Two-grid Domain Decomposition Method for Coupling of Fluid Flow with Porous Media Flow

Hao Zheng, Liyun Zuo

***Abstract*—This paper introduces a hybrid approach, merging the two-grid and domain decomposition strategies, to address the coupled Navier-Stokes-Darcy challenge, which is then elaborated and examined. First, the current Robin boundary condition-based domain decomposition technique is used to get the approximate solution on the coarse grid. Following the substitution of certain interface elements with coarse mesh-based functions, an improved fine grid problem is obtained.**

***Index Terms*—Darcy's law, Navier-Stokes equations, Domain decomposition method, Two-grid technique.**

I. Introduction

THE merging of fluid dynamics subject to Navier-Stokes principles with the movement of fluids in porous systems as dictated by Darcys law, mediated by specific interface conditions, plays a crucial role in a variety of fields. These include hydrogeomechanics, the simulation of soil contamination, biofluid mechanics, oil drilling and production engineering, as well as industrial filtration, among others. Due to its wide-ranging applications, this interdisciplinary area of study has garnered increasing interest in recent years.

Numerous studies have focused on advancing and examining numerical methods for resolving the combined Navier-Stokes (or Stokes) and Darcy flow issue. These methods encompass a broad spectrum of approaches, including: two-grid and multi-grid methods [1], [2], [3], local and parallel finite element methods [4], [5], domain decomposition method [6], [7], [8], [13], lagrange multiplier method [9], discontinuous Galerkin method [10], [11], coupled finite element method [12], and several other ways [14], [15], [16]. These methodologies embody the cutting edge of computational techniques designed to tackle the intricate interactions between fluid flows and porous media, enhancing our capacity to model and comprehend phenomena spanning a multitude of engineering and scientific disciplines.

Among the various methods mentioned, the two-grid method stands out for its ability to surmount the challenges posed by the tight coupling of distinct models within separate domains. Similarly, the domain decomposition method has proven to be highly effective for problems involving multiple domains and the coupling of different physical phenomena, as it benefits from the existence of efficient solvers for the individual, decoupled problems.

Recently, Sun and colleagues introduced a novel approach known as the two-grid domain decomposition method [13] to tackle the coupled Stokes-Darcy problem. This technique leverages Robin-type domain decomposition techniques in conjunction with a two-grid strategy, thereby enhancing the solutions accuracy and efficiency.

We delve into the intricate coupling of the Navier-Stokes-Darcy flow. Leveraging the Robin-Robin domain decomposition method [6] and two-grid domain decomposition method [13], we introduce an innovative two-grid domain decomposition method for the Navier-Stokes-Darcy model. For the Navier-Stokes-Darcy model, we consider the Beavers-Joseph-Saffman interface condition. The introduced approach initially employs a Robin boundary condition-based domain decomposition technique to obtain the coarse grid's estimated solution. Subsequently, the interface conditions are substituted with the coarse grid's solution, thereby obtaining a corrected fine grid problem. The method combines the advantages of both domain decomposition and two-grid techniques. An error analysis has been conducted to demonstrate the convergence of the method.

The structure of the remainder of this paper is organized as follows. In section II, the coupled Navier-Stokes-Darcy problem is presented. In section III, the domain decomposition technique is outlined. In section IV, two-grid domain decomposition approach is put forth. Section V delves into the error analysis of the two-grid domain decomposition method, elucidating its superiorities compared to alternative algorithms. In section VI, we discuss the prospective applications and future advancements of the two-grid domain decomposition method.

II. Coupled Navier-Stokes-Darcy Problem

In this part, we present the coupled Navier-Stokes-Darcy problem within a confined domain $\Omega \subset \mathbb{R}^d$ (where d = 2 or 3). The domain is divided into a fluid flow area Ω_f and a porous medium area Ω_p, with the interface Γ demarcating the boundary between the two, defined as $\Gamma = \partial\Omega_f \cap \partial\Omega_p$. Here, $\Omega_f \cap \Omega_p = \emptyset$, $\overline{\Omega}_f \cup \overline{\Omega}_p = \overline{\Omega}$. Let $\Gamma_f = \Omega_f \setminus \Gamma$, $\Gamma_p = \Omega_p \setminus \Gamma$.

Within the fluid compartment Ω_f, the flow of the fluid is regulated by the Navier-Stokes equations:

$$\begin{cases} -\nabla \cdot (\boldsymbol{T}(\overrightarrow{u}_f, p_f)) + \overrightarrow{u}_f \cdot \nabla \overrightarrow{u}_f = \overrightarrow{f}_1, \\ \nabla \cdot \overrightarrow{u}_f = 0, \end{cases} \tag{1}$$

where

$$\boldsymbol{T}(\overrightarrow{u}_f, p_f) = -p_f \boldsymbol{I} + 2\nu \boldsymbol{D}(\overrightarrow{u}_f),$$
$$\boldsymbol{D}(\overrightarrow{u}_f) = \frac{1}{2}(\nabla \overrightarrow{u}_f + \nabla^T \overrightarrow{u}_f),$$

$\overrightarrow{u}_f$ and p_f are construed respectively as the velocity of the fluid and the dynamic pressure within Ω_f. Besides, $\overrightarrow{f}_1$ represents the applied body force, and $\boldsymbol{T}(\overrightarrow{u}_f, p_f)$ is the stress tensor with the identity matrix $\boldsymbol{I}$ and the fluid's kinematic viscosity is denoted by ν, with $\nu > 0$.

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Within the porous medium domain Ω_p, the fluid movement is dictated by Darcy's law:

$$\begin{cases} \nabla \cdot \overrightarrow{u}_p = f_2, \\ \overrightarrow{u}_p = -\boldsymbol{K}\nabla\phi_p, \end{cases} \tag{2}$$

here, $\overrightarrow{u}_p$ represents the fluid flux within the porous region Ω_p. Without loss of generality, the tensor for hydraulic conductivity is presumed to exhibit isotropy which is represented by $\boldsymbol{K}$. The piezometric head, represented by ϕ_p, is calculated as the sum of the elevation z and the ratio of dynamic pressure p_p to the product of fluid density ρ and gravitational acceleration g. In addition, the term f_2 meets the criteria for solvability.

$$\int_{\Omega_p} f_2 = 0.$$

Upon integrating Darcy's law with (2), we derive the subsequent elliptic partial differential equation:

$$-\nabla \cdot (\boldsymbol{K}\nabla\phi_p) = f_2. \tag{3}$$

Assume the $\overrightarrow{u}_f$ and ϕ_p satisfying uniform Dirichlet boundary specifications:

$$\overrightarrow{u}_f = 0 \quad \text{on } \Gamma_f, \qquad \phi_p = 0 \quad \text{on } \Gamma_p.$$

Across the interface Γ, the following interface conditions are enforced:

$$\begin{cases} \overrightarrow{u}_f \cdot \overrightarrow{n}_f - \boldsymbol{K}\nabla\phi_p \cdot \overrightarrow{n}_p = 0, \\ -(\boldsymbol{T}(\overrightarrow{u}_f, p_f) \cdot \overrightarrow{n}_f) \cdot \overrightarrow{n}_f + \frac{1}{2}\overrightarrow{u}_f \cdot \overrightarrow{u}_f \\ \qquad = g\phi_p - gz, \\ -(\boldsymbol{T}(\overrightarrow{u}_f, p_f) \cdot \overrightarrow{n}_f) \cdot \overrightarrow{\tau}_i \\ \qquad = \frac{\alpha\nu\sqrt{d}}{\sqrt{trace(\prod)}} \cdot \overrightarrow{\tau}_i \cdot \overrightarrow{u}_f. \end{cases} \tag{4}$$

In the aforementioned equation, the index i takes the values $1, 2, 3, \ldots$. Where $\overrightarrow{n}_f$ and $\overrightarrow{n}_p$ are construed respectively as the outward-facing unit normal for Ω_f and Ω_p at the boundary Γ. Let $\vec{\tau}_i$ represent the set of orthogonal unit vectors tangent to the interface Γ, α is a fixed parameter, and $\Pi = \frac{\mathbf{K}\nu}{g}$.

To present the weak form of the hybrid model, we define as follows:

$$\begin{aligned} H_f &= \{\overrightarrow{v}_f \in (H^1(\Omega_f))^d : \overrightarrow{v}_f = 0 \ \text{ on } \Gamma_f\}, \\ Q_f &= L^2(\Omega_f), \\ H_p &= \{\psi_p \in H^1(\Omega_p) : \psi_p = 0 \ \text{ on } \Gamma_p\}. \end{aligned}$$

We use $(\cdot,\cdot)_{\Omega_X}$ and $\|\cdot\|_{L^2(\Omega_X)}$ to denote the standard L^2 inner product for the spaces $L^2(\Omega_X)$ (where X can be f or p) and the corresponding L^2 norms of functions in $L^2(\Omega_X)$, respectively.

III. Domain Decomposition Method

This part provide an overview of procedures detailed as described in reference [6]. This approach breaks down the coupled problem into two distinct sub-problems. These are computed concurrently in the domains Ω_f and Ω_p. Through the use of domain decomposition, the computational problem is effectively downsized, allowing for the leveraging of established software packages to solve each subproblem independently.

We will present the key Robin-type interface conditions. For two predetermined positive constants ξ_f and ξ_p, there exist corresponding functions g_f and g_p on the interface Γ which satisfy the following relationship:

$$(\boldsymbol{T}(\overrightarrow{u}_f, p_f) \cdot \overrightarrow{n}_f) \cdot \overrightarrow{n}_f - \frac{1}{2}\overrightarrow{u}_f \cdot \overrightarrow{u}_f + \xi_f \overrightarrow{u}_f \cdot \overrightarrow{n}_f = g_f, \tag{5}$$

$$\xi_p \boldsymbol{K}\nabla\phi_p \cdot \overrightarrow{n}_p + g\phi_p = g_p. \tag{6}$$

By (4), we can get

$$g_f = \xi_f \overrightarrow{u}_f \cdot \overrightarrow{n}_f - g\phi_p + gz \qquad \text{on } \Gamma, \tag{7}$$

$$g_p = \xi_p \overrightarrow{u}_f \cdot \overrightarrow{n}_f + g\phi_p \qquad \text{on } \Gamma. \tag{8}$$

It is easy to verify that the interface conditions (4) are equivalent to the aforementioned Robin-type conditions (5)-(6) if and only if the functions g_f and g_p fulfill the compatibility requirements on the interface Γ.

The weak form for the coupled problem is outlined as: two given functions g_f, g_p and two normal numbers ξ_f, ξ_p, find $(\overrightarrow{u}_f, p_f) \in H_f \times Q_f$, $\phi_p \in H_p$ under the condition that

$$a_p(\phi_p, \psi_p) + \langle \frac{g\phi_p}{\xi_p}, \psi_p \rangle = \langle \frac{g_p}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \quad \forall\ \psi_p \in H_p, \tag{9}$$

$$\begin{aligned} &\delta b_f(\overrightarrow{u}_f, \overrightarrow{u}_f, \overrightarrow{v}_f) + a_f(\overrightarrow{u}_f, \overrightarrow{v}_f) - \delta d_f(\overrightarrow{v}_f, p_f) \\ &+ \delta d_f(\overrightarrow{u}_f, q_f) + \delta\xi_f \langle \overrightarrow{u}_f \cdot \overrightarrow{n}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle \\ &+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}} \langle P_\tau \overrightarrow{u}_f, P_\tau \overrightarrow{v}_f \rangle \\ &= \delta(\overrightarrow{f}_1, \overrightarrow{v}_f)_{\Omega_f} + \delta\langle g_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle, \\ &\qquad \forall\ (\overrightarrow{v}_f, q_f) \in H_f \times Q_f, \end{aligned} \tag{10}$$

with $\delta = 1/\nu$, and $P_\tau \vec{u}_f = \sum_{j=1}^{d-1} (\vec{u}_f \cdot \vec{\tau}_j)\vec{\tau}_j$ representing the mapping onto the tangent plane of Γ. Here is an introduction to the bilinear form.

$$\begin{aligned} a_f(\overrightarrow{u}_f, \overrightarrow{v}_f) &= (\nabla \overrightarrow{u}_f, \nabla \overrightarrow{v}_f)_{\Omega_f}, \\ a_p(\phi_D, \psi_D) &= (\boldsymbol{K}\nabla\phi_p, \nabla\psi_p)_{\Omega_p}, \\ d_f(\overrightarrow{v}_f, q_f) &= (\nabla \cdot \overrightarrow{v}_f, q_f)_{\Omega_f}, \end{aligned}$$

and the trilinear form is

$$\begin{aligned} b_f(\overrightarrow{u}_f, \overrightarrow{u}_f, \overrightarrow{v}_f) &= (\overrightarrow{u}_f \cdot \nabla \overrightarrow{u}_f, \overrightarrow{v}_f)_{\Omega_f} \\ &+ \frac{1}{2}((\nabla \cdot \overrightarrow{u}_f)\overrightarrow{u}_f, \overrightarrow{v}_f)_{\Omega_f} \\ &- \frac{1}{2}\langle \overrightarrow{u}_f \cdot \overrightarrow{u}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle. \end{aligned}$$

Because $b_f(\overrightarrow{u}_f, \overrightarrow{u}_f, \overrightarrow{v}_f)$ is continuous on the space triplet $H_f \times H_f \times H_f$, we have

$$\begin{aligned} &b_f(\overrightarrow{u}_f, \overrightarrow{u}_f, \overrightarrow{v}_f) \\ &= \frac{1}{2}((\nabla \cdot \overrightarrow{u}_f)\overrightarrow{u}_f, \overrightarrow{v}_f)_{\Omega_f} - \frac{1}{2}\langle \overrightarrow{u}_f \cdot \overrightarrow{u}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle \\ &+ (\overrightarrow{u}_f \cdot \nabla \overrightarrow{u}_f, \overrightarrow{v}_f)_{\Omega_f} \\ &= \frac{1}{2}(\overrightarrow{u}_f \cdot \nabla \overrightarrow{u}_f, \overrightarrow{v}_f)_{\Omega_f} - \frac{1}{2}(\overrightarrow{u}_f \cdot \nabla \overrightarrow{v}_f, \overrightarrow{u}_f)_{\Omega_f} \\ &+ \frac{1}{2}\langle \overrightarrow{u}_f \cdot \overrightarrow{v}_f, \overrightarrow{u}_f \cdot \overrightarrow{n}_f \rangle - \frac{1}{2}\langle \overrightarrow{u}_f \cdot \overrightarrow{u}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle \\ &\qquad \forall \overrightarrow{u}_f, \overrightarrow{v}_f \in H_f. \end{aligned}$$

According $((\nabla \cdot \overrightarrow{u}_f)\overrightarrow{u}_f, \overrightarrow{v}_f)_{\Omega_f}$ equals zero, the trilinear form is suitable for the current problem (1)-(4). Also, it satisfy the following identity:

$$b_f(\overrightarrow{u}_f, \overrightarrow{u}_f, \overrightarrow{u}_f) = 0 \qquad \forall \overrightarrow{u}_f \in H_f.$$

It has been demonstrated by Xiaoming He and others in [6] that the aforementioned weak formulation (9)-(10) is well-posed.

The Robin-Robin domain decomposition approach necessitates consideration of the following finite element discretization methodology. Consider $\mathcal{T}_h$ as a uniform quasi-uniform tessellation of $\overline{\Omega}$ with a mesh parameter $h > 0$. Furthermore, $\mathcal{B}_h$ denotes the segmentation of Γ derived by $\mathcal{T}_h$.

Let $H_{f,h} \subset H_f$, $Q_{f,h} \subset Q_f$ and $H_{p,h} \subset H_p$ be the finite element subspaces. The P2-P1 to Navier-Stokes problem finite element, Darcy by matching the P2 finite element problem.

$$\begin{aligned} H_{f,h} &= \{\overrightarrow{v_{f,h}} \in (H^1(\Omega_f))^d : \overrightarrow{v_{f,h}} \mid_T \in (\mathbb{P}_2(T))^d \\ &\qquad \forall T \in \mathcal{T}_{f,h}, \overrightarrow{v_{f,h}} \mid_{\Gamma_f} = 0\}, \\ H_{p,h} &= \{\psi_{p,h} \in H^1(\Omega_p) : \psi_{p,h} \mid_T \in \mathbb{P}_2(T) \\ &\qquad \forall T \in \mathcal{T}_{p,h}, \psi_{p,h} \mid_{\Gamma_p} = 0\}, \\ Q_{f,h} &= \{q_{f,h} \in L^2(\Omega_f) : q_{f,h} \mid_T \in \mathbb{P}_1(T) \ \ \forall T \in \mathcal{T}_{f,h}\}. \end{aligned}$$

The spaces $H_{f,h}$ and $Q_{f,h}$ fulfill the inf-sup condition.

In particular, the definition of the discrete trace space on Γ is given as:

$$\begin{aligned} X_h &= \{g_h \in L^2(\Gamma) : \ g_h \mid_\tau \in \mathbb{P}_2(\tau) \ \ \forall \tau \in \mathcal{B}_h, \ g_h \mid_{\partial\Gamma} = 0\} \\ &= H_{f,h} \mid_\Gamma \cdot \overrightarrow{n}_f = H_{p,h} \mid_\Gamma . \end{aligned}$$

Drawing upon the Robin conditions for the coupled equation and compatibility conditions (7)-(8), we can outline the method, as described in [6]:

1) Initial values of g_f^0 and g_p^0 are guessed.
2) For n=0,1,2, find $\phi_{p,h}^n \in H_{p,h}$ satisfy

$$a_p(\phi_{p,h}^n, \psi_p) + \langle \frac{g\phi_{p,h}^n}{\xi_p}, \psi_p \rangle = \langle \frac{g_{p,h}^n}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \qquad \forall \ \psi_p \in H_{p,h}, \quad (11)$$

and $(\overrightarrow{u}_{f,h}^n, p_{f,h}^n) \in H_{f,h} \times Q_{f,h}$ satisfy

$$\begin{aligned} &\delta b_f(\overrightarrow{u}_{f,h}^n, \overrightarrow{u}_{f,h}^n, \overrightarrow{v}_f) + a_f(\overrightarrow{u}_{f,h}^n, \overrightarrow{v}_f) - \delta d_f(\overrightarrow{v}_f, p_{f,h}^n) \\ &+ \delta d_f(\overrightarrow{u}_{f,h}^n, q_f) + \delta \xi_f \langle \overrightarrow{u}_{f,h}^n \cdot \overrightarrow{n}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle \\ &+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}} \langle P_\tau \overrightarrow{u}_{f,h}^n, P_\tau \overrightarrow{v}_f \rangle \\ &= \delta(\overrightarrow{f}_1, \overrightarrow{v}_f)_{\Omega_f} + \delta \langle g_{f,h}^n, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle, \\ &\qquad \forall \ (\overrightarrow{v}_f, q_f) \in H_{f,h} \times Q_{f,h}, \quad (12) \end{aligned}$$

respectively.

3) Update $g_{f,h}^{n+1}, g_{p,h}^{n+1}$ by the following way:

$$\begin{aligned} g_{f,h}^{n+1} &= \frac{\xi_f}{\xi_p} g_{p,h}^n - (1 + \frac{\xi_f}{\xi_p}) g\phi_{p,h}^n + gz, \\ g_{p,h}^{n+1} &= -g_{f,h}^n + (\xi_f + \xi_p)\overrightarrow{u}_{f,h}^n \cdot \overrightarrow{n}_f + gz. \end{aligned}$$

The convergence analysis of aforementioned method has been obtained in [6]. Additionally, the method attains an error estimate that is not influenced by the mesh size h when $\xi_f < \xi_p$, provided that the parameters ξ_f and ξ_p are selected judiciously, in accordance with specific control criteria. It also offers a finite element approximation of the decoupled Navier-Stokes-Darcy problem with (5)-(6). Specifically, for given functions $g_{f,h}$, $g_{p,h}$ and two normal numbers ξ_f, ξ_p, the method seeks to determine $\vec{u}_{f,h} \in H_{f,h}, p_{f,h} \in Q_{f,h}$ and $\phi_{p,h} \in H_{p,h}$ satisfying

$$a_p(\phi_{p,h}, \psi_p) + \langle \frac{g\phi_{p,h}}{\xi_p}, \psi_p \rangle = \langle \frac{g_{p,h}}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \qquad \forall \ \psi_p \in H_{p,h}, \quad (13)$$

$$\begin{aligned} &\delta b_f(\overrightarrow{u}_{f,h}, \overrightarrow{u}_{f,h}, \overrightarrow{v}_f) + a_f(\overrightarrow{u}_{f,h}, \overrightarrow{v}_f) - \delta d_f(\overrightarrow{v}_f, p_{f,h}) \\ &+ \delta d_f(\overrightarrow{u}_{f,h}, q_f) + \delta \xi_f \langle \overrightarrow{u}_{f,h} \cdot \overrightarrow{n}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle \\ &+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}} \langle P_\tau \overrightarrow{u}_{f,h}, P_\tau \overrightarrow{v}_f \rangle \\ &= \delta(\overrightarrow{f}_1, \overrightarrow{v}_f)_{\Omega_f} + \delta \langle g_{f,h}, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle, \\ &\qquad \forall \ (\overrightarrow{v}_f, q_f) \in H_{f,h} \times Q_{f,h}, \quad (14) \end{aligned}$$

with the compatibility conditions:

$$g_{f,h} = \xi_f \overrightarrow{u}_{f,h} \cdot \overrightarrow{n}_f - g\phi_{p,h} + gz \qquad \text{on } \ \Gamma, \quad (15)$$

$$g_{p,h} = \xi_p \overrightarrow{u}_{f,h} \cdot \overrightarrow{n}_f + g\phi_{p,h} \qquad \text{on } \ \Gamma. \quad (16)$$

IV. The two-grid domain decomposition method

This section, we concentrate on a decoupling way to solve the coupled equations. Drawing inspiration from the method [6] and [13], we introduce a two-grid domain decomposition method that is specifically designed for the Navier-Stokes-Darcy equations and incorporates the BJS interface condition.

The resolution of the coupled Navier-Stokes-Darcy problem via the two-grid domain decomposition method involves the execution of two successive steps, which are outlined as follows.

1) Utilizing a grid with a coarse resolution of size H, we recall domain decomposition method to solve problems (13)-(14). Then we obtain the coarse grid result $g_{f,H}$, $g_{p,H}$.

2) Refined grid problem is formulated and resolved by determining $(\vec{u}_f^h, p_f^h) \in H_{f,h} \times Q_{f,h}$, $\phi_p^h \in H_{p,h}$, satisfying

$$a_p(\phi_p^h, \psi_p) + \langle \frac{g\phi_p^h}{\xi_p}, \psi_p \rangle = \langle \frac{g_{p,H}}{\xi_p}, \psi_p \rangle + (f_2, \psi_p)_{\Omega_p}, \qquad \forall \ \psi_p \in H_{p,h}, \quad (17)$$

$$\begin{aligned} &\delta b_f(\overrightarrow{u}_f^h, \overrightarrow{u}_f^h, \overrightarrow{v}_f) + a_f(\overrightarrow{u}_f^h, \overrightarrow{v}_f) - \delta d_f(\overrightarrow{v}_f, p_f^h) \\ &+ \delta d_f(\overrightarrow{u}_f^h, q_f) + \delta \xi_f \langle \overrightarrow{u}_f^h \cdot \overrightarrow{n}_f, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle \\ &+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}} \langle P_\tau \overrightarrow{u}_f^h, P_\tau \overrightarrow{v}_f \rangle \\ &= \delta(\overrightarrow{f}_1, \overrightarrow{v}_f)_{\Omega_f} + \delta \langle g_{f,H}, \overrightarrow{v}_f \cdot \overrightarrow{n}_f \rangle, \\ &\qquad \forall \ (\overrightarrow{v}_f, q_f) \in H_{f,h} \times Q_{f,h}. \quad (18) \end{aligned}$$

The two-grid domain decomposition method leverages the strengths of the two-grid approach and the domain decomposition technique. It adeptly manages the robust coupling between disparate models within distinct domains. Also this method is suitable for multi-domain coupling problems. For decoupled solutions, it can further enhance the efficiency of the operation.

V. Error analysis

This part will echoes the rationale presented from [6] to illustrate the convergence of the proposed two-grid domain decomposition algorithm. In the interest of brevity, we introduce the notation $x \lesssim y$ to signify that x is less than or comparable to Cy, where C is an arbitrary constant whose value may take on various values depending on the context. The following are the error estimates for the algorithm as [6] discussed:

$$
\begin{aligned}
&\|\overrightarrow{u}_f - \overrightarrow{u}_{f,h}\|_1 \lesssim h^2, && \|\overrightarrow{u}_f - \overrightarrow{u}_{f,h}\| \lesssim h^3,\\
&\|\phi_p - \phi_{p,h}\|_1 \lesssim h^2, && \|\phi_p - \phi_{p,h}\| \lesssim h^3,\\
&\|p_f - p_{f,h}\| \lesssim h^2.
\end{aligned}
$$

For the finite element approximation given by equations (13)-(14), we express the error functions, which are related to the discrepancies between the solution elements on both the coarse and refined meshes, as follows:

$$
\begin{aligned}
&\sigma_{f,H} = g_{f,h} - g_{f,H}, && \sigma_{p,H} = g_{p,h} - g_{p,H},\\
&\overrightarrow{\theta}_{f,H} = \overrightarrow{u}_{f,h} - \overrightarrow{u}_{f,H}, && \theta_{p,H} = \phi_{p,h} - \phi_{p,H},\\
&\zeta_{f,H} = p_{f,h} - p_{f,H}.
\end{aligned}
$$

Then, by means of the triangle inequality, we can easily obtain several basic error estimates about the numerical solution of (13)-(14) on coarse and fine meshes

$$
\begin{aligned}
&\|\overrightarrow{\theta}_{f,H}\|_1 \lesssim H^2, && \|\overrightarrow{\theta}_{f,H}\| \lesssim H^3,\\
&\|\theta_{p,H}\|_1 \lesssim H^2, && \|\theta_{p,H}\| \lesssim H^3,\\
&\|\zeta_{f,H}\| \lesssim H^2.
\end{aligned} \tag{19}
$$

To implement the error estimation, the following lemma is essential:

Lemma 1: Across the boundary Γ, there are two error bounds for $\sigma_{f,H}$ and $\sigma_{p,H}$ that are associated with the interface constraints, which are expressed as

$$\|\sigma_{f,H}\|_\Gamma \lesssim (\xi_f + g)H^{\frac{5}{2}}, \tag{20}$$

$$\|\sigma_{p,H}\|_\Gamma \lesssim (\xi_p + g)H^{\frac{5}{2}}. \tag{21}$$

Proof: Based on the $\sigma_{f,H}$, $\sigma_{p,H}$, and (15)-(16), the following formula can be obtained.

$$
\begin{aligned}
\sigma_{f,H} &= \xi_f \overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f - g\theta_{p,H},\\
\sigma_{p,H} &= \xi_p \overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f + g\theta_{p,H}.
\end{aligned}
$$

Using the Young inequality we can launch

$$
\begin{aligned}
\|\sigma_{f,H}\|_\Gamma &= \|\xi_f \overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f - g\theta_{p,H}\|_\Gamma\\
&\le \xi_f \|\overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f\|_\Gamma + g\|\theta_{p,H}\|_\Gamma.
\end{aligned}
$$

Based on the trace inequality, we are aware that a constant C is guaranteed to exist satisfying that

$$
\begin{aligned}
&\|\overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f\|_\Gamma \le C\|\theta_{f,H}\|^{\frac{1}{2}}\|\theta_{f,H}\|_1^{\frac{1}{2}},\\
&\|\theta_{p,H}\|_\Gamma \le C\|\theta_{p,H}\|^{\frac{1}{2}}\|\theta_{p,H}\|_1^{\frac{1}{2}},
\end{aligned}
$$

then we can conclude that

$$
\begin{aligned}
\xi_f\|\overrightarrow{\theta}_{f,H} \cdot \overrightarrow{n}_f\|_\Gamma + g\|\theta_{p,H}\|_\Gamma &\le \xi_f C\|\theta_{f,H}\|^{\frac{1}{2}}\|\theta_{f,H}\|_1^{\frac{1}{2}}\\
&\quad + gC\|\theta_{p,H}\|^{\frac{1}{2}}\|\theta_{p,H}\|_1^{\frac{1}{2}}\\
&\le \xi_f C H^{\frac{3}{2}} H + gCH^{\frac{3}{2}} H\\
&\lesssim (\xi_f + g)H^{\frac{5}{2}}.
\end{aligned}
$$

Inequality (21) can be obtained in the same way. ■

Building on the aforementioned groundwork, we can derive the error evaluation for the two-grid domain decomposition method as follows:

Theorem 1: Suppose $(\overrightarrow{u}_{f,h}, p_{f,h}, \phi_{p,h})$ as the solution derived from domain decomposition method, and assume that $(\overrightarrow{u}_f^h, p_f^h, \phi_p^h)$ is the solution derived from two-grid domain decomposition method, we arrive at

$$\|\phi_{p,h} - \phi_p^h\|_1 \lesssim \frac{\xi_p + g}{\boldsymbol{K}\xi_p} H^{\frac{5}{2}}, \tag{22}$$

$$\|\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h\|_1 \lesssim R_1(\xi_f + g)H^{\frac{5}{2}}, \tag{23}$$

$$\|p_{f,h} - p_f^h\| \lesssim R_2(\xi_f + g)H^{\frac{5}{2}}, \tag{24}$$

where

$$
\begin{aligned}
R_1 &= \frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu} - C_0^2\delta},\\
R_2 &= \Big(\frac{1}{\sqrt{2\nu}} + C_1\xi_f + \frac{C_2^2\alpha\sqrt{d}}{\delta\sqrt{trace(\prod)}}\Big)\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu} - C_0^2\delta}.
\end{aligned}
$$

Proof: On the fine grid, subtracting (17)-(18) from (13)-(14) yields

$$
\begin{aligned}
a_p(\phi_{p,h} - \phi_p^h, \psi_p) &+ \langle \frac{g(\phi_{p,h} - \phi_p^h)}{\xi_p}, \psi_p\rangle\\
&= \langle \frac{g_{p,h} - g_{p,H}}{\xi_p}, \psi_p\rangle, \quad \forall\ \psi_p \in H_{p,h},
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&\delta b_f(\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h, \overrightarrow{u}_{f,h}, \overrightarrow{v}_f) + \delta b_f(\overrightarrow{u}_{f,h}, \overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h, \overrightarrow{v}_f)\\
&- \delta b_f(\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h, \overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h, \overrightarrow{v}_f)\\
&+ a_f(\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h, \overrightarrow{v}_f) - \delta d_f(\overrightarrow{v}_f, p_{f,h} - p_f^h)\\
&+ \delta d_f(\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h, q_f)\\
&+ \delta\xi_f\langle(\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f, \overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle\\
&+ \frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{u}_{f,h} - \overrightarrow{u}_f^h), P_\tau\overrightarrow{v}_f\rangle\\
&= \delta\langle(g_{f,h} - g_{f,H}), \overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle.\\
&\qquad\qquad \forall\ (\overrightarrow{v}_f, q_f) \in H_{f,h}\times Q_{f,h}.
\end{aligned} \tag{26}
$$

Let $\psi_p = \phi_{p,h} - \phi_p^h \in H_{p,h}$ in (25), get

$$
\begin{aligned}
a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h) &+ \langle \frac{g(\phi_{p,h} - \phi_p^h)}{\xi_p}, (\phi_{p,h} - \phi_p^h)\rangle\\
&= \langle \frac{g_{p,h} - g_{p,H}}{\xi_p}, (\phi_{p,h} - \phi_p^h)\rangle.
\end{aligned} \tag{27}
$$

Utilizing the Cauchy-Schwarz inequality in conjunction with the trace inequality, we can conclude

$$
\begin{aligned}
&\|\phi_{p,h} - \phi_p^h\|_1^2 \le \frac{1}{\boldsymbol{K}} a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h),\\
&a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h)\\
&\qquad \le a_p(\phi_{p,h} - \phi_p^h, \phi_{p,h} - \phi_p^h) + \frac{g}{\xi_p}\|\phi_{p,h} - \phi_p^h\|_\Gamma^2,
\end{aligned}
$$

then, from Lemma 1 and (27), we get the following inequality,

$$\begin{aligned}\|\phi_{p,h}-\phi_p^h\|_1^2 &\le \frac{1}{\boldsymbol{K}}[a_p(\phi_{p,h}-\phi_p^h,\phi_{p,h}-\phi_p^h)\\ &+\langle\frac{g(\phi_{p,h}-\phi_p^h)}{\xi_p},(\phi_{p,h}-\phi_p^h)\rangle]\\ &\le \frac{1}{\boldsymbol{K}}\langle\frac{g_{p,h}-g_{p,H}}{\xi_p},(\phi_{p,h}-\phi_p^h)\rangle\\ &\le \frac{1}{\boldsymbol{K}\xi_p}\|\sigma_{p,H}\|_\Gamma\|\phi_{p,h}-\phi_p^h\|_\Gamma\\ &\lesssim \frac{\xi_p+g}{\boldsymbol{K}\xi_p}H^{\frac{5}{2}}\|\phi_{p,h}-\phi_p^h\|_1. \qquad (28)\end{aligned}$$

We can get (22) by eliminating $\|\phi_{p,h}-\phi_p^h\|_1$ from (28).

Setting $\overrightarrow{v}_f=\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h, q_f=p_{f,h}-p_f^h$. Substituting into (26), we have

$$\begin{aligned}&\delta b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\\ &+\delta b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\\ &+a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\\ &-\delta d_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,p_{f,h}-p_f^h)\\ &+\delta d_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,p_{f,h}-p_f^h)\\ &+\delta\xi_f\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f,(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f\rangle\\ &+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h),P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\rangle\\ &=\delta\langle(g_{f,h}-g_{f,H}),(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f\rangle, \qquad (29)\end{aligned}$$

for the trilinear terms in (29), we have

$$\begin{aligned}&b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\\ &+b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\\ &\lesssim\frac{1}{\sqrt{2\nu}}\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1^2,\end{aligned}$$

by the $Korn's$ inequality, there exists C_0 that makes

$$\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1^2\le\frac{C_0^2}{2}a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h),$$

then we can get

$$\begin{aligned}&\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1^2\\ &\le\frac{C_0^2}{2}[a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\\ &+\delta\xi_f\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f,(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f\rangle\\ &+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\langle P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h),P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\rangle]\\ &\le\frac{C_0^2}{2}[\delta\langle(g_{f,h}-g_{f,H}),(u_{f,h}-u_f^h)\cdot n_f\rangle\\ &+\delta|b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)|\\ &+\delta|b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)|]\\ &\lesssim\frac{C_0^2\delta}{2}[\|\sigma_{f,H}\|_\Gamma\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_\Gamma\\ &+\frac{1}{\sqrt{2\nu}}\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1^2]. \qquad (30)\end{aligned}$$

Thanks to Lemma 1 we know

$$\|\sigma_{f,H}\|_\Gamma\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_\Gamma\lesssim\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1(\xi_f+g)H^{\frac{5}{2}},$$

by subdividing (30) and simplifying it, we get

$$\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1\lesssim\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu}-C_0^2\delta}(\xi_f+g)H^{\frac{5}{2}}.$$

Let $q_f=p_{f,h}-p_f^h\in Q_{f,h}$, there exist $\overrightarrow{v}_f\in H_{f,h}$ such that

$$\|p_{f,h}-p_f^h\|\le\frac{d_f(\overrightarrow{v}_f,p_{f,h}-p_f^h)}{\|\overrightarrow{v}_f\|_1}.$$

It can also be inferred from (26) that

$$\begin{aligned}\|p_{f,h}-p_f^h\|&\le\frac{1}{\|\overrightarrow{v}_f\|_1\delta}\Big[\delta|\langle(g_{f,h}-g_{f,H}),\overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle|\\ &+\delta|b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{v}_f)|\\ &+\delta|b_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{u}_{f,h},\overrightarrow{v}_f)|\\ &+\delta|b_f(\overrightarrow{u}_{f,h},\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{v}_f)|\\ &+|a_f(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h,\overrightarrow{v}_f)|\\ &+\delta\xi_f|\langle(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h)\cdot\overrightarrow{n}_f,\overrightarrow{v}_f\cdot\overrightarrow{n}_f\rangle|\\ &+\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}|\langle P_\tau(\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h),P_\tau\overrightarrow{v}_f\rangle|\Big]\\ &\lesssim\|\sigma_{f,H}\|_\Gamma+\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1^2\\ &+(\frac{1}{\sqrt{2\nu}}+C_1\xi_f)\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1\\ &+\frac{C_2^2}{\delta}\frac{\alpha\sqrt{d}}{\sqrt{trace(\prod)}}\|\overrightarrow{u}_{f,h}-\overrightarrow{u}_f^h\|_1.\end{aligned}$$

Thus,

$$\begin{aligned}\|p_{f,h}-p_f^h\|&\lesssim(\frac{1}{\sqrt{2\nu}}+C_1\xi_f)\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu}-C_0^2\delta}(\xi_f+g)H^{\frac{5}{2}}\\ &+\frac{C_2^2\alpha\sqrt{d}}{\delta\sqrt{trace(\prod)}}\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu}-C_0^2\delta}(\xi_f+g)H^{\frac{5}{2}}\\ &+\big(\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu}-C_0^2\delta}(\xi_f+g)H^{\frac{5}{2}}\big)^2\\ &\lesssim(\frac{1}{\sqrt{2\nu}}+C_1\xi_f)\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu}-C_0^2\delta}(\xi_f+g)H^{\frac{5}{2}}\\ &+\frac{C_2^2\alpha\sqrt{d}}{\delta\sqrt{trace(\prod)}}\frac{C_0^2\delta\sqrt{2\nu}}{2\sqrt{2\nu}-C_0^2\delta}(\xi_f+g)H^{\frac{5}{2}}\end{aligned}$$

which completes the proof of (24). ■

Leveraging Theorem 1 alongside the triangle inequality, we can construct an estimate for the solution's error obtained by the two-grid domain decomposition method in relation to the exact solution as detailed below.

Corollary 1: Let $(\overrightarrow{u}_f^h,p_f^h)\in H_{f,h}\times Q_{f,h},\phi_p^h\in H_{p,h}$, and $(\overrightarrow{u}_f,p_f)\in H_f\times Q_f,\phi_p\in H_p$ be the solution of two-grid domain decomposition method and (9)-(10), respectively. Choosing $H=h^{\frac{2}{5}}$, we have

$$\|\phi_p-\phi_p^h\|_1\lesssim h,\qquad\|\overrightarrow{u}_f-\overrightarrow{u}_f^h\|_1+\|p_f-p_f^h\|\lesssim h.$$

The algorithm accuracy of two-grid domain decomposition method is obtained. Moreover, the framework can be broadened to encompass higher-order elements provided the continuous solution exhibits high regularity. At last, the accuracy of the algorithm can be further improved by some rigorous analysis in further research.

VI. Conclusion

This paper proposes a novel algorithm designed to solve the coupled Navier-Stokes-Darcy system. A rigorous convergence analysis of the algorithm has been conducted, and the corresponding convergence accuracy has been obtained. The analysis demonstrates that the new algorithm possesses good convergence accuracy and is an effective method for solving coupled problems. In the future, this approach can be extended to solve other distinct and even more complex coupled problems,such as multi-physics coupling problems.

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