Soft Topological Transformation Groups

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Abstract—**In this paper, the notion of a soft topological transformation group is defined and studied. For a soft topological transformation group, it is proven that a map from a soft topological space onto itself is soft homeomorphism. The collection of all soft homeomorphisms of the given soft topological space onto itself constitutes soft topological group under composition. Subsequently, it is proved that there is a homomorphism between soft topological group and the group structure on the collection of all soft homeomorphisms of given topological space. Subsequently, it is shown that the mapping space Map(Y,Y) is soft Hausdorff and verified that any subspace of the mapping space is soft Hausdorff. Additionally, it is proved that the set of all soft homeomorphisms on Y forms a soft discrete space, soft extremally disconnected space, soft Moscow space and a soft Moscow topological group. Later, it is shown that the map from a soft topological group to a mapping space is soft continuous. Finally, it is proved that distinct group structure generates distinct collection of all soft homeomorphisms of the specified soft topological space onto itself is a soft isomorphism.**

Index Terms—**Topological group, soft topological space, soft topological group, soft*continuous, soft homeomorphism.**

I. Introduction

Topology and algebra are pivotal branches of mathematics that mutually complement each other. Topology focuses on concepts like continuity and convergence, while algebra scrutinizes various operations. Despite often evolving autonomously, they naturally intertwine with other mathematical domains like functional analysis, representation theory, and dynamical systems. Notably, many significant mathematical entities embody a fusion of topological and algebraic attributes. Topological transformation groups exemplify this synergy, embodying both topological and algebraic structures.

The investigation into the topological transformation group was initiated by Montogomery and Zippin [37], concentrating in particular on Hilbert's fifth problem. A topological transformation group is formed by combining a topological group and a topological space by a continuous action. This structure has been extensively studied in various concepts.

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A topological transformation group with a fixed end point was examined by William J. Gray [15] in 1966. J. De Vries [13] explored the concept of a universal topological transformation group by examining the actions of an infinite locally compact group. The dimension of a topological transformation group was provided by H. Ku et al., [22] in 1976. In 1990, David B. Ellis conducted a study on the suspensions of topological transformation groups. R. A. Alsabeh et al., [4] provided a definition for distal in the context of topological transformation groups and utilized the concept of automorphism to derive the notion of strongly distal. In 2017, E.Y. Abdullah [1] explored extensive set in topological transformation group. Keerthana Dhanasekar et al., [21] studied various structures of topological transformation groups such as Irr-topological transformation groups, Irr*-topological transformation groups, I-topological transformation groups, I*-topological transformation groups and explored the interrelations between them in 2024.

$\mathcal{S}$-topological transformation group [28], fixed point set and equivariant map of $\mathcal{S}$-topological transformation groups [29] was explored by C. Rajapandiyan et al., in 2024. $\mathcal{S}$-topological transformation group is a structure formed by connecting a topological group and a topological space with a semi totally continuous action. $\mathcal{S}$-topological transformation group implies a topological transformation group but the converse is not necessarily true. The collection of all semi totally continuous functions of X onto itself, denoted by $STC_G(X)$ constitutes a paratopological group under composition. The extremally disconnectedness property of $STC_G(X)$ creates a Moscow topological group structure on $STC_G(X)$ and it contains an open boolean subgroup.

Soft set theory can be traced back to the development of fuzzy set theory in the 1960s by L.A. Zadeh [36]. Fuzzy set theory introduced the concept of partial membership, allowing objects to belong to sets with varying degrees of membership rather than strictly belonging or not belonging. This departure from traditional set theory laid the groundwork for the development of soft set theory.

The paper also dives into soft set theory, which Molodtsov [24] established as a contemporary mathematical paradigm for dealing with vagueness and uncertainty. This idea has generated a wave of research in a variety of disciplines, including engineering, medicine, economics, and environmental sciences. In recent years, it has been combined with fundamental mathematical theories such as algebra and topology, resulting in substantial advances. Numerous research have investigated the algebraic features of soft sets, with fundamental contributions from Maji, Aktas, Jun, and Feng [[3], [14], [19], [23]]. Maji et al., [23] proposed new operations on soft sets, while Aktas et al., [3] revised the definitions of soft groups and subgroups, revealing new features. Additional algebraic properties of soft sets have been investigated and studied in [[6], [8], [26]]. Shabir and Naz [31] laid the groundwork for subsequent research on soft topological spaces. Later, Nazmul and Samanta [25] broadened their investigation by defining soft topological groups. Several other investigations have recently appeared, broadening our understanding of soft topological structures [[2], [5], [7], [9], [12], [20], [32], [33], [35]]. Separation Axioms in Soft L-topological Spaces was explored by Sandhya et al., [30] in 2023.

Since its introduction, soft set theory has seen various extensions and applications in different fields such as decision-making, data analysis, image processing, pattern recognition, and more. Researchers have explored different aspects of soft sets, including operations on soft sets, approximation techniques, and the relationship between soft sets and other mathematical structures like rough sets and fuzzy sets. Over the years, researchers continued their contributions to develop and refine soft set theory, introducing new concepts, exploring its theoretical foundations, and investigating its practical applications. Soft set theory remains an active area of research, with ongoing efforts to extend its scope, address its limitations, and integrate it with other theories and methodologies in mathematics and computer science.

In this paper, the main section is focused on the study of soft topological transformation group. Soft topological transformation group is a structure moulded by interconnecting soft topological group and a soft topological space with a soft*continuous action. For any soft topological transformation group, it is proved that the set of all soft homeomorphisms on Y forms a group structure and is denoted by $SFhomeo_H(Y)$. Later, the map ζ forms a homomorphism between the topological group H and $SFhomeo_H(Y)$. The homomorphism ζ provokes a H-action $\delta' : H \times Y \to Y$ given by $\zeta(h)(y) = \delta'(h, y)$. The existence of soft*continuous map δ' leads to have a continuous map from a soft topological group to a subset of a mapping space. Subsequently, it is ascertained that the set of all soft homeomorphisms on Y forms a soft topological group. Additionally, it is shown that the mapping space $\text{Map}(Y, Y)$ is soft Hausdorff, for a given soft Hausdorff space Y and verified that any subspace of the mapping space is soft Hausdorff. Later, it is proved that the set of all soft homeomorphisms on Y is a soft Hausdorff and also verified that $SFhomeo_H(Y) \times SFhomeo_H(Y)$ is a soft Hausdorff. Furthermore, it has been demonstrated that the collection of all soft homeomorphisms on Y constitutes a soft discrete space, soft extremally disconnected space, soft Moscow space, and a soft Moscow topological group. Finally, it is proved that the map between $SFhomeo_H(Y)$ and $SFhomeo_{H'}(Y)$ is a soft isomorphism.

In this paper, the soft topological transformation group is defined in the sense of T. Hida's [17] soft topological group.

The paper is sectioned as follows. The first section pertains to the introduction. Preliminaries were discussed in the second section. Soft topological transformation group is explored in section three and section four is the conclusion.

II. Preliminaries

The basic definitions of this article are defined in this section.

Definition 2.1. [27] A nonempty set H is called a topological group, if
(1) H is a group.
(2) H is a topological space.
(3) Both multiplication $\sigma : H \times H \to H$ and inversion $\gamma : H \to H$ maps are continuous.

Definition 2.2. [11] Let H be a group, Y be a set and the map $\delta : H \times Y \to Y$ given by $\delta(h, y) = hy$ such that the following conditions hold,
(1) $\delta(e, y) = y$, for every $y \in Y$, where e is the identity element of H.
(2) $\delta(h_2, \delta(h_1, y)) = \delta(h_2h_1, y)$, for all $h_1, h_2 \in H$ and $y \in Y$.

The triplet (H, Y, δ) is said to be a transformation group or H-action on Y and Y is called a H-set.

Definition 2.3. [11] Let H be a topological group, Y a topological space and the map $\delta : H \times Y \to Y$ is continuous. Then the triplet (H, Y, δ) forms a topological transformation group such that the following conditions hold,
(1) $\delta(e, y) = y$, for every $y \in Y$, where e is the identity element of H.
(2) $\delta(h_2, \delta(h_1, y)) = \delta(h_2h_1, y)$, for all $h_1, h_2 \in H$ and $y \in Y$.

The space Y, along with a given action δ of H, is called a H-space.

Definition 2.4. [16] A set is said to be a G_δ set if it is a countable intersection of open sets.

Definition 2.5. [24] Let U represent the initial universe and E denote the parameter set. A soft set over U is defined as the function $\mathcal{F} : E \to \mathcal{P}(U)$.

Definition 2.6. [10] Let $\mathcal{F}$ and $\mathcal{F}'$ be soft sets over U. Then
(1) $\mathcal{F}$ is a soft subset of $\mathcal{F}'$, denoted by $\mathcal{F} \tilde{\subset} \mathcal{F}'$, if $\mathcal{F}(e) \subset \mathcal{F}'(e)$ for all $e \in E$.
(2) $\mathcal{F}$ is soft equal to $\mathcal{F}'$, denoted by $\mathcal{F} \tilde{=} \mathcal{F}'$, if both

$\mathcal{F} \mathbin{\widetilde{\subset}} \mathcal{F}'$ and $\mathcal{F}' \mathbin{\widetilde{\subset}} \mathcal{F}$ hold.

The use of $\widetilde{(\cdot)}$ is to distinguish soft objects from usual ones.

Definition 2.7. [10] Let $\mathcal{F}$ and $\mathcal{F}'$ be soft sets over U. Then
(1) The soft intersection of $\mathcal{F}$ and $\mathcal{F}'$, denoted by $\mathcal{F} \mathbin{\widetilde{\cap}} \mathcal{F}'$, is defined by $(\mathcal{F} \mathbin{\widetilde{\cap}} \mathcal{F}')(e) = \mathcal{F}(e) \cap \mathcal{F}'(e)$ for every $e \in E$.
(2) The soft union of $\mathcal{F}$ and $\mathcal{F}'$, denoted by $\mathcal{F} \mathbin{\widetilde{\cup}} \mathcal{F}'$, is defined by $(\mathcal{F} \mathbin{\widetilde{\cup}} \mathcal{F}')(e) = \mathcal{F}(e) \cup \mathcal{F}'(e)$ for every $e \in E$.

Definition 2.8. [31] A soft topology on U is defined by a family $\mathcal{T}$ of soft sets that satisfy the following conditions,
(1) $\widetilde{\emptyset} \in \mathcal{T}$ and $\widetilde{U} \in \mathcal{T}$,
(2) $\mathcal{T}$ is closed under finite soft intersection,
(3) $\mathcal{T}$ is closed under (arbitrary) soft union.
Then a soft topological space is denoted by the triplet $(U, \mathcal{T}, E)$. Every element in $\mathcal{T}$ is referred to as a soft open set.

Definition 2.9. [17] Consider the element x in the universe U. A soft set $\mathcal{F}$ is called a soft neighborhood of x if there is a soft open set $\mathcal{F}'$ such that $x \mathbin{\widetilde{\in}} \mathcal{F}' \mathbin{\widetilde{\subset}} \mathcal{F}$.

Definition 2.10. [17] $\varphi : U \to U'$ is called a soft continuous function from $(U, \mathcal{T}, E)$ to $(U', \mathcal{T}', E)$ if for every $x \in U$ and for every soft neighborhood $\mathcal{F}'$ of $\varphi(x)$, there exists a soft neighborhood $\mathcal{F}$ of x such that $\varphi(F) \mathbin{\widetilde{\subset}} \mathcal{F}'$.

Definition 2.11. [17] A bijection $\varphi : U \to U'$ is called a soft homeomorphism between $(U, \mathcal{T}, E)$ and $(U', \mathcal{T}', E)$ if both φ and φ^{-1} are soft continuous.

Definition 2.12. [18] Let U be an initial universe set, E be the set of parameters and let $\mathcal{T}$ be the collection of all soft sets which can be defined over U. Then $\mathcal{T}$ is called the soft discrete topology on U and $(U, \mathcal{T}, E)$ be a soft discrete space over U.

Definition 2.13. [31] A soft topological space $(U, \mathcal{T}, E)$ is called a soft Hausdorff space, if for every pair of distinct points $x, x' \in U$, there exists soft open sets $\mathcal{F}, \mathcal{F}' \mathbin{\widetilde{\in}} \mathcal{T}$ with $x \mathbin{\widetilde{\in}} \mathcal{F}$ and $x' \mathbin{\widetilde{\in}} F'$ and $\mathcal{F} \mathbin{\widetilde{\cap}} \mathcal{F}' = \widetilde{\emptyset}$.

Definition 2.14. [17] For any soft topological spaces $(U, \mathcal{T}, E)$ and $(U', \mathcal{T}', E)$, the set $\{F \times F' | F \in \mathcal{T}, F' \in \mathcal{T}'\}$ generates a soft topology $\mathcal{T}_\times$ on $U \times U'$. The soft space $(U \times U', \mathcal{T}_\times, E)$ is called the soft product of $(U, \mathcal{T}, E)$ and $(U', \mathcal{T}', E)$.

Definition 2.15. [34] The soft product of any two soft Hausdorff spaces is also a soft Hausdorff space.

Definition 2.16. [17] A family C of soft open sets over U is called soft open cover of U if for every $x \in U$ there is $\mathcal{F} \in C$ such that $x \mathbin{\widetilde{\in}} \mathcal{F}$.

Definition 2.17. [17] A soft topological space $(U, \mathcal{T}, E)$ is called a soft compact space, if for every soft open cover C of $(U, \mathcal{T}, E)$, there is $\mathcal{F}_1, ..., \mathcal{F}_n$ in C such that $\{\mathcal{F}_1, ..., \mathcal{F}_n\}$ is a soft open cover.

Definition 2.18. [17] Let H be a group and $\mathcal{T}$ a soft topology on H with a parameter set E. Then the triplet $(H, \mathcal{T}, E)$ is called a soft topological group if the following conditions are satisfied,
(1) For every $(h, y) \in H \times Y$ and a soft neighborhood $\mathcal{F}$ of $h.y \in Y$ there exists soft neighborhoods $\mathcal{F}_h$ and $\mathcal{F}_y$ of h and y such that $\mathcal{F}_h \cdot \mathcal{F}_y \mathbin{\widetilde{\subset}} \mathcal{F}$.
(2) $\beta : H \to H$ is soft continuous.

III. Soft Topological Transformation Groups

Soft topological transformation group is defined and some of its basic properties are studied. This section aims to obtain a soft topological group structure on the set of all soft homeomorphisms on Y.

Definition 3.1. A map $\delta : H \times Y \to Y$ is soft*continuous if for every $(h, y) \in H \times Y$ and a soft neighborhood $\mathcal{F}$ of $h \cdot y \in Y$ there exists soft neighborhoods $\mathcal{F}_h$ and $\mathcal{F}_y$ of h and y such that $\mathcal{F}_h \cdot \mathcal{F}_y \mathbin{\widetilde{\subset}} \mathcal{F}$.

Definition 3.2. (H, Y, δ, E) is called a soft topological transformation group if $(H, \mathcal{T}, E)$ is a soft topological group, $(Y, \mathcal{T}, E)$ is a soft topological space and the map $\delta : H \times Y \to Y$ is soft*continuous such that it satisfies the following conditions,
(1) $\delta(e, y) = y$, for every $y \in Y$, where e is the identity element of H.
(2) $\delta(h_2, \delta(h_1, y)) = \delta(h_2 h_1, y)$, for every $h_1, h_2 \in H$ and $y \in Y$.

Example 3.1. Let $H = \{e, a = (12)(34), b = (13)(24), c = (14)(23)\}$, $E_1 = \{e_{11}, e_{12}\}$ and $\mathcal{T}_H = \{\emptyset, \{e_{11}, a\}, E_1 \times H\}$ be a topology of H, $(H, \mathcal{T}, E)$ is a soft topological group and $Y = \{1, 2, 3, 4\}$, $E_2 = \{e_{21}, e_{22}\}$ and $\mathcal{T}_Y = \{\emptyset, \{e_{22}, 1\}, E_2 \times Y\}$ be a topology of Y, $(Y, \mathcal{T}, E)$ is a soft topological space. Then $\delta : H \times Y \to Y$ forms a soft topological transformation group.

Theorem 3.1. For any soft topological transformation group $(H, Y, \delta, E), h \in H$, a map $\delta_h : Y \to Y$ be defined by $\delta_h(y) = \delta(h, y)$ is a soft homeomorphism of Y, such that the following diagram commutes,

$$\begin{tikzcd}
H \times Y \arrow[r, "\delta"] & Y \\
Y \arrow[u, "i_h"] \arrow[ur, "\delta_h"'] &
\end{tikzcd}$$

Fig. 1. Commutative Diagram of δ_h

Proof: Given $h \in H$, let $i_h : Y \to H \times Y$ be the map defined by $i_h(y) = (h, y)$ and $\delta_h = \delta \circ i_h$. Let $\delta_h : Y \to Y$ be given by $\delta_h(y) = \delta(h, y) = hy$. Now we show that δ_h is bijective.

1) The injectivity condition : Let $y_1, y_2 \in Y$,

$$\begin{aligned} \delta_h(y_1) &= \delta_h(y_2) \\ hy_1 &= hy_2 \\ h^{-1}(hy_1) &= h^{-1}(hy_2) \\ (h^{-1}h)y_1 &= (h^{-1}h)y_2 \\ ey_1 &= ey_2 \\ y_1 &= y_2. \end{aligned}$$

Therefore δ_h is injective.

2) The surjectivity condition : Let $y_2 \in Y$,

$$\begin{aligned} y_2 &= ey_2 \\ &= (hh^{-1})y_2 \\ &= h(h^{-1}xy_2) \\ &= hy_1 \\ &= \delta_h(y_1) \end{aligned}$$

Therefore, δ_h is surjective. Hence δ_h is bijective and inverse exists, $(\delta_h)^{-1} = \delta_{h^{-1}}$.

Now, the commutativity of the diagram is given by,

$$\begin{aligned} \delta \circ i_h(y) &= \delta(i(h, y)) \\ &= \delta(h, y) \\ &= \delta_h(y) \end{aligned}$$

Since δ is soft*continuous, for every $(h, y) \in H \times Y$ and a soft neighborhood $\mathcal{F}$ of $h \cdot y \in Y$, there exists soft neighborhoods $\mathcal{F}_h$ and $\mathcal{F}_y$ of h and y respectively such that $\mathcal{F}_h \cdot \mathcal{F}_y \tilde{\subset} \mathcal{F}$. Since $i_h(y) = (h, y), i_{h^{-1}}(h, y) \in \mathcal{T}_y$ in Y. Thus δ_h is soft continuous. Since, for every $h^{-1} \in H$ and $y \in Y$, $(\delta_h)^{-1} = \delta_{h^{-1}}$, the inverse soft continuous of δ_h follows. Thus δ_h is soft homeomorphism.

■

Note 3.1. For a soft topological space $(Y, \mathcal{T}, E)$ and a mapping space Map(Y, Y), the collection of all soft homeomorphisms of Y onto itself, denoted by $SFhomeo_H(Y)$ forms a group under composition. Now, for a subset $F \subset$ Map(Y, Y), let $W(SC, SO) = \left\{\delta \in F \middle| \delta(SC) \subset SO\right\}$ where SC and SO are the given soft compact subset and soft open subset of Y. The soft compact-soft open topology on $\mathcal{F}$ has a sub-basis, $W(SC, SO)$ for a given soft compact $SC \subset Y$ and a soft open $SO \subset Y$. It is possible to have a mapping space with the soft compact-soft open topology.

Proposition 3.1. Let $\zeta : H \to SFhomeo_H(Y)$ defined by $h \mapsto \delta_h$. Then H is homomorphic to $SFhomeo_H(Y)$.

Proof: For any $h_1, h_2 \in H$, we have

$$\begin{aligned} \zeta(h_1h_2) &= \zeta(h_1h_2)(y) \\ &= \delta_{h_1h_2}(y) \\ &= \delta \circ i_{h_1h_2}(y) \\ &= \delta(h_1h_2, y) \\ &= \delta(h_1, \delta(h_2, y)) \\ &= \delta(h_1, \delta_{h_2})(y) \\ &= \delta(h_1, \delta \circ i_{h_2})(y) \\ &= \delta_{h_1}(\delta \circ i_{h_2})(y) \\ &= (\delta_{h_1} \circ \delta_{h_2})(y) \\ &= \zeta(h_1) \circ \zeta(h_2). \end{aligned}$$

Therefore H is homomorphic to $SFhomeo_H(Y)$. ■

Proposition 3.2. Given a homomorphism $\zeta : H \to SFhomeo_H(Y)$ defined by $\zeta(h)(y) = \delta'(h, y)$, where $\delta' : H \times Y \to Y$. Then δ' is a H-action.

Proof: Let $e \in H$ and $y \in Y$, we have

$$\begin{aligned} \delta'(e, y) &= \zeta(e)(y) \\ &= \delta_e(y) \\ &= \delta(e, y) \\ &= y. \end{aligned}$$

For any $h_1, h_2 \in H$ and $y \in Y$, we have,

$$\begin{aligned} \delta'(h_2, h_1y) &= \zeta(h_2)(h_1y) \\ &= \delta_{h_2}(h_1, y) \\ &= \delta(h_2, h_1y) \\ &= h_2h_1y \\ \delta'(h_2h_1, y) &= \zeta(h_2h_1)(y) \\ &= \delta_{h_2h_1}(y) \\ &= \delta(h_2h_1, y) \\ &= h_2h_1y \\ \Rightarrow \delta'(h_2, h_1y) &= \delta'(h_2h_1, y). \end{aligned}$$

■

Theorem 3.2. Let F be a subset of Map (Y, Y) where Y is a soft compact topological space. For a map ζ from a soft topological group H into F, we define a map $\delta' : H \times Y \to Y$ by $\delta'(h, y) = \zeta(h)(y)$. If δ' is soft*continuous then ζ is soft continuous.

Proof: Given a soft*continuous function δ', for every $(h, y) \in H \times Y$ and a soft neighborhood $\mathcal{F}$ of $h \cdot y \in Y$ there exists soft neighborhoods $\mathcal{F}_h$ and $\mathcal{F}_y$ of h and y such that $\mathcal{F}_h \cdot \mathcal{F}_y \tilde{\subset} \mathcal{F}$. Let h be any arbitrary point of $\zeta^{-1}\Big(W(SC, SO)\Big)$. For $y \in SC$, $\delta'(h, y) = \zeta(h)(y) \in SO$. Since δ' is soft*continuous, there exists soft neighborhoods $\mathcal{F}_h$ and $\mathcal{F}'_y$ of h and y such that $\delta'(\mathcal{F}_h \times \mathcal{F}'_y) \tilde{\subset} SO$. Now, for each $y \in SC$, $\bigcup_{y \in SC} \mathcal{F}_y \tilde{\supset} SC$ is trivial. Since Y is soft compact, there exists a finite soft open sets $\{y_1, ..., y_n\} \subset SC$ with $\bigcup_{r=1}^{n} \mathcal{F}_{y_r} \tilde{\supset} SC$. Put, $\mathcal{F}_h = \bigcap_{r=1}^{n} \mathcal{F}_{y_r}$ and $\mathcal{F}_y = \bigcup_{r=1}^{n} \mathcal{F}'_{y_r}$. Then $\mathcal{F}$ is a soft neighborhood of h and $\mathcal{F}'$ is a soft neighborhood of y containing SC. Now, for any point $(h', y') \in \mathcal{F} \times \mathcal{F}'$, Choose r with $y' \in \mathcal{F}'_{y_r}$. Since $h' \in$

$\mathcal{F} \mathrel{\widetilde{\subset}} \mathcal{F}_{y_r}, \delta'(h', y') \in \delta'(\mathcal{F}_h \times \mathcal{F}_y') \mathrel{\widetilde{\subset}} SO$. Hence $\delta'(\mathcal{F}_y \times \mathcal{F}_y') \mathrel{\widetilde{\subset}} SO$ and $\zeta(\mathcal{F}_y) \mathrel{\widetilde{\subset}} W(SC, SO)$. Therefore, ζ is soft continuous. ■

Theorem 3.3. $SFhomeo_H(Y)$ is a soft topological group.

Proof: Since $SFhomeo_H(Y)$ is a group and a soft topological space, it is enough to prove the maps, $\alpha : SFhomeo_H(Y) \times SFhomeo_H(Y) \rightarrow SFhomeo_H(Y)$ defined by $\alpha(\delta_{h_1}, \delta_{h_2}) = \alpha(\delta_{h_1} \circ \delta_{h_2}) = \delta_{h_1 h_2}$ is soft*continuous and $\beta : SFhomeo_H(Y) \rightarrow SFhomeo_H(Y)$ defined by $\beta(\delta_h) = \delta_h{}^{-1}$ is soft continuous. Now, for every $(\delta_{h_1}, \delta_{h_2}) \in SFhomeo_H(Y) \times SFhomeo_H(Y)$ and soft neighborhood $W(SC, SO)$ containing $\delta_{h_1} \circ \delta_{h_2} \in SFhomeo_H(Y)$, $\delta_{h_1} \circ \delta_{h_2}(SC) \subset SO$, where SC is a soft compact set and SO is a soft open set, there exists soft neighborhoods $W(SC_1, SO_1)$ and $W(SC_2, SO_2)$ containing δ_{h_1} and δ_{h_2} such that $W(SC_1, SO_1) \circ W(SC_2, SO_2) \mathrel{\widetilde{\subset}} W(SC, SO)$. Thus α is soft* continuous.

Also, for every $\delta_h \in SFhomeo_H(Y)$ and a soft neighborhood $W(SC, SO)$ containing $\beta(\delta_h) \in SFhomeo_H(Y)$, there exists soft neighborhood $W(SC_1, SO_1)$ containing δ_h such that $\delta_h(SC_1) \subset SO_1$, where SC_1 is a soft compact set and SO_1 is a soft open set such that $\beta(W(SC_1, SO_1)) \mathrel{\widetilde{\subset}} W(SC, SO)$. Thus β is soft continuous. ■

Corollary 3.1. Any subgroup of $SFhomeo_H(Y)$ is a soft topological group.

Proof: Let $SFhomeo_K(Y)$ be a subgroup of $SFhomeo_H(Y)$. Then $SFhomeo_K(Y)$ forms a group and a soft topological space. Now, by Theorem 3.3, $\alpha^* : SFhomeo_K(Y) \times SFhomeo_K(Y) \rightarrow SFhomeo_K(Y)$ defined by $\alpha^*(\delta_{h_1}, \delta_{h_2}) = \delta_{h_1 h_2}$ is soft*continuous and $\beta^* : SFhomeo_K(Y) \rightarrow SFhomeo_K(Y)$ defined by $\beta^*(\delta_h) = \delta_h{}^{-1}$ is soft continuous. Thus $SFhomeo_K(Y)$ forms a soft topological group. ■

Proposition 3.3. If Y is soft Hausdorff, then the Map (Y, Y) is soft Hausdorff and hence any subspace of Map (Y, Y) is soft Hausdorff.

Proof: Let $\delta_{h_1}, \delta_{h_2} \in$ Map (Y, Y). Then there exists y of Y with $\delta_{h_1}(y) \neq \delta_{h_2}(y)$. Since Y is Hausdorff, there are soft open sets SO_1 and SO_2 of Y such that $\delta_{h_1}(y) \in SO_1, \delta_{h_2}(y) \in SO_2, SO_1 \mathrel{\widetilde{\cap}} SO_2 = \widetilde{\emptyset}$ and $W\Big(\{y\}, SO_1\Big), W\Big(\{y\}, SO_2\Big)$ are soft open sets of Map (Y, Y) containing δ_{h_1} and δ_{h_2} respectively. Clearly $W\Big(\{y\}, SO_1\Big) \mathrel{\widetilde{\cap}} W\Big(\{y\}, SO_2\Big) = \widetilde{\emptyset}$, which implies that Map (Y, Y) is soft Hausdorff. ■

Corollary 3.2. $SFhomeo_H(Y)$ is soft Hausdorff.

Proof: Since $SFhomeo_H(Y) \subset$ Map(Y, Y) and by Proposition 3.3, $SFhomeo_H(Y)$ is soft Hausdorff. ■

Corollary 3.3. $SFhomeo_H(Y) \times SFhomeo_H(Y)$ is soft Hausdorff.

Proof: Since the soft product of any two soft Hausdorff spaces is also a soft Hausdorff space. Then, by Corollary 3.2 $SFhomeo_H(Y)$ is soft Hausdorff and hence $SFhomeo_H(Y) \times SFhomeo_H(Y)$ is soft Hausdorff. ■

Proposition 3.4. For a finite soft topological group H of order m and for all $h_1 \in H$, $\delta_{h_1} \in SFhomeo_H(Y)$. Then there exist atleast one soft compact set SC and a soft open set SO such that $W(SC, SO) = \{\delta_{h_1}\}$.

Proof: Since δ_{h_1} is soft homeomorphism, for all $h_1 \in H, \delta_{h_1} \in SFhomeo_H(Y)$. Then there exists a soft compact set SC_1 and a soft open set SO_1 such that $\delta_{h_1} \in W(SC_1, SO_1)$. Assume that $W(SC_1, SO_1) = \{\delta_{h_1}, \delta_{h_2}, ..., \delta_{h_j}\}$. Suppose $\delta_{h_1} \neq \delta_{h_2}$, then there exists a soft compact set SC_2 and a soft open set SO_2 such that $\delta_{h_1} \in W(SC_2, SO_2)$, $\delta_{h_2} \notin W(SC_2, SO_2)$ and this implies $\delta_{h_1} \in W(SC_1, SO_1) \cap W(SC_2, SO_2)$. Next, if $\delta_{h_3} \in W(SC_1, SO_1) \cap W(SC_2, SO_2)$ and $\delta_{h_1} \neq \delta_{h_3}$, then there exists a soft compact set SC_3 and a soft open set SO_3 such that $\delta_{h_1} \in W(SC_3, SO_3)$, $\delta_{h_3} \notin W(SC_3, SO_3)$ and so, $\delta_{h_1} \in W(SC_1, SO_1) \cap W(SC_2, SO_2) \cap W(SC_3, SO_3)$. Proceeding in the same way, we have $W(SC_1, SO_1) \cap W(SC_2, SO_2) \cap W(SC_3, SO_3) \cap ... \cap W(SC_j, SO_j) = \{\delta_{h_1}\}$. ■

Theorem 3.4. $SFhomeo_H(Y)$ is a soft discrete space.

Proof: By Proposition 3.4, we have $W(SC, SO) = \{\delta_{h_1}\}$, for all $h_1 \in H$ and hence $SFhomeo_H(Y)$ is a soft discrete space. ■

Definition 3.3. A soft topological space $(U, \mathcal{T}, E)$ is called soft extremally disconnected, if the soft closure of a soft open set is soft open.

Corollary 3.4. $SFhomeo_H(Y)$ is soft extremally disconnected.

Proof: Since every soft discrete space is soft extremally disconnected, $SFhomeo_H(Y)$ is soft extremally disconnected. ■

Definition 3.4. A soft set is said to be a soft G_δ set if it is a countable intersection of soft open sets.

Definition 3.5. A soft Hausdorff space $(U, \mathcal{T}, E)$ is called soft Moscow space, if for each soft open subset $\mathcal{F}$ of $(U, \mathcal{T}, E)$, the soft closure of $\mathcal{F}'$ in $(U, \mathcal{T}, E)$ is the soft union of a family of soft G_δ-subset of $(U, \mathcal{T}, E)$.

Corollary 3.5. $SFhomeo_H(Y)$ is a soft Moscow space.

Proof: Since every soft extremally disconnected space is soft Moscow space, $SFhomeo_H(Y)$ is a soft Moscow space. ■

Corollary 3.6. $SFhomeo_H(Y)$ is a soft Moscow topological group.

Proof: Since $SFhomeo_H(Y)$ is a soft Moscow space and a soft topological group, $SFhomeo_H(Y)$ is a soft Moscow topological group. ■

Theorem 3.5. Let $\Gamma : H \rightarrow H'$ be a map given by $\Gamma(h) = h'$ is a soft isomorphism of soft topological group. Then $\Phi' : SFhomeo_H(Y) \rightarrow SFhomeo_{H'}(Y)$ given by $\Phi'(\delta_h) = \delta_{\Gamma(h)}$ is also a soft isomorphism of soft topological group.

Proof: To prove Φ' is a bijective homomorphism and a soft homeomorphism.

1) The injectivity condition : Let δ_h and $\delta_k \in$

$SFhomeo_H(Y)$.

$$\begin{aligned}\Phi'(\delta_h) &= \Phi'(\delta_k)\\ \delta_{\Gamma(h)} &= \delta_{\Gamma(k)}\\ \delta(\Gamma(h),x) &= \delta(\Gamma(k),x)\\ \Gamma(h)x &= \Gamma(k)x\\ \delta_h &= \delta_k\end{aligned}$$

Therefore Φ' is injective.

2) The surjectivity condition : For every $\delta_{h'_i} \in SFhomeo_{H'}(Y)$, there exists $\delta_{h_i} \in SFhomeo_H(Y)$ such that $\Phi'(\delta h_i) = \delta_{\Gamma(h_i)} = \delta_{h'_i}$. Therefore, Φ' is surjective.

3) Homomorphism: Let δ_h and $\delta_k \in SFhomeo_H(Y)$

$$\begin{aligned}\Phi'(\delta_h \circ \delta_k) &= \Phi'(\delta_h \circ \delta_k)(y)\\ &= \delta_{\Gamma(hk)}(y)\\ &= \delta_{\Gamma(h)\Gamma(k)}(y)\\ &= \delta_{h'k'}(x)\\ &= \Phi'(\delta_h) \circ \Phi'(\delta_k)\end{aligned}$$

Therefore Φ' is a homomorphism.

Now, let $\Phi' : SFhomeo_H(Y) \to SFhomeo_{H'}(Y)$ defined by $\Phi'(\delta_h) = \delta_{\Gamma(h)}$. For every $\delta_h \in SFhomeo_H(Y), W(SC,SO)$ be a soft neighborhood containing $\Phi'(\delta_h) \in SFhomeo_{H'}(Y)$. Then there exists soft neighborhood $W(SC,SO)$ containing $\delta_h(SC_1) \subset SO_1$, where SC_1 is a soft compact set and SO_1 is a soft open set such that $\Phi'\Big(W(SC_1,SO_1)\Big) \widetilde{\subset} W(SC,SO)$. Thus Φ' is soft continuous. Also, let $\eta : SFhomeo_{H'}(Y) \to SFhomeo_{H'}(Y)$ given by $\eta(\delta_{h'}) = \delta_{\Gamma(h')}$. Now, for every $\delta_{h'} \in SFhomeo_{H'}(Y), W(SC,SO)$ be a soft neighborhood containing $\eta(\delta_{h'}) \in SFhomeo_H(Y)$. Then there exists soft neighborhood $W(SC',SO')$ containing $\delta_{h'}$ such that $\delta_{h'}(SC') \subset SO'$, where where SC' is a soft compact set and SO' is a soft open set such that $\eta\Big(W(SC',SO')\Big) \widetilde{\subset} W(SC,SO)$. Thus η is soft continuous. ■

IV. Conclusion

A new structure called soft topological transformation group is defined. Basic algebraic and topological properties of soft topological transformation group have been discussed. It is proved that there is a homomorphism between a soft topological group and the collection of all soft homeomorphisms of the given soft topological space. Also, it is established that the set of all soft homeomorphisms involved in soft topological transformation groups acts as a group under composition as well as a soft topological group. Later, it is proved that the map from a soft topological group to a mapping space is soft continuous. Additionally, it is shown that the mapping space $\mathrm{Map}(Y,Y)$ is soft Hausdorff, for a given soft Hausdorff space Y and it is proved that the set of all soft homeomorphisms on Y is a soft Hausdorff and also verified that $SFhomeo_H(Y) \times SFhomeo_H(Y)$ is a soft Hausdorff. Later, the set of all soft homeomorphisms on Y creates a soft discrete space, soft extremally disconnected space, soft Moscow space and a soft Moscow topological group. Finally, it is shown that there is a soft isomorphism between two distinct collection of soft homeomorphisms. In a future work, we explore some other properties of soft topological transformation groups.

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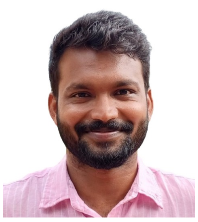

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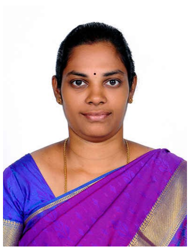

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