Modeling Pollen Tube Polar Growth Pattern under Asymmetric Consideration and Creating Game-theoretical Model for Ecotoxicity Assessment

Boontida Uapipatanakul, Jong-Chin Huang, Kelvin H.-C. Chen, Sirawit Ngammuangpak and Yu-Hsien Liao

***Abstract*—The project aims to develop a predictive model for assessing the effects of environmental toxins on lily pollen tube growth. By integrating principles from biology, mathematics, and game theory, the project will simulate asymmetric polar growth patterns of pollen tubes in response to chemical stressors and determine the ecotoxicological impacts of new chemicals or emerging materials. To address this, this study first proposes a game-theoretical method for balancing various efficacy under multiple-considerations. Additionally, considering that different conditional impacts result from varying factors, this study also presents several asymmetric generalizations relative to the factors and its behavior. Concurrently, several axiomatic processes are utilized to demonstrate the mathematical correctness and practicality for these measuring methods.**

***Index Terms*—Asymmetry, game-theoretical method, multiple-consideration, axiomatic process.**

I. Introduction

Pollen tube growth is a critical phase in plant reproduction, and its sensitivity to chemical agents makes it an excellent bioindicator for environmental toxicity. Related objectives can be considered as follows.

- To observe and quantify the effects of various chemical agents on lily pollen tube germination and growth.
- To create a mathematical model simulating the polar growth of pollen tubes, incorporating asymmetric factors.
- To integrate asymmetric game theory to understand the strategic biological responses to environmental stressors.
- To develop a predictive model for ecotoxicity assessment based on the observed growth patterns of lily pollen tubes.

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Related research processes will utilize the *Semi-In Vivo Germination Method* to replicate natural conditions closely and apply asymmetric game theory to capture the adaptive strategies of pollen tubes under chemical stress. The outcome will be a comprehensive model for ecotoxicity assessment that accounts for the asymmetrical growth patterns triggered by chemical exposure. Modeling pollen tube polar growth patterns under asymmetric considerations involves considering directional cues or gradients influencing pollen tube growth. Additionally, creating a model for ecotoxicity assessment would require incorporating parameters related to the toxic effects of substances on pollen tube growth. Related factors considered throughout this research are as follows.

1) Modified Semi-In Vivo Germination Method.
   - Materials: Freshly collected lily pollen, germination medium, lily flowers' stigma and style tissues, petri dishes.
   - Procedure: In vitro germination of pollen on a medium that includes stigma and style tissues from lily flowers to simulate natural conditions.
   - Pollen Tube Growth Parameters: Growth rate; Directional sensitivity; Apical dominance.
2) Preparation and Application of Chemicals:
   - Chemical Solutions: Prepare a series of dilutions of the chemicals in a non-toxic solvent.
   - Pollen Treatment: Treat the pollen with these solutions, including a range from non-toxic to potentially toxic concentrations.
   - Ecotoxicity Parameters: Concentration of toxic substance; Exposure duration; Potential impacts on cell viability and growth

Assessing related effects caused by various factors typically requires addressing multiple-considerations simultaneously, which may sometimes conflict. For instance, achieving highly complete pollution reduction through certain measures or equipment without consuming excessive energy or resources, and without generating other types of pollution or waste, necessitates considering these multiple facets simultaneously in an optimal or balanced state. In the field of mathematics, multiple-considerations optimization or equilibrium aims to achieve such benefits within any operational system. Related researches can be found in Bednarczuk et al. [1], Cheng et al. [2], Goli et al. [4], Guarini et al. [5], Mustakerov et al. [13], Tirkolaee et al. [19], and so on. Under traditional game-theoretical assessing notions, units are often considered in binary conditions of participation or non-participation. Utilizing marginal notion, the *equal allocation of non-separable costs* (EANSC, Ransmeier [17]) and the normalized index are introduced for assessing efficacy under traditional conditions, as per the respective proposals. Moulin [14] defined the concept of complement-reduction to illustrate that the EANSC could offer an equitable method for assessing efficacy. Considering multiple-considerations conditions, it is logical for units under the conditions to have varying operating behavior of involvement, necessitating a multi-choice assessing conditions where each unit has different partaking behavior of involvement. Under game-theoretical multi-choice conditions, Hwang and Liao [8], Liao [9], [10], [12], and Nouweland et al. [15] proposed several generalized extensions for the EANSC. Inspired by related axiomatic notion due to Moulin [14], Hwang and Liao [8] and Liao [9], [10], [12] also considered an extended complement-reduction to characterize these generalized EANSC.

The findings mentioned above prompt a key inquiry:

- whether the marginal notion and its associated outcomes could be expanded to the framework of multiple-considerations and multi-choice behavior simultaneously.

To investigate this question, we aim to establish different necessary mathematical foundations of multiple-considerations assessing methods to analyze balance problems with multiple-considerations and multi-choice behavior simultaneously. Departing from the frameworks of traditional and multi-choice TU conditions, this study considers the framework of *multiple-considerations multi-choice conditions*, and further introduces new assessing methods. These assessing methods generalize the concept of average marginal behavior-efficacy to account for multiple-considerations and multi-choice behavior conditions.

- By extending related assessing notion of the EANSC to multiple-considerations multi-choice conditions, the *uniform assessing of indistinguishable efficacy* (UMIE) are introduced in Section 2. The UMIE concept involves units receiving average marginal behavior-efficacy from the grand coalition, and then assessing the remaining efficacy uniformly.
- By incorporating the concept of unit-weighted emphasis into the UMIE, the *1-weighted assessing of indistinguishable efficacy* (1-WMIE) is defined in Section 2. In brief, the assessing notion of the 1WMIE involves units first assessing its average marginal behavior-efficacy, followed by assessing the remaining efficacy based on unit-weighted proportions.
- By integrating the concept of behavior-weighted emphasis into the UMIE, the *2-weighted assessing of indistinguishable efficacy* (2-WMIE) is defined in Section 2. In essence, the assessing notion of the 2WMIE entails units first assessing its weighted marginal behavior-efficacy, and then uniformly assessing the remaining efficacy.
- Combining the assessing notions of the 1WMIE and the 2WMIE gave rise to the *bi-weighted assessing of indistinguishable efficacy* (BWMIE) in Section 2. Briefly, the assessing notion of the BWMIE involves units first assessing its weighted marginal behavior-efficacy, and then assessing the remaining efficacy based on unit-weighted proportions.
- However, both the unit-weighted and behavior-weighted mechanisms appear somewhat subjective or artificial. In Section 4, the *interior assessing of indistinguishable efficacy* (IMIE) is derived as an alternative to weighted concepts, utilizing average marginal behavior-efficacy.

To analyze the mathematical correctness and the practicality for these assessing methods, we introduce an extended reduction and related characteristics of consistency, discussed in Sections 3 and 4.

- The UMIE is the only assessing method satisfying the characteristics of *standard for multiple-considerations conditions* and *multiple-considerations bilateral consistency*.
- The 1-WMIE is the only assessing method satisfying the characteristics of *1-weighted standard for multiple-considerations conditions* and *multiple-considerations bilateral consistency*.
- The 2-WMIE is the only assessing method satisfying the characteristics of *1-weighted standard for multiple-considerations conditions* and *multiple-considerations bilateral consistency*.
- The BWMIE is the only assessing method satisfying the characteristics of *bi-weighted standard for multiple-considerations conditions* and *multiple-considerations bilateral consistency*.
- While the IMIE violates multiple-considerations bilateral consistency, it adheres to the characteristics of *interior standard for multiple-considerations conditions* and *multiple-considerations revised-consistency*.

Throughout the study, additional interpretations and discussions regarding these characteristics and axiomatic results are stated to further elucidate its implications.

II. Preliminaries

A. Definitions and notations

Let $\mathbb{UU}$ denote the universe of units, for instance, the set comprised of all units of the Earth. Any $t \in \mathbb{UU}$ is identified as an unit of $\mathbb{UU}$, such as a unit in an ecological condition. For $t \in \mathbb{UU}$ and $\xi_t \in \mathbb{N}$, we define $\mathbb{OB}_t = \{0, 1, \cdots, \xi_t\}$ to represent the set of operating behavior for unit t, and $\mathbb{OB}_t^+ = \mathbb{OB}_t \setminus \{0\}$, where 0 indicates no operation.

Consider $\mathbb{U} \subseteq \mathbb{UU}$ as the largest set encompassing all units of an interactive condition within $\mathbb{UU}$, like all employees of a company in a country. Let $\mathbb{OB}^{\mathbb{U}} = \prod_{t\in\mathbb{U}} \mathbb{OB}_t$ be the product set of operating behavior sets for every unit in $\mathbb{U}$. For every $H \subseteq \mathbb{U}$, an unit alliance H corresponds, in a standard manner, to the multi-choice alliance $\hat{y}^H \in \mathbb{OB}^{\mathbb{U}}$, which is a vector indicating $\hat{y}^H_k = 1$ if $k \in H$, and $\hat{y}^Q_K = 0$ if $K \in \mathbb{U} \setminus H$. Denote $0_{\mathbb{U}}$ as the zero vector in $\Re^{\mathbb{U}}$. For $m \in \mathbb{N}$, also define 0_m as the zero vector in $\Re^m$ and $\mathbb{C}_m = \{1, 2, \cdots, m\}$.

A **multi-choice condition** is denoted as $(\mathbb{U}, \xi, \nu)$, where $\mathbb{U} \neq \emptyset$ is a finite set of elements, $\xi = (\xi_k)_{k\in\mathbb{U}} \in \mathbb{OB}^{\mathbb{U}}$ is a vector indicating the number of operating behavior for each element, and $\nu : \mathbb{OB}^{\mathbb{U}} \to \Re$ is a mapping with $\nu(0_{\mathbb{U}}) = 0$ that assigns to each operating behavior vector $\rho = (\rho_k)_{k\in\mathbb{U}} \in \mathbb{OB}^{\mathbb{U}}$ the benefit that elements can receive when each element k operating at degree ρ_k. A **multiple-considerations multi-choice condition** is denoted by $(\mathbb{U}, \xi, \mathbb{V}^m)$, where $m \in \mathbb{N}$, $\mathbb{V}^m = (\nu^t)_{t\in\mathbb{C}_m}$ and $(\mathbb{U}, \xi, \nu^t)$

represents a multi-choice condition for each $t \in \mathbb{C}_m$. The family of all multiple-considerations multi-choice conditions is denoted as $\mathbb{MCM}$.

A **method** is defined as a mapping ψ that assigns to each $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$ an efficacy vector

$$\psi\big(\mathbb{U}, \xi, \mathbb{V}^m\big) = \big(\psi^t\big(\mathbb{U}, \xi, \mathbb{V}^m\big)\big)_{t\in\mathbb{C}_m},$$

where $\psi^t\big(\mathbb{U}, \xi, \mathbb{V}^m\big) = \big(\psi^t_k\big(\mathbb{U}, \xi, \mathbb{V}^m\big)\big)_{k\in\mathbb{U}} \in \Re^{\mathbb{U}}$ and $\psi^t_k\big(\mathbb{U}, \xi, \mathbb{V}^m\big)$ represents the efficacy of element k if k operates in $\big(\mathbb{U}, \xi, \nu^t\big)$. Let $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, $H \subseteq \mathbb{U}$, and $\rho \in \Re^{\mathbb{U}}$. We define $NB(\rho) = \{k \in \mathbb{U} | \rho_k \neq 0\}$ and $\rho_H \in \Re^H$ as the restriction of ρ to H. Given $k \in \mathbb{U}$, we also define ρ_{-k} to represent $\rho_{\mathbb{U}\setminus\{k\}}$. Additionally, $\iota = (\rho_{-k}, j) \in \Re^{\mathbb{U}}$ is defined by $\iota_{-k} = \rho_{-k}$ and $\iota_k = j$.

Based on the claim of assessing how to balance various efficacy during operational processes, this study introduces derivative the concepts of the EANSC within the framework of multiple-considerations multi-choice conditions.

Definition 1: The **uniform assessing of indistinguishable efficacy (UMIE)**, $\overline{\lambda}$, is defined by

$$\begin{aligned}\overline{\lambda^t_b}(\mathbb{U}, \xi, \mathbb{V}^m) &= \lambda^t_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ &\quad + \tfrac{1}{|\mathbb{U}|} \cdot \big[\nu^t(\xi) - \sum_{k\in\mathbb{U}} \lambda^t_k(\mathbb{U}, \xi, \mathbb{V}^m)\big]\end{aligned}$$

for every $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, for every $t \in \mathbb{C}_m$ and for every $b \in \mathbb{U}$. The value $\lambda^t_b(\mathbb{U}, \xi, \mathbb{V}^m) = \frac{1}{\xi_b} \sum_{q\in\mathbb{OB}^+_b} \{\nu^t(\xi) - \nu^t(\xi_{-b}, q-1)\}$ is the **average marginal behavior-efficacy** among all operating behavior of element b in $(\mathbb{U}, \xi, \nu^t)$.[1] Under the concept of $\overline{\lambda}$, all elements firstly measure its average marginal behavior-efficacy, and further measure the rest of efficacy uniformly.

As mentioned in the introduction, the concept of weighting often becomes a consideration in various assessing processes. For example, weight proportions may be related to drug allocating, where weight can represent the relative pollution risk of various drugs used in different ecological conditions. Similarly, weighting can be applied to mitigating measures for toxins, where different mitigating measures may incur varying weighted implementation costs under different ecological conditions. Even if the implementation items and ecological conditions of a certain mitigating measure are fixed, the implementation costs of the mitigating item relative to different mitigated areas under the condition may vary in weighted proportions. Therefore, assigning weights to "units" or its "operating behavior" to differentiate relative differences is worth considering.

Let $\hat{w} : \mathbb{UU} \to \mathbb{R}^+$ be a positive mapping. Then $\hat{w}$ is treated as a **weight function for elements**. Similarly, let $\check{w} : \cup_{k\in\mathbb{UU}} \mathbb{OB}^+_k \to \mathbb{R}^+$ be a positive mapping. Then $\check{w}$ is regarded as a **weight function for operating behavior**. Based on these two kinds of weight functions, three weighted analogues of the UMIE could be generated as follows.

Definition 2:

- The **1-weighted assessing of indistinguishable efficacy (1-WMIE)**, $\lambda^{\hat{w}}$, is defined as follows: For every $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every $t \in \mathbb{C}_m$, and for every element $b \in \mathbb{U}$,

$$\begin{aligned}&\lambda^{\hat{w},t}_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ =\ &\lambda^t_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ &+ \tfrac{\hat{w}(b)}{\sum_{k\in\mathbb{U}} \hat{w}(k)} \cdot \big[\nu^t(\xi) - \sum_{k\in\mathbb{U}} \lambda^t_k(\mathbb{U}, \xi, \mathbb{V}^m)\big].\end{aligned}$$

According to the definition of $\lambda^{\hat{w}}$, all elements initially measure its average marginal behavior-efficacy, and the remaining efficacy are measured proportionally via weights for elements.

- The **2-weighted assessing of indistinguishable efficacy (2-WMIE)**, $\lambda^{\check{w}}$, is defined as follows: For every $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$, and for every element $b \in \mathbb{U}$,

$$\begin{aligned}&\lambda^{\check{w},t}_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ =\ &\gamma^{\check{w},t}_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ &+ \tfrac{1}{|\mathbb{U}|} \cdot \big[\nu^t(\xi) - \sum_{k\in\mathbb{U}} \gamma^{\check{w},t}_k(\mathbb{U}, \xi, \mathbb{V}^m)\big],\end{aligned}$$

where $\gamma^{\check{w},t}_b(\mathbb{U}, \xi, \mathbb{V}^m) = \frac{1}{\sum_{q\in\mathbb{OB}^+_b} \check{w}(q)} \sum_{q\in\mathbb{OB}^+_b} \{\check{w}(q) \cdot [\nu^t(\xi) - \nu^t(\xi_{-b}, q-1)]\}$ is the weighted marginal behavior-efficacy among all operating behavior of element b. By definition of $\lambda^{\check{w},t}$, all elements initially measure its weighted marginal behavior-efficacy, and the remaining efficacy are measured equally.

- The **bi-weighted assessing of indistinguishable efficacy (BWMIE)**, $\lambda^{\hat{w},\check{w}}$, is defined by for every $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$ and for every element $b \in \mathbb{U}$,

$$\begin{aligned}&\lambda^{\hat{w},\check{w},t}_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ =\ &\gamma^{\check{w},t}_b(\mathbb{U}, \xi, \mathbb{V}^m) \\ &+ \tfrac{\hat{w}(b)}{\sum_{k\in\mathbb{U}} \hat{w}(k)} \cdot \big[\nu^t(\xi) - \sum_{k\in\mathbb{U}} \gamma^{\check{w},t}_k(\mathbb{U}, \xi, \mathbb{V}^m)\big].\end{aligned}$$

Based on the definition of $\lambda^{\hat{w},\check{w}}$, all elements initially measure its weighted marginal behavior-efficacy, and the remaining efficacy are measured proportionally via weights for elements.

B. Motivating and practical examples

As mentioned in introduction, pollen tube growth is a critical phase in plant reproduction, and its sensitivity to chemical agents makes it an excellent bioindicator for environmental toxicity. Related purposes can be considered as follows.

- To observe and quantify the effects of various chemical agents on lily pollen tube germination and growth.
- To create a mathematical model simulating the polar growth of pollen tubes, incorporating asymmetric factors.
- To integrate asymmetric game theory to understand the strategic biological responses to environmental stressors.
- To develop a predictive model for ecotoxicity assessment based on the observed growth patterns of lily pollen tubes.

[1] This study utilizes bounded multi-choice conditions, treated as the conditions $(\mathbb{U}, \xi, \nu^t)$ such that, there exists $B^t_\nu \in \Re$ such that $\nu^t(\rho) \leq B^t_\nu$ for every $\rho \in \mathbb{OB}^{\mathbb{U}}$. It could be utilized to assure that $\lambda^t_b(\mathbb{U}, \xi, \nu^t)$ is well-defined.

Related research processes will utilize the *Semi-In Vivo Germination Method* to replicate natural conditions closely and apply asymmetric game theory to capture the adaptive strategies of pollen tubes under chemical stress. The outcome will be a comprehensive model for ecotoxicity assessment that accounts for the asymmetrical growth patterns triggered by chemical exposure. Modeling pollen tube polar growth patterns under asymmetric considerations involves considering directional cues or gradients influencing pollen tube growth. Additionally, creating a model for ecotoxicity assessment would require incorporating parameters related to the toxic effects of substances on pollen tube growth. Related considerations, factors and methods applied throughout this study are as follows.

1) Modified Semi-In Vivo Germination Method (Dickinson et al. [3]; Park et al. [16])
   - Materials: Freshly collected lily pollen, germination medium, lily flowers' stigma and style tissues, petri dishes.
   - Procedure: In vitro germination of pollen on a medium that includes stigma and style tissues from lily flowers to simulate natural conditions.
   - Pollen Tube Growth Parameters: Growth rate; Directional sensitivity; Apical dominance.
2) Preparation and Application of Chemicals:
   - Chemical Solutions: Prepare a series of dilutions of the chemicals in a non-toxic solvent.
   - Pollen Treatment: Treat the pollen with these solutions, including a range from non-toxic to potentially toxic concentrations.
   - Ecotoxicity Parameters: Concentration of toxic substance; Exposure duration; Potential impacts on cell viability and growth
3) Data Collection and Analysis.
   - Microscopic Analysis: Observe and record the germination rates and pollen tube growth.
   - Statistical Analysis: Use statistical tools to analyze the data and identify patterns of growth disruption.
4) Model for Ecotoxicity Assessment.
   - Model Creation: Based on the empirical data, develop a comprehensive model that predicts the ecotoxicological impact of chemicals.
   - Parameterization for Ecotoxicity Assessment:
   - Extend the model to include parameters relevant to ecotoxicity assessment.
   - Integrate toxicological data, including concentration-response relationships and exposure-response relationships, into the model.
   - Validation: Validate the model with independent datasets and iteratively refine it.
   - Validate the model by comparing its predictions with experimental data from studies on pollen tube growth under various conditions.
   - Refine the model based on the validation results.
5) Sensitivity Analysis.
   - Conduct sensitivity analyses to identify key parameters significantly influencing model outcomes.
   - Assess how changes in ecotoxicity parameters impact the model predictions.
6) Implementation for Ecotoxicity Assessment.
   - Use the validated model as a tool for ecotoxicity assessment.
   - Input relevant concentrations and exposure durations of toxic substances to predict their effects on pollen tube growth.
7) Scenarios and Predictions.
   - Simulate different exposure scenarios to predict the potential impacts of various concentrations and exposure durations on pollen tube growth.
   - Generate dose-response curves to quantify the ecotoxicity of different substances.
8) Visualization and Interpretation.
   - Create visual representations of model predictions, such as graphs or spatial maps, to facilitate interpretation.
   - Interpret the results in the context of ecotoxicity and provide insights into the potential risks of the tested substances.
9) Iterative Refinement.
   - Iterate the model based on new data or insights from experimental studies or real-world observations.

In order to elucidate the application concept of the multi-faceted multi-choice framework, an applied example related to this study could be modeled concisely. Let $\mathbb{U}$ denote the set of all participating factors in ecotoxicity assessment based on the observed growth patterns of lily pollen tubes $(\mathbb{U}, \xi, \mathbb{V}^m)$. The function ν^t is regarded as the effect assessing function, which can assess relative effect generated by any overall operational behavior vector $\rho = (\rho_f)_{f\in\mathbb{U}} \in \mathbb{OB}^{\mathbb{U}}$ in one of the consideration aspects, presenting relative effect generated by each operating factor $f \in \mathbb{U}$ when adopting a specific operating behavior $\rho_f \in \mathbb{OB}_f$ under the overall environmental consideration aspect $(\mathbb{U}, \xi, \nu^t)$. Utilizing the aforementioned correspondence framework, an ecotoxicity assessment based on the observed growth patterns of lily pollen tubes can be relatively coincided with a multiple-considerations multi-choice condition $(\mathbb{U}, \xi, \mathbb{V}^m)$. In the following statements, we will further demonstrate how the assessing methods proposed in this study are applied to effect assessing under practical application situations.

Under the example related to ecotoxicity assessment based on the observed growth patterns of lily pollen tubes mentioned earlier, all participating factors may not only cause its own particular effect to the overall circumstance but may also lead to reciprocal prevention, contamination depravation, or even unpredictable effect due to reactions with other participating factors. As presented in above example, the mapping ν^t can be defined as the assessing function for generated effects, where the collaborative operating behavior of all participating factors can be expressed holistically by vector $\rho = (\rho_f)_{f\in\mathbb{U}} \in \mathbb{OB}^{\mathbb{U}}$.

- Following the assessing notion of the UAIE considered in Definition 1, the average marginal behavior-efficacy caused by the operating behavior of all participating factors are initially assessed, and the remaining efficacy are collectively assessed by all participating factors.
- However, since each participating factor may exhibit distinct relative effect during reactive processes due to

different contaminated circumstances or relevant objective requirements, it is necessary to assign relative weights to the participating factors. The average marginal behavior-efficacy caused by the operating behavior of all participating factors are initially assessed, and the remaining efficacy are assessed proportionally by means of the relative weights of each participating factor, resulting the assessing notion of the 1-WAIE in Definition 2.
- From the perspective of operating behavior, since the operating behavior of each participating factor may exhibit distinct effect during reactive processes due to different contaminated circumstances or relevant objective requirements, these operating behavior should naturally possess various forms of relative impacts. Thence, assigning weights via the weighting rule is also very rational. The weighted marginal behavior-efficacy caused by all operating behavior of each participating factor are initially assessed, and the remaining effect are collectively assessed by all participating factors, resulting the assessing concept of the 2-WAIE in Definition 2.
- By further combining the assessing concepts of the 1-WAIE and the 2-WAIE, the weighted marginal behavior-efficacy caused by all operating behavior of each participating factor are initially assessed, and the remaining effect are assessed proportionally by means of the relative weights of each participating factor, forming the assessing concept of the BWAIE in Definition 2.

III. Axiomatic processes

A. Axiomatizations for the UMIE and its weighted extensions

By simultaneously considering the axiomatic notions and related proof techniques proposed by Hart and Mas-Colell [6] and Moulin [14], this section will utilize several axiomatizations of the UMIE, the 1-WMIE, the 2-WMIE, and the BWMIE to demonstrate the mathematical correctness and practical applicability for these methods.

A method ψ fits the **multiple-considerations effectiveness (MCEES)** requirement if, for every $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$ and for every $t \in \mathbb{C}_m$, the sum of remunerations measured via ψ to all elements in $\mathbb{U}$ coincides with the overall efficacy $\nu^t(\xi)$, i.e., $\sum_{b\in\mathbb{U}} \psi_b^t(\mathbb{U}, \xi, \mathbb{V}^m) = \nu^t(\xi)$. The MCEES requirement ensures that all elements measure whole the efficacy entirely. Further, a method ψ fits the **multiple-considerations individual effectiveness (MCIEES)** requirement if ψ fits MCEES under all $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$ with $|\mathbb{U}| = 1$.

Remark 1: Based on definitions of MCEES and MCIEES, it is easy to have that a method fits MCIEES absolutely if it fits MCEES.

Lemma 1: The methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit MCEES.

Proof of Lemma 1: Let $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, $t \in \mathbb{C}_m$, $\hat{w}$ be weight function for elements and $\check{w}$ be weight function for operating behavior. By Definition 2,

$$\begin{aligned}
& \sum_{b\in\mathbb{U}} \lambda_b^{\hat{w},\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) \\
= & \sum_{b\in\mathbb{U}} \gamma_b^{\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) \\
& + \sum_{b\in\mathbb{U}} \Big[\frac{\hat{w}(b)}{\sum_{k\in\mathbb{U}} \hat{w}(k)} \cdot \big[\nu^t(\xi) - \sum_{k\in\mathbb{U}} \gamma_k^{\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) \big] \Big] \\
= & \sum_{b\in\mathbb{U}} \gamma_b^{\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) \\
& + \frac{\sum_{b\in\mathbb{U}} \hat{w}(b)}{\sum_{k\in\mathbb{U}} \hat{w}(k)} \cdot \big[\nu^t(\xi) - \sum_{k\in\mathbb{U}} \gamma_k^{\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) \big] \\
= & \sum_{b\in\mathbb{U}} \gamma_b^{\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) + \nu^t(\xi) - \sum_{k\in\mathbb{U}} \gamma_k^{\check{w},t}(\mathbb{U}, \xi, \mathbb{V}^m) \\
= & \nu^t(\xi).
\end{aligned}$$

The proof is done. If all the weights for elements are set to 1 in the above proof process, the MCEES requirement of 2-WMIE can be verified. Similarly, if all the weights for operating behavior are set to 1 in the above proof process, the MCEES requirement of 1-WMIE can be completed. Furthermore, if all the weights for both elements and operating behavior are set to 1 in the above proof process, the MCEES requirement of UMIE can be finished. ■

To axiomatize the EANSC, Moulin[14] introduced the concept of a specific reduction: if any units within any interactive group in the organizing condition do not yield the expected benefits, a mechanism can be implemented to initiate a re-interaction under the complete cooperation of all units that achieve the expected benefits. The derived definition of the Moulin's reduction under the multiple-considerations multi-choice condition is defined as follows.

Let $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, $K \subseteq \mathbb{U}$, and ψ be a method. The **reduced condition** $(K, \xi_K, \mathbb{V}^m_{K,\psi})$ is defined by $\mathbb{V}^m_{K,\psi} = (\nu^t_{K,\psi})_{t\in\mathbb{C}_m}$, and for every $\rho \in \mathbb{OB}^K$,

$$\begin{aligned}
& \nu^t_{K,\psi}(\rho) \\
= & \begin{cases} 0 & \text{if } \rho = 0_K, \\ \nu^t\big(\rho, \xi_{\mathbb{U}\setminus K}\big) - \sum_{b\in\mathbb{U}\setminus K} \psi_b^t(\mathbb{U}, \xi, \mathbb{V}^m) & \text{otherwise,} \end{cases}
\end{aligned}$$

Moreover, a method ψ satisfies the **multiple-considerations bilateral consistency (MCBCSY)** requirement if $\psi_b^t(K, \xi_K, \mathbb{V}^m_{K,\psi}) = \psi_b^t(\mathbb{U}, \xi, \mathbb{V}^m)$ for every $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, for every $t \in \mathbb{C}_m$, for every $K \subseteq \mathbb{U}$ with $|K| = 2$, and for every $b \in K$.

Lemma 2: The methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit MCBCSY.

Proof of Lemma 2: Let $(\mathbb{U}, \xi, \mathbb{V}^m) \in \mathbb{MCM}$, $K \subseteq \mathbb{U}$, $t \in \mathbb{C}_m$, $\hat{w}$ be weight function for elements and $\check{w}$ be weight function for operating behavior. Let $|\mathbb{U}| \geq 2$ and $|K| = 2$. By Definition 2,

$$\begin{aligned}
& \lambda_b^{\hat{w},\check{w},t}(K, \xi_K, \mathbb{V}^m_{K,\lambda^{\hat{w},\check{w}}}) \\
= & \gamma_b^{\check{w},t}(K, \xi_K, \mathbb{V}^m_{K,\lambda^{\hat{w},\check{w}}}) \\
& + \frac{\hat{w}(b)}{\sum_{k\in K} \hat{w}(k)} \cdot \big[\nu^t_{K,\lambda^{\hat{w},\check{w}}}(\xi_K) \\
& \qquad - \sum_{k\in K} \gamma_k^{\check{w},t}(K, \xi_K, \mathbb{V}^m_{K,\lambda^{\hat{w},\check{w}}}) \big]
\end{aligned} \tag{1}$$

for every $b \in K$ and for every $t \in \mathbb{C}_m$. By definitions of

$\gamma^{\check{w},t}$ and $\nu^t_{K,\lambda^{\hat{w},\check{w}}}$,

$$\begin{aligned}
& \gamma_b^{\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\lambda^{\hat{w},\check{w}}}) \\
= & \frac{1}{\sum\limits_{q\in\mathbb{OB}_b^+}\check{w}(q)}\sum_{q\in\mathbb{OB}_b^+}\{\check{w}(q)\cdot[\nu^t_{K,\lambda^{\hat{w},\check{w}}}(\xi_K) \\
& \qquad -\nu^t_{K,\lambda^{\hat{w},\check{w}}}(\xi_{K\setminus\{b\}},q-1)]\} \\
= & \frac{1}{\sum\limits_{q\in\mathbb{OB}_b^+}\check{w}(q)}\sum_{q\in\mathbb{OB}_b^+}\{\check{w}(q)\cdot[\nu^t(\xi)-\nu^t(\xi_{-b},q-1)]\} \\
= & \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m).
\end{aligned} \tag{2}$$

Based on equations (1), (2) and definitions of $\nu^t_{K,\lambda^{\hat{w},\check{w}}}$ and $\lambda^{\hat{w},\check{w}}$,

$$\begin{aligned}
& \lambda_b^{\hat{w},\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\lambda^{\hat{w},\check{w}}}) \\
= & \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& +\frac{\hat{w}(b)}{\sum\limits_{k\in K}\hat{w}(k)}\Big[\nu^t_{K,\lambda^{\hat{w},\check{w}}}(\xi_K)-\sum_{k\in K}\gamma_k^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\Big] \\
= & \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& +\frac{\hat{w}(b)}{\sum\limits_{k\in K}\hat{w}(k)}\Big[\nu^t(\xi)-\sum_{k\in\mathbb{U}\setminus K}\lambda_k^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& \qquad -\sum_{k\in K}\gamma_k^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\Big] \\
= & \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& +\frac{\hat{w}(b)}{\sum\limits_{k\in K}\hat{w}(k)}\Big[\sum_{k\in K}\lambda_k^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& \qquad -\sum_{k\in K}\gamma_k^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\Big] \\
& \textbf{(MCEES of } \lambda^{\hat{w},\check{w}}\textbf{)} \\
= & \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& +\frac{\hat{w}(b)}{\sum\limits_{k\in K}\hat{w}(k)}\Big[\frac{\sum\limits_{k\in K}\hat{w}(k)}{\sum\limits_{p\in\mathbb{U}}\hat{w}(p)}\big[\nu^t(\xi)-\sum_{p\in\mathbb{U}}\gamma_p^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\big]\Big] \\
= & \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\
& +\frac{\hat{w}(b)}{\sum\limits_{p\in\mathbb{U}}\hat{w}(p)}\big[\nu^t(\xi)-\sum_{p\in\mathbb{U}}\gamma_p^t(\mathbb{U},\xi,\mathbb{V}^m)\big] \\
= & \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)
\end{aligned}$$

for every $b\in K$ and for every $t\in\mathbb{C}_m$. If all the weights for elements are set to 1 in the above proof process, the MCBCSY requirement of 2-WMIE can be verified. Similarly, if all the weights for operating behavior are set to 1 in the above proof process, the MCBCSY requirement of 1-WMIE can be completed. Furthermore, if all the weights for both elements and operating behavior are set to 1 in the above proof process, the MCBCSY requirement of UMIE can be finished. ■

A method ψ satisfies the **standard for multiple-considerations conditions (SMCC)** requirement if $\psi(\mathbb{U},\xi,\mathbb{V}^m)=\overline{\lambda}(\mathbb{U},\xi,\mathbb{V}^m)$ for every $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ with $|\mathbb{U}|\leq 2$. A method ψ satisfies the **1-weighted standard for multiple-considerations conditions (1WSMCC)** if $\psi(\mathbb{U},\xi,\mathbb{V}^m)=\lambda^{\hat{w}}(\mathbb{U},\xi,\mathbb{V}^m)$ for every $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ with $|\mathbb{U}|\leq 2$ and for every weight function $\hat{w}$ for elements. A method ψ satisfies the **2-weighted standard for multiple-considerations conditions (2WSMCC)** if $\psi(\mathbb{U},\xi,\mathbb{V}^m)=\lambda^{\check{w}}(\mathbb{U},\xi,\mathbb{V}^m)$ for every $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ with $|\mathbb{U}|\leq 2$ and for every weight function $\check{w}$ for degrees. A method ψ fits **bi-weighted standard for multiple-considerations conditions (BWSMCC)** if $\psi(\mathbb{U},\xi,\mathbb{V}^m)=\lambda^{\hat{w},\check{w}}(\mathbb{U},\xi,\mathbb{V}^m)$ for every $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ with $|\mathbb{U}|\leq 2$, for every weight function for elements $\hat{w}$ and for every weight function for operating behavior $\check{w}$. The axioms of the SMCC, the 1WSMCC, the 2WSMCC, and the BWSMCC are extended analogues in conditions involving only two units interacting, proposed by Hart and Mas-Colell [6] in characterizing the Shapley value.

Lemma 3: A method ψ fit MCIEES if is fits SMCC, 1WSMCC, 2WSMCC and BWSMCC respectively.

Proof of Lemma 3: By Lemma 1, it is shown that the methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit MCEES simultaneously. By further assuming one of participating factors in all two-factors conditions under requirements of SMCC, 1WSMCC, 2WSMCC and BWSMCC, the proof is finished. ■

Lemma 4: A method ψ fit MCEES if is fits SMCC (1WSMCC, 2WSMCC, BWSMCC) and MCBCSY.

Proof of Lemma 4: Let ψ be a method fitting SMCC and MCBCSY. By Lemma 3, ψ fits MCIEES. Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ and $t\in\mathbb{C}_m$. It is trivial for $|\mathbb{U}|=1$ by MCIEES. Suppose that $|\mathbb{U}|\geq 2$. Consider the reduction $\big(\{a,b\},\xi_{\{a,b\}},\mathbb{V}^m_{\{a,b\},\psi}\big)$ with $a,b\in\mathbb{U}$. Therefore,

$$\nu^t_{\{a,b\},\psi}(\xi_{\{a,b\}})=\nu^t(\xi)-\sum_{j\in\mathbb{U}\setminus\{a,b\}}\psi_j^t(\mathbb{U},\xi,\mathbb{V}^m).$$

Since ψ fits MCBCSY,

$$\psi_s^t\big(\{a,b\},\xi_{\{a,b\}},\mathbb{V}^m_{\{a,b\},\psi}\big)=\psi_s^t(\mathbb{U},\xi,\mathbb{V}^m)$$

for all $s\in\{a,b\}$. Then,

$$\begin{aligned}
& \psi_a(\mathbb{U},\xi,\mathbb{V}^m)+\psi_b(\mathbb{U},\xi,\mathbb{V}^m) \\
= & \nu^t(\xi)-\sum_{j\in\mathbb{U}\setminus\{a,b\}}\psi_j^t(\mathbb{U},\xi,\mathbb{V}^m).
\end{aligned}$$

So, $\sum\limits_{j\in\mathbb{U}}\psi_j^t(\mathbb{U},\xi,\mathbb{V}^m)=\nu^t(\xi)$, i.e., ψ fits MCEES. Since the proofs under cases for 1WSMCC, 2WSMCC and BWSMCC are similar, it is could be omitted. ■

Considering the axiomatic notions and relevant proof techniques proposed due to Hart and Mas-Colell [6] and Moulin [14], the axiom of the MABCY is adopted to axiomatize these methods as follows.

Theorem 1:

1) On $\mathbb{MCM}$, the UMIE is the unique method fitting SMCC and MCBCSY.
2) On $\mathbb{MCM}$, the 1-WMIE is the unique method fitting 1WSMCC and MCBCSY.
3) On $\mathbb{MCM}$, the 2-WMIE is the unique method fitting 2WSMCC and MCBCSY.
4) On $\mathbb{MCM}$, the BWMIE is the unique method fitting BWSMCC and MCBCSY.

Proof of Theorem 1: By Lemma 2, the methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit MCBCSY. Clearly, the methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit SMCC, 1WSMCC, 2WSMCC and BWSMCC respectively.

To present the uniqueness of result 4, suppose that ψ fits BWSMCC and MCBCSY. By BWSMCC and MCBCSY of ψ, it is easy to clarify that ψ also fits MCEES based on Lemma 4. Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$, $\hat{w}$ be weight function for elements and $\check{w}$ be weight function for operating behavior. By BWSMCC of ψ, $\psi(\mathbb{U},\xi,\mathbb{V}^m)=\lambda^{\hat{w},\check{w}}(\mathbb{U},\xi,\mathbb{V}^m)$ if $|\mathbb{U}|\leq 2$. The situation $|\mathbb{U}|>2$: Let $b\in\mathbb{U}$, $t\in\mathbb{C}_m$ and $K=\{b,p\}$

with $p \in \mathbb{U} \setminus \{b\}$.

$$\begin{array}{rl} & \psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\ = & \psi_b^t(K,\xi_K,\mathbb{V}^m_{K,\psi}) - \lambda_b^{\hat{w},\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\lambda^{\hat{w},\check{w}}}) \\ & \textbf{(MCBCSY of } \lambda^{\hat{w},\check{w},t} \textbf{ and } \psi\textbf{)} \\ = & \lambda_b^{\hat{w},\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\psi}) - \lambda_b^{\hat{w},\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\lambda^{\hat{w},\check{w},}}). \\ & \textbf{(BWSMCC of } \psi\textbf{)} \end{array} \tag{3}$$

Similar to equation (2)

$$\begin{array}{rl} \gamma_b^{\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\psi}) & = \gamma_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\ & = \gamma_b^{\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\gamma^{\lambda,\check{w},}}). \end{array} \tag{4}$$

By equations (3) and (4),

$$\begin{array}{rl} & \psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) \\ = & \lambda_b^{\hat{w},\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\psi}) - \lambda_b^{\hat{w},\check{w},t}(K,\xi_K,\mathbb{V}^m_{K,\lambda^{\hat{w},\check{w},}}) \\ = & \frac{\hat{w}(b)}{\hat{w}(b)+\hat{w}(p)}\big[\nu^t_{K,\psi}(\xi_K) - \nu^t_{K,\lambda^{\hat{w},\check{w}}}(\xi_K)\big] \\ = & \frac{\hat{w}(b)}{\hat{w}(b)+\hat{w}(p)}\big[\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) + \psi_{\mathbb{U}}^t(\mathbb{U},\xi,\mathbb{V}^m) \\ & \quad -\lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_{\mathbb{U}}^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\big]. \end{array}$$

Thus,

$$\begin{array}{rl} & \hat{w}(p)\cdot\big[\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\big] \\ = & \hat{w}(b)\cdot\big[\psi_{\mathbb{U}}^t(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_{\mathbb{U}}^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\big]. \end{array}$$

By MCEES of $\lambda^{\hat{w},\check{w},t}$ and ψ,

$$\begin{array}{rl} & \big[\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\big]\cdot\sum\limits_{p\in\mathbb{U}}\hat{w}(p) \\ = & \hat{w}(b)\cdot\sum\limits_{p\in\mathbb{U}}\big[\psi_{\mathbb{U}}^t(\mathbb{U},\xi,\mathbb{V}^m) - \lambda_{\mathbb{U}}^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)\big] \\ = & \hat{w}(b)\cdot\big[\nu^t(\xi) - \nu^t(\xi)\big] \\ = & 0. \end{array}$$

Hence, $\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) = \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m)$ for every $b \in \mathbb{U}$ and for every $t \in \mathbb{C}_m$. If all the weights for elements are set to 1 in the above proof process, the proof of outcome 3 could be verified. Similarly, if all the weights for operating behavior are set to 1 in the above proof process, the proof of outcome 2 could be completed. Furthermore, if all the weights for both elements and operating behavior are set to 1 in the above proof process, the proof of outcome 1 could be presented. ■

In the following some instances are exhibited to display that every of the requirements applied in Theorem 1 is independent of the rest of requirements.

Example 1: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$ and for every element $b \in \mathbb{U}$,

$$\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) = \begin{cases} \lambda_b^{\hat{w},\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) & \text{if } |\mathbb{U}| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ψ fits BWSMCC, but it does not fit MCBCSY.

Example 2: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$ and for every element $b \in \mathbb{U}$,

$$\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) = \begin{cases} \lambda_b^{\check{w},t}(\mathbb{U},\xi,\mathbb{V}^m) & \text{if } |\mathbb{U}| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ψ fits 2WSMCC, but it does not fit MCBCSY.

Example 3: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$ and for every element $b \in \mathbb{U}$,

$$\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) = \begin{cases} \lambda_b^{\hat{w},t}(\mathbb{U},\xi,\mathbb{V}^m) & \text{if } |\mathbb{U}| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ψ fits 1WSMCC, but it does not fit MCBCSY.

Example 4: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$ and for every element $b \in \mathbb{U}$,

$$\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) = \begin{cases} \overline{\lambda_b^t}(\mathbb{U},\xi,\mathbb{V}^m) & \text{if } |\mathbb{U}| \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, ψ fits SMCC, but it does not fit MCBCSY. $\lambda^{\hat{w}}$

Example 5: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for every weight function for elements $\hat{w}$, for every weight function for operating behavior $\check{w}$, for every $t \in \mathbb{C}_m$ and for every element $b \in \mathbb{U}$, $\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m) = 0$. Clearly, ψ fits MCBCSY, but it does not fit SMCC, 1WSMCC, 2WSMCC and BWSMCC.

B. Different generalization and revised consistency

Throughout Section 2 and Section 3.1, this study proposes corresponding weight functions for units and its relevant operating behavior to measure the corresponding interaction weights. However, the validity or representativeness of these weight functions may be questioned, as the relative weighting of units or its related operating behavior may appear somewhat artificial. Therefore, it seems more natural and reasonable to replace the weight functions with relative average marginal behavior-efficacy under different conditions.

By using "average marginal behavior-efficacy" instead of "weighting," it is possible to define a concept of efficacy assessing that is different from previous ones in a natural manner.

Definition 3: The **interior assessing of indistinguishable efficacy (IMIE)**, λ^I, is defined as follows: for every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}^*$, for every $t \in \mathbb{C}_m$, and for every element $b \in \mathbb{U}$,

$$\begin{array}{rl} & \lambda_b^{I,t}(\mathbb{U},\xi,\mathbb{V}^m) \\ = & \lambda_b^t(\mathbb{U},\xi,\mathbb{V}^m) \\ & +\frac{\lambda_b^t(\mathbb{U},\xi,\mathbb{V}^m)}{\sum\limits_{k\in\mathbb{U}}\lambda_k^t(\mathbb{U},\xi,\mathbb{V}^m)}\big[\nu^t(\xi) - \sum\limits_{k\in\mathbb{U}}\lambda_k^t(\mathbb{U},\xi,\mathbb{V}^m)\big], \end{array}$$

where $\mathbb{MCM}^* = \{(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM} \mid \sum\limits_{k\in\mathbb{U}}\lambda_k^t(\mathbb{U},\xi,\mathbb{V}^m) \neq 0 \text{ for every } t \in \mathbb{C}_m\}$. Based on definition of λ^I, all elements initially measure their average marginal behavior-efficacy, and the remaining efficacy then assessed proportionally based on these average marginal behavior-efficacy.

Next, one would like to axiomatize the IMIE using related notion of consistency. A method ψ fits the **interior standard for multiple-considerations conditions (ISMCC)** if $\psi(\mathbb{U},\xi,\mathbb{V}^m) = \lambda^I(\mathbb{U},\xi,\mathbb{V}^m)$ for every $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$ with $|\mathbb{U}| \le 2$.

It is straightforward to verify that $\sum\limits_{k\in K}\lambda_k^t(\mathbb{U},\xi,\mathbb{V}^m) = 0$ for some $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for some $K \subseteq \mathbb{U}$, and for some $t \in \mathbb{C}_m$, i.e., $\lambda^{I,t}(K,\xi_K,\mathbb{V}^m_{K,\lambda})$ doesn't exist for some $(\mathbb{U},\xi,\mathbb{V}^m) \in \mathbb{MCM}$, for some $K \subseteq \mathbb{U}$, and for some $t \in \mathbb{C}_m$. Therefore, we focus on the *multiple-considerations revised-consistency* as follows. A method

ψ fits the **multiple-considerations revised-consistency (MCRCSY)** if $(K,\xi_K,\mathbb{V}^m_{K,\psi})$ and $\psi(K,\xi_K,\mathbb{V}^m_{K,\psi})$ exist for some $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$, for some $K\subseteq\mathbb{U}$, and for some $t\in\mathbb{C}_m$, and it holds that $\psi_b(K,\xi_K,\mathbb{V}^m_{K,\psi})=\psi_b(\mathbb{U},\xi,\mathbb{V}^m)$ for every $b\in K$.

Similar to Lemmas 1, 2, 3, 4 and Theorem 1, related axiomatic results for λ^I can also be presented as follows.

Lemma 5: The method λ^I fits MCEES on $\mathbb{MCM}^*$.

Proof of Lemma 5: Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ and $t\in\mathbb{C}_m$. By Definition 3,

$$\begin{aligned}
&\sum_{b\in\mathbb{U}}\lambda^{I,t}_b(\mathbb{U},\xi,\mathbb{V}^m)\\
=\ &\sum_{b\in\mathbb{U}}\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\sum_{b\in\mathbb{U}}\Big[\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\sum_{k\in\mathbb{U}}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)}\big[\nu^t(\xi)-\sum_{k\in\mathbb{U}}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)\big]\Big]\\
=\ &\sum_{b\in\mathbb{U}}\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\tfrac{\sum_{b\in\mathbb{U}}\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\sum_{k\in\mathbb{U}}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)}\cdot\big[\nu^t(\xi)-\sum_{k\in\mathbb{U}}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)\big]\\
=\ &\sum_{b\in\mathbb{U}}\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)+\nu^t(\xi)-\sum_{k\in\mathbb{U}}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)\\
=\ &\nu^t(\xi).
\end{aligned}$$

The proof is done. ■

Lemma 6: The method λ^I fits MCRCSY on $\mathbb{MCM}^*$.

Proof of Lemma 6: Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$, $K\subseteq\mathbb{U}$ and $t\in\mathbb{C}_m$. Let $|\mathbb{U}|\geq 2$ and $|K|=2$. By Definition 3,

$$\begin{aligned}
&\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\\
=\ &\lambda^t_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\\
&\quad+\tfrac{\hat{w}(b)}{\sum_{k\in K}\hat{w}(k)}\cdot\big[\nu^t_{K,\lambda^I}(\xi_K)\\
&\qquad-\sum_{k\in K}\lambda^t_k(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\big]
\end{aligned}\tag{5}$$

for every $b\in K$ and for every $t\in\mathbb{C}_m$. By definitions of λ^t and ν^t_{K,λ^I},

$$\begin{aligned}
&\lambda^t_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\\
=\ &\tfrac{1}{\xi_b}\sum_{q\in\mathbb{OB}^+_b}\{\nu^t_{K,\lambda^I}(\xi_K)-\nu^t_{K,\lambda^I}(\xi_{K\setminus\{b\}},q-1)\}\\
=\ &\tfrac{1}{\xi_b}\sum_{q\in\mathbb{OB}^+_b}\{\nu^t_{K,\lambda^I}(\xi)-\nu^t_{K,\lambda^I}(\xi_{-b},q-1)\}\\
=\ &\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m).
\end{aligned}\tag{6}$$

Based on equations (5), (6) and definitions of ν^t_{K,λ^I} and λ^I,

$$\begin{aligned}
&\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\\
=\ &\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\sum_{k\in K}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)}\big[\nu^t_{K,\lambda^I}(\xi_K)-\sum_{k\in K}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)\big]\\
=\ &\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\sum_{k\in K}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)}\big[\nu^t(\xi)-\sum_{k\in\mathbb{U}\setminus K}\lambda^{I,t}_k(\mathbb{U},\xi,\mathbb{V}^m)\\
&\qquad-\sum_{k\in K}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)\big]\\
=\ &\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\sum_{k\in K}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)}\big[\sum_{k\in K}\lambda^{I,t}_k(\mathbb{U},\xi,\mathbb{V}^m)\\
&\qquad-\sum_{k\in K}\lambda^t_k(\mathbb{U},\xi,\mathbb{V}^m)\big]\\
&\textbf{(MCEES of } \lambda^I\textbf{)}\\
=\ &\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\sum_{p\in\mathbb{U}}\lambda^t_p(\mathbb{U},\xi,\mathbb{V}^m)}\big[\nu^t(\xi)-\sum_{p\in\mathbb{U}}\gamma^t_p(\mathbb{U},\xi,\mathbb{V}^m)\big]\\
=\ &\lambda^{I,t}_b(\mathbb{U},\xi,\mathbb{V}^m)
\end{aligned}$$

for every $b\in K$ and for every $t\in\mathbb{C}_m$. ■

Remark 2: Based on definitions of MCBCSY and MCRCSY, it is easy to see that a method fits MCRCSY if it fits MCBCSY. Since it is shown that the methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit MCBCSY, these methods also fit MCRCSY.

Lemma 7: On $\mathbb{MCM}^*$, a method ψ fit MCIEES if is fits ISMCC.

Proof of Lemma 7: By Lemma 5, it is shown that the method λ^I fits MCEES on $\mathbb{MCM}^*$. By further assuming one of participating factors in all two-factors conditions under requirement of ISMCC, the proof is finished. ■

Lemma 8: On $\mathbb{MCM}^*$, a method ψ fit MCEES if is fits ISMCC and MCRCSY.

Proof of Lemma 8: Let ψ be a method fitting ISMCC and MCRCSY on $\mathbb{MCM}^*$. By Lemma 7, ψ fits MCIEES on $\mathbb{MCM}^*$. Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}^*$ and $t\in\mathbb{C}_m$. It is trivial for $|\mathbb{U}|=1$ by MCIEES. Suppose that $|\mathbb{U}|\geq 2$. Consider the reduction $\big(\{a,b\},\xi_{\{a,b\}},\mathbb{V}^m_{\{a,b\},\psi}\big)$ with $a,b\in\mathbb{U}$. Therefore,

$$\nu^t_{\{a,b\},\psi}(\xi_{\{a,b\}})=\nu^t(\xi)-\sum_{j\in\mathbb{U}\setminus\{a,b\}}\psi^t_j(\mathbb{U},\xi,\mathbb{V}^m).$$

Since ψ fits MCRCSY,

$$\psi^t_s\big(\{a,b\},\xi_{\{a,b\}},\mathbb{V}^m_{\{a,b\},\psi}\big)=\psi^t_s(\mathbb{U},\xi,\mathbb{V}^m)$$

for all $s\in\{a,b\}$. Then,

$$\begin{aligned}
&\psi_a(\mathbb{U},\xi,\mathbb{V}^m)+\psi_b(\mathbb{U},\xi,\mathbb{V}^m)\\
=\ &\nu^t(\xi)-\sum_{j\in\mathbb{U}\setminus\{a,b\}}\psi^t_j(\mathbb{U},\xi,\mathbb{V}^m).
\end{aligned}$$

So, $\sum_{j\in\mathbb{U}}\psi^t_j(\mathbb{U},\xi,\mathbb{V}^m)=\nu^t(\xi)$, i.e., ψ fits MCEES. ■

Theorem 2: On $\mathbb{MCM}^*$, the IMIE is the only method fitting ISMCC and MCRCSY.

Proof of Theorem 2: By Lemma 4, the method λ^I fits MCRCSY on $\mathbb{MCM}^*$. Clearly, the method λ^I fits ISMCC.

To present the uniqueness, suppose that ψ fits ISMCC and MCRCSY. By ISMCC and MCRCSY of ψ, it is easy to clarify that ψ also fits MCEES based on Lemma 8. Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$. By ISMCC of ψ, $\psi(\mathbb{U},\xi,\mathbb{V}^m)=\lambda^I(\mathbb{U},\xi,\mathbb{V}^m)$ if $|\mathbb{U}|\leq 2$. The situation $|\mathbb{U}|>2$: Let $b\in\mathbb{U}$, $t\in\mathbb{C}_m$ and $K=\{b,p\}$ with $p\in\mathbb{U}\setminus\{b\}$.

$$\begin{aligned}
&\psi^t_b(\mathbb{U},\xi,\mathbb{V}^m)-\lambda^{I,t}_b(\mathbb{U},\xi,\mathbb{V}^m)\\
=\ &\psi^t_b(K,\xi_K,\mathbb{V}^m_{K,\psi})-\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\\
&\textbf{(MCRCSY of } \lambda^{I,t} \textbf{ and } \psi\textbf{)}\\
=\ &\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\psi})-\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I}).\\
&\textbf{(ISMCC of } \psi\textbf{)}
\end{aligned}\tag{7}$$

Similar to equation (6)

$$\begin{aligned}
\lambda^t_b(K,\xi_K,\mathbb{V}^m_{K,\psi})&=\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&=\lambda^t_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I}).
\end{aligned}\tag{8}$$

By equations (7) and (8),

$$\begin{aligned}
&\psi^t_b(\mathbb{U},\xi,\mathbb{V}^m)-\lambda^{I,t}_b(\mathbb{U},\xi,\mathbb{V}^m)\\
=\ &\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\psi})-\lambda^{I,t}_b(K,\xi_K,\mathbb{V}^m_{K,\lambda^I})\\
=\ &\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)+\lambda^t_p(\mathbb{U},\xi,\mathbb{V}^m)}\big[\nu^t_{K,\psi}(\xi_K)-\nu^t_{K,\lambda^I}(\xi_K)\big]\\
=\ &\tfrac{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)}{\lambda^t_b(\mathbb{U},\xi,\mathbb{V}^m)+\lambda^t_p(\mathbb{U},\xi,\mathbb{V}^m)}\big[\psi^t_b(\mathbb{U},\xi,\mathbb{V}^m)\\
&\quad+\psi^t_p(\mathbb{U},\xi,\mathbb{V}^m)-\lambda^{I,t}_b(\mathbb{U},\xi,\mathbb{V}^m)-\lambda^{I,t}_p(\mathbb{U},\xi,\mathbb{V}^m)\big].
\end{aligned}$$

Thus,

$$\begin{aligned} & \lambda_p^t(\mathbb{U},\xi,\mathbb{V}^m)\cdot\left[\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m)-\lambda_b^{I,t}(\mathbb{U},\xi,\mathbb{V}^m)\right] \\ = \; & \lambda_b^t(\mathbb{U},\xi,\mathbb{V}^m)\cdot\left[\psi_p^t(\mathbb{U},\xi,\mathbb{V}^m)-\lambda_p^{I,t}(\mathbb{U},\xi,\mathbb{V}^m)\right]. \end{aligned}$$

By MCEES of $\lambda^{I,t}$ and ψ,

$$\begin{aligned} & \left[\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m)-\lambda_b^{I,t}(\mathbb{U},\xi,\mathbb{V}^m)\right]\cdot\sum_{p\in\mathbb{U}}\lambda_p^t(\mathbb{U},\xi,\mathbb{V}^m) \\ = \; & \lambda_b^t(\mathbb{U},\xi,\mathbb{V}^m)\cdot\sum_{p\in\mathbb{U}}\left[\psi_p^t(\mathbb{U},\xi,\mathbb{V}^m)-\lambda_p^{I,t}(\mathbb{U},\xi,\mathbb{V}^m)\right] \\ = \; & \lambda_b^t(\mathbb{U},\xi,\mathbb{V}^m)\cdot\left[\nu^t(\xi)-\nu^t(\xi)\right] \\ = \; & 0. \end{aligned}$$

Hence, $\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m)=\lambda_b^{I,t}(\mathbb{U},\xi,\mathbb{V}^m)$ for every $b\in\mathbb{U}$ and for every $t\in\mathbb{C}_m$. ■

In the following some examples are exhibited to display that every of the properties applied in Theorem 2 is independent of the rest of properties.

Example 6: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}^*$, for every $t\in\mathbb{C}_m$ and for every element $b\in\mathbb{U}$,

$$\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m)=\begin{cases}\lambda_b^{I,t}(\mathbb{U},\xi,\mathbb{V}^m) & \text{if } |\mathbb{U}|\leq 2,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, ψ fits ISMCC, but it does not fit MCRCSY.

Example 7: Consider the method ψ as follows. For every $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}^*$, for every $t\in\mathbb{C}_m$ and for every element $b\in\mathbb{U}$, $\psi_b^t(\mathbb{U},\xi,\mathbb{V}^m)=0$. Clearly, ψ fits MCRCSY, but it does not fit ISMCC.

Based on Remark 2 and Theorem 1, the axiom of the MCRCSY is adopted to axiomatize these methods as follows.

Theorem 3:

1) On $\mathbb{MCM}$, the UMIE is the unique method fitting SMCC and MCRCSY.
2) On $\mathbb{MCM}$, the 1-WMIE is the unique method fitting 1WSMCC and MCRCSY.
3) On $\mathbb{MCM}$, the 2-WMIE is the unique method fitting 2WSMCC and MCRCSY.
4) On $\mathbb{MCM}$, the BWMIE is the unique method fitting BWSMCC and MCRCSY.

Proof of Theorem 3: By Remark 2, the methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit MCRCSY. Clearly, the methods $\overline{\lambda}$, $\lambda^{\hat{w}}$, $\lambda^{\check{w}}$, $\lambda^{\hat{w},\check{w}}$ fit SMCC, 1WSMCC, 2WSMCC and BWSMCC respectively. Similar to Theorem 1, the remaining proofs of the uniqueness for all results could be finished. ■

In the following, an instance is provide to present **(*)** how the new methods would distribute efficacy differently than the previous methods and **(**)** differently from each other. Let $(\mathbb{U},\xi,\mathbb{V}^m)\in\mathbb{MCM}$ with $\mathbb{U}=\{a,b,c\}$, $m=2$, $\xi=(2,1,1)$, $\mathbb{OB}_a=\{0,1_a,2_a\}$, $\mathbb{OB}_b=\{0,1_b\}$, $\mathbb{OB}_c=\{0,1_c\}$, $\hat{w}(a)=3$, $\hat{w}(b)=2$, $\hat{w}(c)=4$, $\check{w}(1_a)=2$, $\check{w}(2_a)=3$, $\check{w}(1_b)=6$, $\check{w}(1_c)=5$.

Further, let $\nu^1(2,1,1)=6$, $\nu^1(1,1,1)=8$, $\nu^1(2,1,0)=4$, $\nu^1(2,0,1)=3$, $\nu^1(2,0,0)=10$, $\nu^1(1,1,0)=4$, $\nu^1(1,0,1)=-5$, $\nu^1(0,1,1)=5$, $\nu^1(1,0,0)=-2$, $\nu^1(0,1,0)=3$, $\nu^1(0,0,1)=-4$, $\nu^2(2,1,1)=10$, $\nu^2(1,1,1)=4$, $\nu^2(2,1,0)=6$, $\nu^2(2,0,1)=7$, $\nu^2(2,0,0)=5$, $\nu^2(1,1,0)=-4$, $\nu^2(1,0,1)=5$, $\nu^2(0,1,1)=4$, $\nu^2(1,0,0)=8$, $\nu^2(0,1,0)=-3$, $\nu^2(0,0,1)=4$ and $\nu^1(0,0,0)=0=\nu^2(0,0,0)$. By Definitions 1–3,

$$\begin{array}{rlrl}
\overline{\lambda_a^1}(\mathbb{U},\xi,\mathbb{V}^m)= & 0, & \overline{\lambda_b^1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{7}{2},\\
\overline{\lambda_c^1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{5}{2}, & \overline{\lambda_a^2}(\mathbb{U},\xi,\mathbb{V}^m)= & 5,\\
\overline{\lambda_b^2}(\mathbb{U},\xi,\mathbb{V}^m)= & 2, & \overline{\lambda_c^2}(\mathbb{U},\xi,\mathbb{V}^m)= & 3,\\
\lambda_a^{\hat{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & 0, & \lambda_b^{\hat{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{10}{3},\\
\lambda_c^{\hat{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{8}{3}, & \lambda_a^{\hat{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & 5,\\
\lambda_b^{\hat{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{7}{3}, & \lambda_c^{\hat{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{8}{3},\\
\lambda_a^{\check{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{1}{5}, & \lambda_b^{\check{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{17}{5},\\
\lambda_c^{\check{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{12}{5}, & \lambda_a^{\check{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & 5,\\
\lambda_b^{\check{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & 2, & \lambda_c^{\check{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & 3,\\
\lambda_a^{\hat{w},\check{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{1}{5}, & \lambda_b^{\hat{w},\check{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{34}{15},\\
\lambda_c^{\hat{w},\check{w},1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{53}{15}, & \lambda_a^{\hat{w},\check{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & 5,\\
\lambda_b^{\hat{w},\check{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{7}{3}, & \lambda_c^{\hat{w},\check{w},2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{8}{3},\\
\lambda_a^{I,1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{-2}{3}, & \lambda_b^{I,1}(\mathbb{U},\xi,\mathbb{V}^m)= & 4,\\
\lambda_c^{I,1}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{8}{3}, & \lambda_a^{I,2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{60}{13},\\
\lambda_b^{I,2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{30}{13}, & \lambda_c^{I,2}(\mathbb{U},\xi,\mathbb{V}^m)= & \frac{40}{13}.
\end{array}$$

IV. Conclusions

1) Differing from existing researches, this study defines different types of weighted functions for elements and its operating behavior in a multiple-considerations multi-choice condition. Consequently, assessing methods due to the UMIE, the 1-WMIE, the 2-WMIE, and the BWMIE along with related axiomatizations are introduced. In contrast to more artificial weight functions, this study reasonably utilizes the average marginal behavior-efficacy to replace weight functions, proposing the UMIE and its related axiomatizations within the framework of multiple-considerations multi-choice conditions.
   - Assessing methods under traditional frameworks mostly focus on the participation of elements.
   - The efficacy assessing concepts of the UMIE, the 1-WMIE, the 2-WMIE, the BWMIE, and the IMIE, along with its related axiomatizations, have not been proposed in traditional frameworks or in research literature related to multiple-considerations multi-choice conditions.
     - Under the assessing concepts of the UMIE and the 2-WMIE, different types of the marginal behavior-efficacy of elements are first measured, and the remaining efficacy are evenly distributed among all elements.
     - Under the assessing concepts of the 1-WMIE and the BWMIE, different types of the marginal behavior-efficacy of elements are first measured, and the remaining efficacy are distributed among all elements based on its relative weighted proportions.
     - Under the assessing concepts of the 2-WMIE and the BWMIE, the marginal behavior-efficacy of all elements are measured considering the weighting of operating behavior, while the UMIE and the 1-WMIE do not consider the weighting of operating behavior.
     - The importance of elements and its related operating behavior under multiple-considerations multi-choice conditions is paramount. Therefore, assigning weights should consider both

elements and its related operating behavior. Under the assessing concept of the BWMIE, the average marginal weighted efficacy of elements are first measured, and then the remaining efficacy are distributed among all elements based on its relative weighted proportions.
   - However, assigning weights via weight functions may lack rationality or representativeness. Therefore, under the assessing concept of the IMIE, the average marginal behavior-efficacy of elements are first measured, and then the remaining efficacy are distributed among all elements based on relative proportions due to its average marginal behavior-efficacy.

2) The approach outlined in the proposal is interdisciplinary, combining biological experimentation with game-theoretical modeling, which is a cutting-edge field of research. The novel aspect of this proposal is integrating asymmetric game theory into the assessment of chemical toxicity on pollen tube growth, which is a more innovative and less commonly reported approach. The research is poised to offer significant advancements in environmental risk assessments, providing a nuanced understanding of the reproductive risks faced by plants. It will serve as a valuable tool for regulators and agricultural stakeholders in assessing new chemical products' environmental impact.
3) The efficacy assessing methods proposed in this paper have several advantages.
   - Related assessing methods under traditional conditions often consider whether elements are involved or not. However, under the framework of multiple-considerations multi-choice conditions considered throughout this study, all elements can adopt different operating behavior of operation depending on different situations.
   - In some studies applying multi-choice conditions, although assessing methods considered that elements have different operating behavior, they measure the interaction efficacy derived from specific elements at specific operating behavior. In contrast, the assessing concepts of this study consider the overall interaction efficacy derived from whole the operating behavior of each element.
   - To comply with real-world situations, the BWMIE simultaneously considers the weighting of elements and its operating behavior for efficacy assessing. Furthermore, considering potential concerns about the rationality or representativeness of weight functions, the IMIE employs relative average marginal efficacy instead of weighting.
4) However, the assessing methods proposed in this study have some limitations. As emphasized by the above advantages, each element can adopt different operating behavior of operation in different situations. Although it is possible to measure the overall interaction efficacy derived from the operating behavior of each element, it is unable to measure the interaction efficacy derived from specific elements at specific operating behavior. Future research directions should focus on developing extended assessing methods that simultaneously consider both overall efficacy and specific operating behavior efficacy.
5) The relevant results of this study have also led to further research motivation.
   - Is it possible to replace the EANSC with other traditional assessing methods in multiple-considerations and multi-choice considerations to derive balance efficacy assessing methods?

   The above motivation can provide avenues for further research.

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