Impulse Synchronization Strategy of Nonlinear Finance Systems

Yang Peng, Yuming Feng

***Abstract*—In recent decades, impulse control has been increasingly applied as a discontinuous control method across diverse domains such as satellite orbit transfers, financial market regulation, chaos synchronization, and communication security. This paper investigates an impulse synchronization strategy specifically tailored for chaotic financial systems. We propose a novel strategy grounded in Lyapunov stability theory and the maximum Lyapunov exponent, designed to synchronize chaotic financial systems effectively. Simulation experiments are conducted to validate the efficacy of our proposed approach.**

***Index Terms*—impulse synchronization, finance chaotic systems, maximum Lyapunov exponent**

I. Introduction

THE financial system encompasses institutions such as banks, insurance companies, and stock exchanges, which facilitate the flow of funds. The interaction among price dynamics, supply and demand dynamics, and interest rates introduces complexity into various financial phenomena. For example, financial markets frequently encounter stagnation, loss of equilibrium, and occasionally trigger financial crises.

In 1985, the identification of chaotic behavior within financial systems markedly influenced subsequent research in the field of finance [3]. This phenomenon demonstrates that macroeconomic dynamics are intrinsically unstable. It is imperative to comprehend the internal structure, elucidate the developmental trajectories, and regulate the chaotic states within these complex financial systems.

Ma and Chen[10] introduced a type of nonlinear financial system by choosing the right coordinate system and allocating appropriate dimensions to each state variable as described below:

$$\begin{cases} \dot{x}_1 = (1/b - a)\,x_1 + x_3 + x_1 x_2 \\ \dot{x}_2 = -b x_2 - x_1^2 \\ \dot{x}_3 = -x_1 - c x_3 \end{cases}, \quad (1)$$

where x_1, x_2, x_3 represent the interest rate, investment demand, and price exponent, respectively. The parameters a, b, c are the amount of savings, unit investment cost, and the elasticity of commodity demand, respectively, where a, b, c are greater than zero. They analyzed various possible situations in the operation of China's macro financial system reflected by this model, including balance, stable periods, fractals, Hopf bifurcation, the relationship between parameters and Hopf bifurcation, and even chaotic movement. They also studied the changes in various parameters in the model through theoretical analysis and numerical simulation calculations. Subsequently, they used this analysis to examine the conditions for complex local behavior in such financial systems, as well as to evaluate the adjustments of macroeconomic policies and the impacts of changes in certain parameters on the behavior of the entire financial system.

System (1) is chaotic for some parameters, such as $a = 0.8, b = 0.2, c = 1.9$. The maximum Lyapunov exponent of system (1) is $L = 3.6000$. Fig 1, Fig 2, Fig 3, Fig 4 and Fig 5 show the chaotic phenomenon of system (1) with the initial condition $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$.

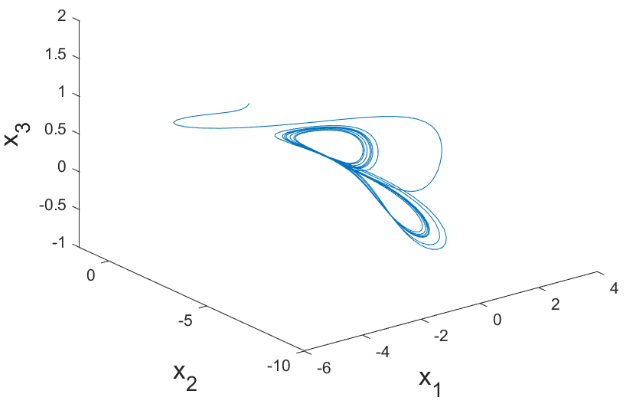

Fig. 1. 3D phase portraits of system (1) with parameters $a = 0.8, b = 0.2, c = 1.9$ and the initial condition $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$.

Recently, many authors have studied the dynamic behavior of system (1). Liao, Li, and Zhou [7] discussed the influence of policy lag on system (1), investigating how delays in policy response affect macroeconomic stability. Cai and Huang [1] discussed the dynamical behavior and slow manifold of system (1), including aspects such as symmetry, dissipation, and equilibrium points. Ma and Yang [9], in their study, they highlighted that the system(1) differs topologically from several well-known chaotic systems like the Lorenz system family, Rössler system, and Chua's system. Additionally, they examined the Hopf bifurcation and topological horseshoe of the system(1).Tirandaz, Aminabadi, and Tavakoli [12] presented an adaptive linear feedback controller capable of achieving synchronization of the chaotic financial system (1). Wu and Xia [13] developed a chaotic

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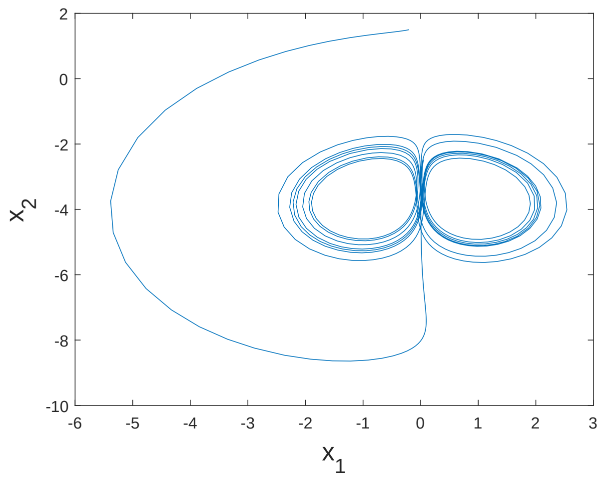

Fig. 2. Chaotic attractors of the finance chaotic (1) with parameters $a = 0.8, b = 0.2, c = 1.9$ and the initial condition $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$.

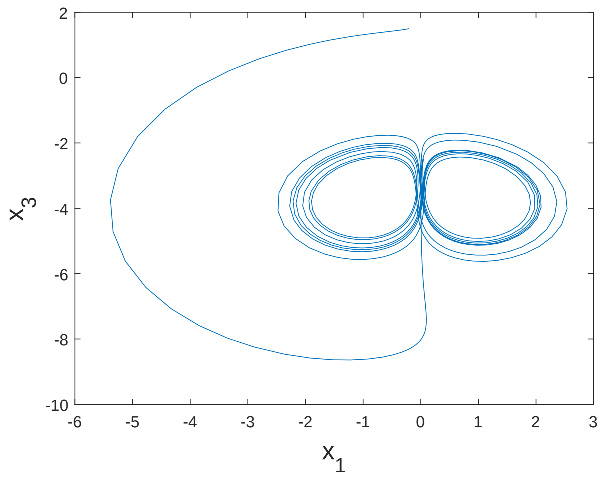

Fig. 3. Chaotic attractors of the finance chaotic system(1) with parameters $a = 0.8, b = 0.2, c = 1.9$ and the initial condition $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$.

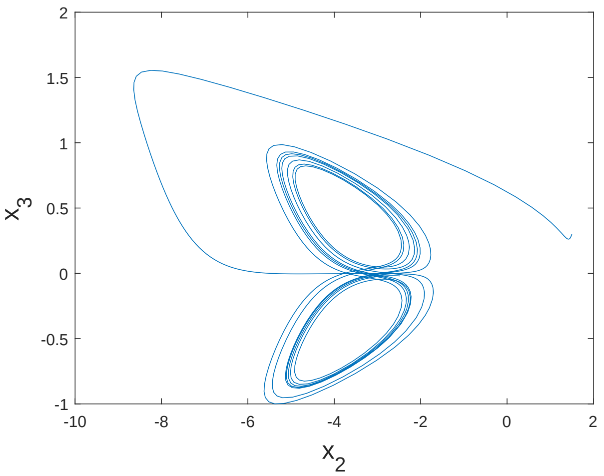

Fig. 4. Chaotic attractors of the finance chaotic system (1) with parameters $a = 0.8, b = 0.2, c = 1.9$ and the initial condition $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$.

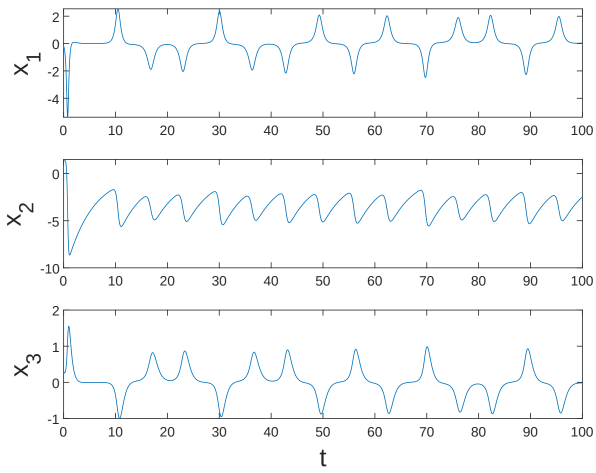

Fig. 5. Time series of system (1) with parameters $a = 0.8, b = 0.2, c = 1.9$ and the initial condition $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$.

financial system and introduced an innovative sliding mode controller, following their analysis of the laws that oversee macroeconomic activities.This controller is designed to alter the dynamics of the original economic system and establish a new spatiotemporal organizational pattern.This controller can help transition the financial system from a state of chaos to one of greater regularity. For additional details on synchronization, please see the cited references[8], [16].

In certain applications, providing continuous control is not feasible, and only impulse control is available as an option.For example, it is unfeasible for the government to adjust the saving rate of the central bank every day [15]. Hence, in this study, due to the practicality of the control strategy, we utilize the impulse control method to synchronize the system(1). For additional details on impulse control methods and their benefits, the reader is referred the cited references[2], [4], [5], [6], [11], [14], [17], [18].

II. Impulse Synchronization of Finance Chaotic Systems

Separating the linear and nonlinear elements of the system in (1), we can reformulate it as follows:

$$\dot{X} = AX + \varphi(X), \tag{2}$$

where

$$A = \begin{pmatrix} 1/b - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{pmatrix}$$

and

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \varphi(X) = \begin{pmatrix} x_1 x_2 \\ -x_1^2 \\ 0 \end{pmatrix}.$$

We now examine impulse synchronization between two chaotic financial systems of identical structure. System ((2)) is driving system, and the corresponding driven system is described as follows:

$$\begin{cases} \dfrac{dY}{dt} = AY + \varphi(Y), t \neq \tau_k, \\ \Delta Y = Be, t = \tau_k, k = 1, 2, \cdots \end{cases} \tag{3}$$

where $Y = (y_1, y_2, y_3)^T$ and B is impulse control gain matrix. The symbol e denotes system errors and $e = (e_1, e_2, e_3)^T = (y_1 - x_1, y_2 - x_2, y_3 - x_3)^T$. There will be a jump in system (3) when $t = \tau_k$ and $t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots, \lim_{k\to\infty} \tau_k = \infty$. Subtracting system (2) from system (3),

$$\begin{cases} \dfrac{de}{dt} = Ae + \psi(X, Y), t \neq \tau_k, \\ \Delta e = Be, t = \tau_k, k = 1, 2, \cdots, \end{cases} \tag{4}$$

where $\psi(X, Y) = \varphi(Y) - \varphi(X)$.

In the following theorem, we present a sufficient condition for the stability of error system (4).

Theorem 2.1 Let L be the maximum Lyapunov exponent of system (2) and suppose that λ is the the largest eigenvalues $\left(I + B^T\right)(I + B)$. If $\log\lambda + 2L(\tau_k - \tau_{k-1}) < 0$ and $\tau_k - \tau_{k-1} < 1/L$ hold for all $k = 1, 2, \cdots$, then the origin of impulse synchronization error system (4) is asymptotically stable.

Proof. It is generally considered that the maximum predictable time for chaotic motion is $1/L$. So, impulse intervals $\tau_k - \tau_{k-1}, k = 1, 2, \cdots$ should satisfy $\tau_k - \tau_{k-1} < 1/L$.

By the definition of the maximum Lyapunov exponent, we know that if system (4) is not controlled, then after a short period of time, we have

$$\|e(t)\| \leq \|e(t_0)\| e^{L(t-t_0)}.$$

Let

$$V(e(t)) = e^T(t) e(t).$$

For $t \in [t_0, \tau_1)$, we obtain

$$\begin{array}{rcl} V(e(t)) & = & \|e(t)\|^2 \leq \|e(t_0)\|^2 e^{2L(t-t_0)} \\ & = & V(e(t_0)) e^{2L(t-t_0)}. \end{array} \tag{5}$$

When $t = \tau_1$, by inequality (5), we have

$$\begin{array}{rcl} V(e(\tau_1)) & = & \left((I + B) e\left(\tau_1^-\right)\right)^T (I + B) e\left(\tau_1^-\right) \\ & = & e^T\left(\tau_1^-\right)\left(I + B^T\right)(I + B) e\left(\tau_1^-\right) \\ & \leq & \lambda e^T\left(\tau_1^-\right) e\left(\tau_1^-\right) \\ & = & \lambda V\left(e\left(\tau_1^-\right)\right) \\ & \leq & \lambda V(e(t_0)) e^{2L(\tau_1 - t_0)}. \end{array} \tag{6}$$

If $t \in [\tau_1, \tau_2)$, we also obtain

$$V(e(t)) \leq V(e(\tau_1)) e^{2L(t-\tau_1)}. \tag{7}$$

It follows from (6) and (7) that

$$V(e(t)) \leq \lambda V(e(t_0)) e^{2L(t-t_0)}. \tag{8}$$

When $t = \tau_2$, using the same method employed to derive inequalities (6) and (8), we obtain

$$V(e(\tau_2)) \leq \lambda V\left(e\left(\tau_2^-\right)\right) \leq \lambda^2 V(e(t_0)) e^{2L(\tau_2 - t_0)}$$

and so

$$\begin{array}{rcl} V(e(t)) & \leq & V(e(\tau_2)) e^{2L(t-\tau_2)} \\ & \leq & \lambda^2 V(e(t_0)) e^{2L(\tau_2 - t_0)} e^{2L(t-\tau_2)} \\ & = & \lambda^2 V(e(t_0)) e^{2L(t-t_0)} \end{array} \tag{9}$$

for $t \in [\tau_2, \tau_3)$.

Therefore, by the repeatability of the proof process, we know that if $t \in [\tau_{k-1}, \tau_k)$, then

$$\begin{array}{rcl} V(e(t)) & \leq & \lambda^{k-1} V(e(t_0)) e^{2L(t-t_0)} \\ & = & V(e(t_0)) e^{2L(t-t_0)} e^{(k-1)\log\lambda} \\ & = & V(e(t_0)) e^{\log\lambda + 2L(\tau_1 - t_0)} e^{\log\lambda + 2L(\tau_2 - \tau_1)} \\ & \times & \cdots e^{2L(t-\tau_{k-1})} \\ & = & \lambda^{-1} V(e(t_0)) e^{\log\lambda + 2L(\tau_1 - t_0)} e^{\log\lambda + 2L(\tau_2 - \tau_1)} \\ & \times & \cdots e^{\log\lambda + 2L(t-\tau_{k-1})}. \end{array}$$

For a chaotic system, we have $L > 0$ and so

$$V(e(t)) \leq \lambda^{-1} V(e(t_0)) e^{\sum\limits_{i=1}^{k} (\log\lambda + 2L(\tau_i - \tau_{i-1}))}.$$

Hence, if $\log\lambda + 2L(\tau_k - \tau_{k-1}) < 0, k = 1, 2, \cdots$, then $V(e(t)) \to 0$. This completes the proof.

III. Simulation experiments

Practice has proven that impulse control is an effective method that is lower in cost, better in performance, and easier to implement compared to continuous control methods. In this section, we will present two numerical simulation experiments to illustrate the application of impulse control strategies introduced in Section 2.

Example 1. As mentioned in Section 1, if we choose $a = 0.8, b = 0.2, c = 1.9$, then the maximum Lyapunov exponent of system (1) is $L = 3.6000$, indicating that system (1) exhibits chaotic behavior.

Now, suppose that the initial values of driving and the driven system are chosen as $(x_1, x_2, x_3)^T = (-0.2, 1.5, 0.3)^T$ and $(y_1, y_2, y_3)^T = (0.2, -1.5, -0.3)^T$.According to Theorem 2.1 and basic computations, we find that the intensity and timing of impulse control must adhere to the inequalities: $\log\lambda + 3.6000(\tau_k - \tau_{k-1}) < 0$ and $\tau_k - \tau_{k-1} < 1/3.6000$, Under these conditions, the driven system, after being subjected to effective impulse control, will synchronize with the driving system. To achieve this, we select the impulsive control gain matrix $B = -0.2I$, which means that the value of each node in the supply chain Y_{i+1} becomes $0.2X_i + 0.8Y_i$. Therefore, the time intervals for adjustments need to comply with $\tau_k - \tau_{k-1} < 0.1240$, Assuming that $\tau_k - \tau_{k-1} = 0.1000$. we present the results of this arrangement in numerical simulations, illustrated in Figure 6, Fig 7 and Fig 8.

From Figure 6, Fig 7 and 8, it is evident that the operational state of system (1) swiftly reverts to the anticipated trajectory.

Note that if the impulse control intensity and time interval are not selected appropriately, synchronization between the two systems cannot be achieved.

For example, if the impulsive intensity is still set to $B = -0.2I$, but the impulsive time interval is set to $\tau_k - \tau_{k-1} = 1.2000$, the two systems cannot achieve synchronization. The results of the simulation are presented in Figure 9, Fig 10 and Fig 11.Therefore, our impulse synchronization strategy proves to be useful.

Example 2. When $a = 0.6, b = 0.3, c = 1.7$, the system also exhibits chaos. The maximum Lyapunov exponent of system (1) is $L = 1.6744$. Fig 12 and Fig 13 depict the chaotic behavior of system (1) with the initial condition $(x_1, x_2, x_3)^T = (1, -2, 0)^T$.

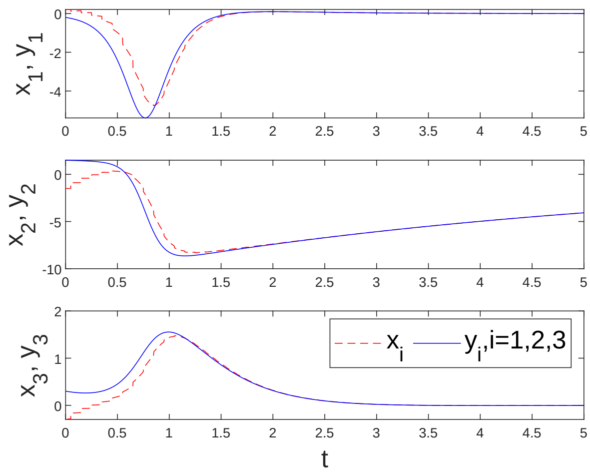

Fig. 6. Time series of $x_i, i = 1, 2, 3$.

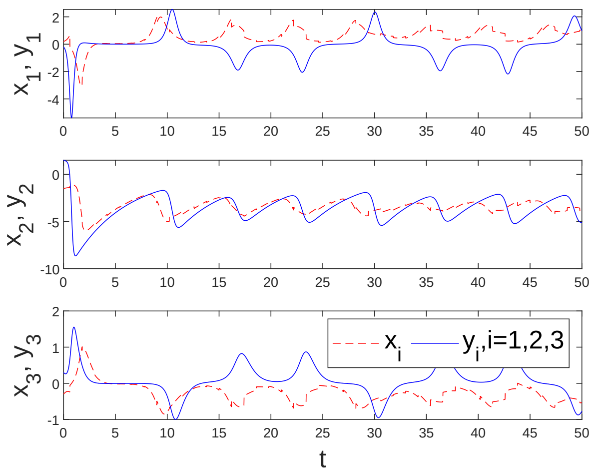

Fig. 9. Time series of $x_i, i = 1, 2, 3$.

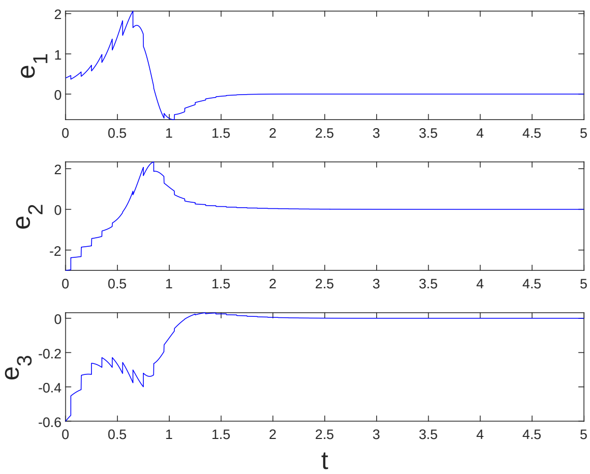

Fig. 7. Errors of each node.

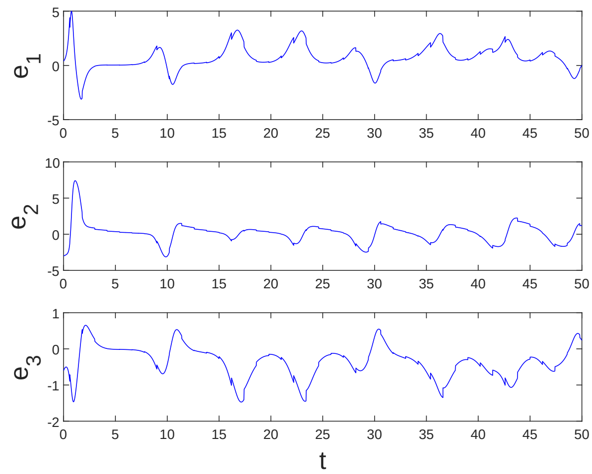

Fig. 10. Errors of each node.

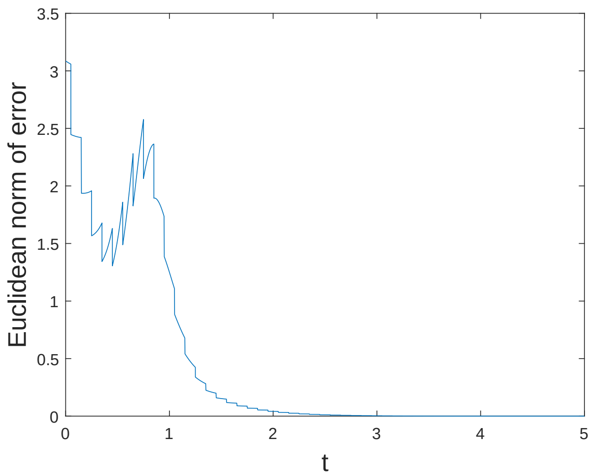

Fig. 8. Euclidean norm of error.

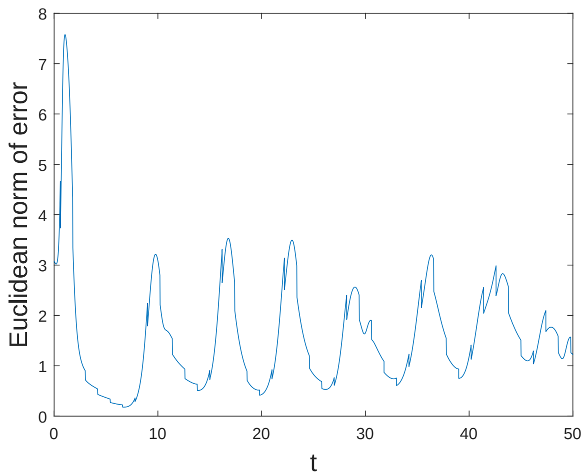

Fig. 11. Euclidean norm of error.

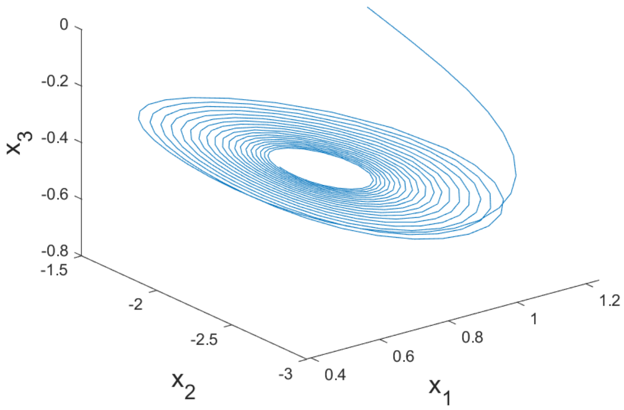

Fig. 12. 3D phase portraits of system (1) with parameters $a = 0.6, b = 0.3, c = 1.7$ and the initial condition $(x_1, x_2, x_3)^T = (1, -2, 0)^T$.

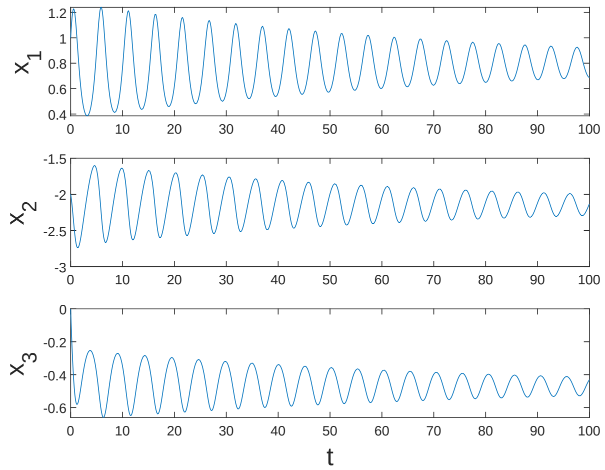

Fig. 13. Time series of system (1) with parameters $a = 0.6, b = 0.3, c = 1.7$ and the initial condition $(x_1, x_2, x_3)^T = (1, -2, 0)^T$.

Now, suppose that the initial values of driving and the driven system are chosen as $(x_1, x_2, x_3)^T = (11, 7, 9)^T$ and $(y_1, y_2, y_3)^T = (1, -2, 0)^T$.

Referencing Theorem 2.1 and simple arithmetic, we can confirm that the strength and timing of impulse control should conform to the following conditions:$\log \lambda + 1.6744\,(\tau_k - \tau_{k-1}) < 0$ and $\tau_k - \tau_{k-1} < 1/1.6744$, When these criteria are met, the driven system will synchronize with the driving system. To fulfill these parameters, we select the impulsive control gain matrix $B = -0.5I$, This configuration results in each node in the supply chain Y_{i+1} becomes $0.5X_i + 0.5Y_i$. Hence, the time interval of regulation should satisfy $\tau_k - \tau_{k-1} < 0.5972$ and let's suppose that $\tau_k - \tau_{k-1} = 0.5000$. we present the results of this arrangement in numerical simulations, illustrated in Figure 14, Fig 15 and Fig 16.

Similarly, from Fig 14, Fig 15 and 16, it can be observed that the operational state of system (1) returns quickly to the expected orbit.

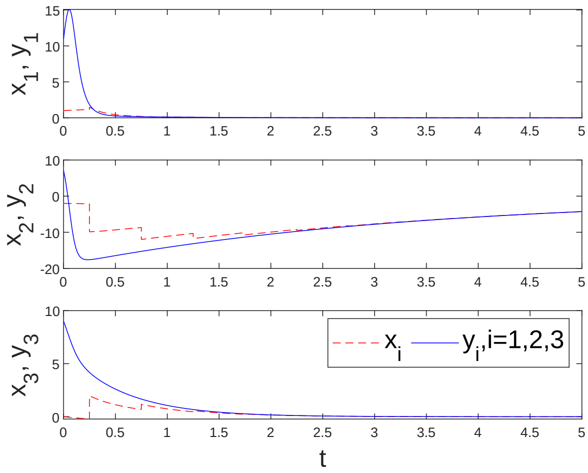

Fig. 14. Time series of $x_i, i = 1, 2, 3$.

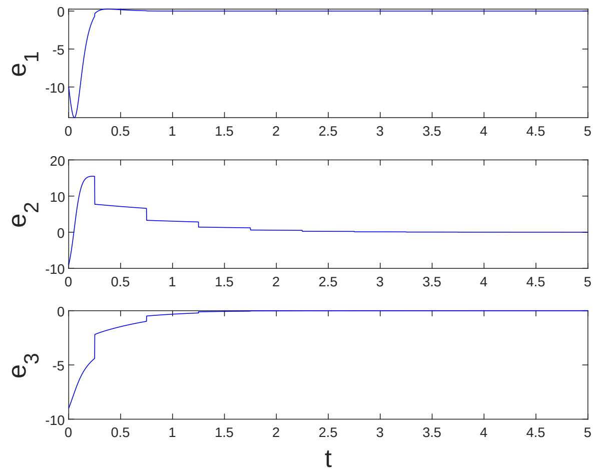

Fig. 15. Errors of each node.

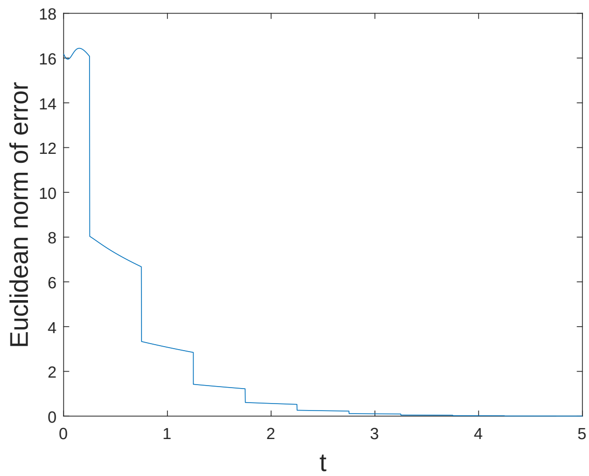

Fig. 16. Euclidean norm of error.

IV. Conclusion

In the past decades, impulse control has garnered widespread attention and application in various fields such as mechanics and communications. As pointed out in [15], impulse control proves to be more effective than continuous control in certain scenarios, such as when a deep-space spacecraft cannot maintain continuous engine operation due to limited fuel supply. In the present study, we explore impulse synchronization of chaotic financial systems, presenting a sufficient condition for synchronization based on Lyapunov stability theory. Simulation experiments demonstrate the effectiveness of the proposed method, offering potential insights for nonlinear finance and economics.

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