq n - 1$.*

Proof: Assume that the Hasse diagram of the vc-poset $(V, \leq)$ is connected. Suppose there are two vertices u and w such that every u-w path in G contains some adjacent vertices of same vc-degree. Then u and w are not comparable in the poset $(V, \leq)$. We shall show that lowerbound and upperbound of u and w do not exist in $(V, \leq)$. In fact, suppose that a lowerbound of u and w exists in V. We choose a lowerbound l of u and w such that l is a maximal element of the set $\{v \in V : v \text{ is a lowerbound of } u \text{ and } w\}$. Then, $l < u$ and $l < w$. This implies that there exist paths $l = u_1, u_2, \ldots, u_k = u$ such that $d_{vc}(u_1) < d_{vc}(u_2) < \cdots < d_{vc}(u_k)$ and $l = v_1, v_2, \ldots, v_n = w$ such that $d_{vc}(v_1) < d_{vc}(v_2) < \cdots < d_{vc}(v_n)$. Then $u = u_k, u_{k-1}, \ldots, u_2, u_1 = l = v_1, v_2, \ldots, v_n = w$ is a u-w path in G such that adjacent vertices have the distinct vc-degrees, which is not possible. Similarly, we can prove that no upperbound u and w exists in V. This implies that u and w do not lie in the same component of the Hasse diagram of $(V, \leq)$, which is a contradiction to our assumption. Thus, there exists a path $u = v_1, v_2, \ldots, v_n = w$ in G such that $d_{vc}(v_i) \neq d_{vc}(v_{i+1})$, for all i, $1 \leq i \leq n - 1$. Conversely, assume that for any $u, w \in V$, there exists a path

$u = v_1, v_2, \ldots, v_n = w$ in G such that $d_{vc}(v_i) \neq d_{vc}(v_{i+1})$, for all i, $1 \leq i \leq n-1$. Consider v_i and v_{i+1}. Then either $d_{vc}(v_i) > d_{vc}(v_{i+1})$ or $d_{vc}(v_i) < d_{vc}(v_{i+1})$. This implies that either $v_i > v_{i+1}$ or $v_i < v_{i+1}$ in $(V, \leq)$. Then v_i and v_{i+1} lie in the same component of the Hasse diagram of $(V, \leq)$, for all i, $1 \leq i \leq n-1$. Therefore, u and w lie in the same component of the Hasse diagram of $(V, \leq)$. Hence, the Hasse diagram of the vc-poset $(V, \leq)$ is connected. ■

Definition 7. *Let $G = (V, E)$ be a graph. Then a vertex $v \in V$ is called a vc-strong (vc-weak) vertex if $d_{vc}(v) \geq d_{vc}(u)$ $(d_{vc}(v) \leq d_{vc}(u))$, for all u adjacent to v. A vertex $v \in V$ is vc-regular if $d_{vc}(v) = d_{vc}(u)$, for all u adjacent to v, and v is vc-balanced if there exist vertices u and w adjacent to v such that $d_{vc}(u) < d_{vc}(v) < d_{vc}(w)$.*

Definition 8. *The vc-strong number of a graph G denoted by $S_{vc}(G)$ is the number of vc-strong vertices in G. Similarly, vc-weak number $(W_{vc}(G))$, vc-regular number $(R_{vc}(G))$ and vc-balanced number $(B_{vc}(G))$ of G are defined.*

Observation 1. *Let G be a graph of order p. Then $S_{vc}(G) + W_{vc}(G) - R_{vc}(G) + B_{vc}(G) = p$.*

Proposition 8. *Let $G = (V, E)$ be a graph and $(V, \leq)$ be the vc-poset of G. Then*

(i) v is a vc-strong vertex of G if and only if v is a maximal element of $(V, \leq)$.
(ii) v is a vc-weak vertex of G if and only if v is a minimal element of $(V, \leq)$.
(iii) v is a vc-regular vertex of G if and only if v is not related to any element $u \in V$ such that $u \neq v$ with respect to $\leq$.
(iv) v is a vc-balanced vertex of G if and only if v is neither a minimal nor a maximal element of $(V, \leq)$.

Proof: (i) Assume that v is a vc-strong vertex of G. Suppose v is not a maximal element of $(V, \leq)$. Then there exists $u \in V$ such that $v < u$. This implies that there is a v-u path in G say $v = v_1, v_2, \ldots, v_n = u$ such that $d_{vc}(v) < d_{vc}(v_2) < \cdots < d_{vc}(u)$. We observe that v_2 is adjacent to v and $d_{vc}(v) < d_{vc}(v_2)$, which is a contradiction to our assumption. Conversely, assume that v is a maximal element of V. Suppose there exists a vertex u adjacent to v in G such that $d_{vc}(u) > d_{vc}(v)$. Then, v, u is a v-u path in G with the property $d_{vc}(v) < d_{vc}(u)$. This implies that $v < u$ in V, which is a contradiction to our assumption. Hence, v is a vc-strong vertex of G. By similar argument *(ii)* can be proved.
(iii) A vertex v of G is vc-regular if and only if v is both vc-strong and vc-weak vertex of G if and only if v is both maximal and minimal element of $(V, \leq)$ if and only if there is no $u, w \in V$ such that $u > v$ and $w < v$ if and only if v is not related to any element of $V - \{v\}$ with respect to $\leq$.
(iv) Let v be a vc-balanced vertex of G. Then there exist vertices u and w adjacent to v such that $d_{vc}(u) < d_{vc}(v) < d_{vc}(w)$. This implies that $u < v < w$ in V. Hence, v is neither a minimal nor a maximal element of $(V, \leq)$. Conversely, suppose there is $v \in V$ such that u is neither a minimal nor a maximal element. Then there exists $u, w \in V$ such that $u < v < w$. Then there is a u-v path in G say $u = v_1, v_2, \ldots, v_n = v$ such that $d_{vc}(u) < d_{vc}(v_2) < \cdots < d_{vc}(v_{n-1}) < d_{vc}(v)$ and a v-w path in G say $v = u_1, u_2, \ldots, u_k = w$ such that $d_{vc}(v) < d_{vc}(u_2) < \cdots < d_{vc}(w)$. We observe that v_{n-1} and u_2 are adjacent to v such that $d_{vc}(v_{n-1}) < d_{vc}(v) < d_{vc}(u_2)$. Hence, v is a vc-balanced vertex of G. ■

Remark 5. *(i) Let $\Delta_{vc}(G)$ and $\delta_{vc}(G)$ denote the maximum vc-degree and minimum vc-degree of a graph G. Then any vertex v of G with $d_{vc}(v) = \Delta_{vc}$ $(d_{vc}(v) = \delta_{vc})$ is a maximal (minimal) element of the vc-poset of G. But, the converse need not be true.*
(ii) Let $G = (V, E)$ be a graph and $l(V)$ be the length of the vc-poset $(V, \leq)$ of G. Then, $l(V) \leq \Delta_{vc}(G) - \delta_{vc}(G)$.
(iii) If two graphs G and H are isomorphic then their vc-posets are order isomorphic. But the converse need not be true. For example, consider the graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ given in the Figure 5.

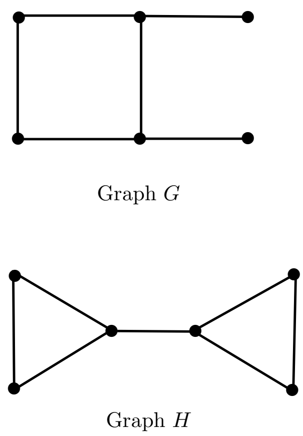

Fig. 5. Non isomorphic graphs G and H

Note that the graphs G and H are not isomorphic, but their vc-posets are order isomorphic. Then, the Hasse diagrams of the vc-posets $(V_1, \leq)$ and $(V_2, \leq)$ are same and it is given in the Figure 6.

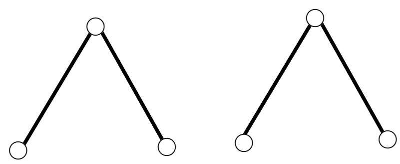

Fig. 6. Hasse diagram of the vc-posets $(V_1, \leq)$ and $(V_2, \leq)$

(iv) If the vc-poset of a graph G is a lattice, then $S_{vc}(G) = 1$, $W_{vc}(G) = 1$, $R_{vc}(G) = 0$ and $B_{vc}(G) = p - 2$.

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